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GEOMETRY OF GENERALIZED AFFINE SPRINGER FIBERS

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ABSTRACT

We study basic geometric properties of Kottwitz-Viehmann varieties, which are certain group analogue of affine Springer fibers. We establish a dimension formula and formulate a conjecture relating the number of irreducible components of Kottwitz-Viehmann varieties to certain weight multiplicities. The conjecture is proved in the case of split conjugacy class.

CHAPTER 1

INTRODUCTION

1.1 Background and motivation

In this article we study certain analogue of affine Springer fibres that we call *Kottwitz-Viehmann varieties* whose underlying set is defined as

$$X_\gamma^\lambda = \{g \in G(F)/G(\mathcal{O}) | g^{-1}\gamma g \in G(\mathcal{O})\varpi^\lambda G(\mathcal{O})\}$$

where

- G is a connected reductive algebraic group over a field k;
- $F = k((\varpi))$ is the field of Laurent series with coefficients in k and $\mathcal{O} = k[[\varpi]]$ is the ring of power series;
- $\gamma \in G(F)$ is a regular semisimple element;
- $\lambda : \mathbb{G}_m \to T$ is a cocharacter of a maximal torus T of G and

$$\varpi^\lambda := \lambda(\varpi) \in G(F).$$

These sets were first studied by Kottwitz and Viehmann in [KV12]. More general versions of them (replacing $G(\mathcal{O})$ by parahoric subgroups of $G(F)$) have also been studied by Lusztig in [Lus15]. When k is a finite field, they arise naturally in the study of orbital integrals of functions in the spherical Hecke algebra $\mathcal{H}(G(F), G(\mathcal{O}))$ consisting of $G(\mathcal{O})$-biinvariant locally constant functions with compact support on $G(F)$.

It turns out that X_γ^λ can be realized as the set of k-rational points of some algebraic variety over k. We view them as group analogue of affine Springer fibers for Lie algebras studied by Kazhdan and Lusztig in [KL88]:

$$X_\gamma = \{g \in G(F)/G(\mathcal{O}) | \mathrm{ad}(g)^{-1}\gamma \in \mathfrak{g}(\mathcal{O})\}.$$

Here $\mathfrak{g}$ is the Lie algebra of G, $\gamma \in \mathfrak{g}(F)$ is a regular semisimple element and "ad" denotes the adjoint action of G on $\mathfrak{g}$.

Basic geometric properties of these affine Springer fibers X_γ have been well understood through the works of Kazhdan and Lusztig [KL88], Bezrukavnikov [Bez96], Ngô [Ngô10]. A key ingredient in their approach is the symmetry on X_γ arising from the centralizer $G_\gamma(F)$. More precisely, the group $G_\gamma(F)$ has a dense open orbit X_γ^{reg} (the "regular locus") and geometric properties of X_γ^λ are reduced to the commutative algebraic group $G_\gamma(F)$ (more precisely certain finite dimensional quotient P_γ of the infinite dimensional loop group $G_\gamma(F)$).

We would like to generalize these methods to study the Kottwitz-Viehmann varieties X_γ^λ. Similar to Lie algebra case, the (connected) centralizer $G_\gamma^0(F)$ acts naturally on X_γ^λ and we consider the open orbits $X_\gamma^{\lambda,\mathrm{reg}}$ (the "regular locus"). However, there are the following notable differences from the Lie algebra situation:

- In general the action of $G_\gamma^0(F)$ on $X_\gamma^{\lambda,\mathrm{reg}}$ is not transitive.
- A more serious problem is that in general the "regular locus" $X_\gamma^{\lambda,\mathrm{reg}}$ is not dense in X_γ and there might be irreducible components disjoint from $X_\gamma^{\lambda,\mathrm{reg}}$.

Thus X_γ^λ may have more irreducible components than $X_\gamma^{\lambda,\mathrm{reg}}$. This makes it more difficult to reduce geometric properties of X_γ^λ to the commutative group $G_\gamma^0(F)$.

1.2 Main results

Our first goal is to prove a dimension formula of X_γ^λ.

Theorem 1.2.1. *X_γ^λ is a k-scheme locally of finite type with dimension*

$$\dim X_\gamma^\lambda = \langle \rho, \lambda \rangle + \frac{1}{2}(d(\gamma) - c(\gamma))$$

where

- *ρ is half sum of the positive roots for G;*
- *$d(\gamma)$ is the discriminant valuation of γ (cf. Definition 3.1.1);*
- *$c(\gamma) = \mathrm{rank}(G) - \mathrm{rank}_F(G_\gamma)$, the difference between the dimension of the maximal torus of G and the dimension of the maximal F-split subtorus of the centralizer G_γ.*

In [Bou15a] and [BC17], this theorem is proved when G is semisimple and simply-connected. In this article we prove it for any split connected reductive group.

As in the Lie algebra case, there are two major steps. First we prove the dimension formula for the regular open subset, this step generalize the method of Kazhdan-Lusztig in [KL88]. The second step is to show that

$$\dim X_\gamma^{\lambda,\mathrm{reg}} = \dim X_\gamma^\lambda$$

For this the argument of Kazhdan-Lusztig in [KL88] does not generalize, since otherwise it would imply that the complement of the regular open subset has strictly smaller dimension (see [Ngô10, Proposition 3.7.1]), which in our situation may not be true due to the possible existence of irregular components. In general, actually most components of X_γ^λ will be irregular, see Remark 3.9.3. Instead, we bypass this diffulty by studying the global analogue of Kottwitz-Viehmann varieties, the Hitchin-Frenkel-Ngô fibration. Similar ideas occured previously in [BC17].

This major difference from Lie algebra case lead us naturally to the question of determining the number of irreducible components of X_γ^λ, which is our second goal. We will formulate a conjecture on the number of irreducible components of X_γ^λ and prove the conjecture in the case where γ is an unramified (or split) conjugacy class. One formulation of the conjecture involves the Newton point $\nu_\gamma \in (X_*(T) \otimes \mathbb{Q})^+$ of γ, which is an element in the dominant rational coweight cone. By the discussion in §3.9, if X_γ^λ is nonempty, there exists a unique *smallest* dominant *integral* coweight μ such that $\nu_\gamma \leq_{\mathbb{Q}} \mu$ and $\mu \leq \lambda$.

Conjecture (Conjecture 3.9.1)**.** Let μ be as above. The number of $G_\gamma^0(F)$-orbits on the set of irreducible components of X_γ^λ equals to $m_{\lambda\mu}$, which is the dimension of μ-weight space in the irreducible representation V_λ of the Langlands dual group $\hat{G}$ with highest weight λ.

We remark that there is a similar conjecture made by Miaofen Chen and Xinwen Zhu on the irreducible components of affine Deligne-Lusztig varieties, see [HV17] and [XZ17] for statements.

In fact we will also give a conceptually better formulation of this Conjecture using the extended Steinberg base of Vinberg monoid. See Conjecture 3.9.1 for more details.

Theorem 1.2.2. *The Conjecture is true if* $\gamma \in G(F)^{\mathrm{rs}}$ *is split.*

This is proved in Corollary 3.5.2.

Remark 1.2.1. Although we restrict to equal characteristic local field, we expect that most results involving only local arguments in this paper could also be generalize to mixed characteristic Kottwitz-Viehmann varieties, which could be defined based on the work of X.Zhu [Zhu17]. However, the dimension formula in full generality involves global argument and currently it's not clear how to generalize this to mixed characteristic case. It would be interesting to see if there is a purely local argument to prove dimension formula.

1.3 Organization of the article

In §2, we review certain facts needed from the theory of reductive monoids. In §3, we prove dimension formula and the conjecture on irreducible components in the unramified case. In § 4, we review basic facts of Hitchin-Frenkel-Ngô fibration. The main result we establish in this chapter is properness of the fibration over anisotropic open subset. In § 5, we relate Kottwitz-Viehmann varieties and Hitchin-Frenkel-Ngô fibrations and finish the prove of dimension formula for X_γ^λ.

1.4 Notations and conventions

1.4.1 Group theoretic notations

We assume throughout the article that k is an algebraically closed field. $F = k((\varpi))$ and $\mathcal{O} = k[[\varpi]]$. We let G be a (split) connected reductive group over k. Assume that either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > 0$ does not divide the order of Weyl group of G.

Denote by G_{der} the derived group of G, a semisimple group of rank r. Let G^{sc} be the simply-connected cover of G_{der} and G_{ad} the adjoint group of G.

Fix a maximal torus T of G and a Borel subgroup B containing G. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots determined by $T \subset B$. Let $\check{\Lambda} := X^*(T)$ (resp. $\Lambda := X_*(T)$) be the weight (resp. coweight) lattice. Let $\check{\Lambda}^+$ (resp. Λ^+) be the set of dominant weights (resp. dominant coweights). Let W be the Weyl group of G and $S \subset W$ the set of simple reflections associated to the simple roots Δ. There is a unique longest element w_0 of W under the Bruhat order determined by S. Then w_0 is a reflection and $-w_0$ defines a bijection on the sets Δ, Λ^+ and $\check{\Lambda}^+$.

Let $\hat{G}$ be the Langlands dual group of G, viewed as a complex reductive group. For each $\lambda \in \Lambda^+$, viewed as a dominant weight for $\hat{G}$, let $V(\lambda)$ be the irreducible representation of $\hat{G}$ with highest weight λ. For any $\mu \in \Lambda^+$ with $\mu \leq \lambda$, let $m_{\lambda\mu}$ be the dimension of μ weight space in $V(\lambda)$.

1.4.2 Scheme theoretic notations

For any k-scheme X, we let L_n^+X be its n-th jet space. Then L_n^+X is the k-scheme whose set of R points is $L_n^+X(R) = X(R[t]/t^n)$ for any k algebra R. Let $L^+X := \varprojlim L_n^+X$ be the arc space and LX the loop space of X. More precisely, LX is the k-functor that associates to any k-algebra R the set $LX(R) = X(R((t)))$.

For any scheme X, we denote by $\mathrm{Irr}(X)$ the set of its irreducible components.

CHAPTER 2

REVIEW ON REDUCTIVE MONOIDS

In this chapter we summarize some results on reductive monoids needed later. We will roughly follow the exposition in [Bou15a], with several modifications and improvements. We refer the reader to [Vin95], [Rit98], [Rit01] for more backgrounds on this subject.

2.1 Construction of Vinberg monoid

In this section, we assume that G is semisimple *simply connected*.

The Vinberg monoid for G is an algebraic monoid Vin_G such that the derived group of its unit group is isomorphic to G, and it is characterized by certain nice universal properties. For our purpose, we construct it in an explicit manner as follows.

Let $\omega_1, \dots, \omega_r \in X_*(T)_+$ be the fundamental weights. For each $1 \le i \le r$, let $\rho_{\omega_i} : G \to \mathrm{GL}(V_{\omega_i})$ be the irreducible representation with highest weight ω_i.

We introduce the extended group $G_+ := (T \times G)/Z$ where Z, the center of G, embeds anti-diagonally in $T \times G$. Then G_+ is a reductive group with center $Z_+ = (T \times Z)/Z \cong T$ and derived group G. Let $T_+ = (T \times T)/Z$ be a maximal torus of G^+. We extend the representations ρ_{ω_i} representations of G_+:

$$\begin{aligned} \rho_i^+ : G_+ &\longrightarrow \mathrm{GL}(V_{\omega_i}) \\ (t, g) &\longmapsto \omega_i(t)\rho_{\omega_i}(g) \end{aligned}$$

For each $1 \le i \le r$, we also extend the simple roots α_i to $\alpha_i^+ : G^+ \to \mathbb{G}_m$ by $\alpha_i^+(t, g) = \alpha_i(t)$. Altogether, we get the following homomorphism

$$(\alpha^+, \rho^+) : G^+ \to \mathbb{G}_m^r \times \prod_{i=1}^{r} \mathrm{GL}(V_{\omega_i})$$

Definition 2.1.1. The *Vinberg monoid* of G, denoted by Vin_G, is the normalization of the closure of G_+ in the product

$$\mathbb{A}^r \times \prod_{i=1}^{r} \mathrm{End}(V_{\omega_i}).$$

Then Vin_G is an algebraic monoid with unit group G_+. It has a smooth dense open subvariety Vin_G^0 defined as the normalization of the closure of G_+ in the product

$$\mathbb{A}^r \times \prod_{i=1}^{r} (\mathrm{End}(V_{\omega_i}) - \{0\}).$$

Definition 2.1.2. The *abelianization* of the monoid Vin_G is the invariant quotient

$$A_G := \mathrm{Vin}_G//(G \times G)$$

Let $\alpha : \mathrm{Vin}_G \to A_G$ be the natural map.

Using the maps α^+ we get a canonical isomorphism $A_G \cong \mathbb{A}^r$. The adjoint torus T_{ad} embeds via the simple roots as the open subset where all the r-coordinates are nonzero.

Note that the fibers of α over points in T_{ad} are isomorphic to G. One can construct a canonical section of the abelianization map α as follows.

Let T_{diag} be the image of the diagonal embedding $T \to T_+$. Then there is a canonical isomorphism $T_{\mathrm{diag}} \cong T_{\mathrm{ad}}$ which extends to an isomorphism $\overline{T_{\mathrm{diag}}} \cong A_G$ between the closure of T_{diag} in Vin_G and A_G. The inverse of this isomorphism defines a section of the abelianization map α, which we denote by

$$\mathfrak{s} : A_G \to \mathrm{Vin}_G \tag{2.1.1}$$

The group $G_+ \times G_+$ acts by left and right multiplication on Vin_G. More precisely, for all $(x, y) \in G_+ \times G_+$ and $\gamma \in \mathrm{Vin}_G$, the action is given by $(x, y) \cdot \gamma = x\gamma y^{-1}$. The $G_+ \times G_+$-orbits on Vin_G corresponds bijectively to pairs (I, J) of subsets of Δ such that no connected component (in the sense of Dynkin diagram) of the complement of J is entirely contained in

I. Each orbit $O_{I,J}$ containes an idempotent $e_{I,J} \in \mathrm{Vin}_G$, defined up to conjugation. We can choose $e_{I,J} \in \overline{T_+}$, the closure of T_+ in Vin_G. Then it is well-defined up to W-conjugation.

Fix such a pair (I, J). Let J^c be the complement of J in Δ and J^0 be the interior of J, i.e. the elements in J that is not connected to any element of J^c in the Dynkin diagram. Let $M := I \cap J^0 \sqcup J^c$. Let $P_+(M)$ the corresponding standard parabolic subgroup of G_+. Let $P_+(M)^-$ be the opposite of $P_+(M)$ and $L(M)$ their common Levi subgroup. Denote by $\delta : P_+(M) \to L_+(M)$ and $\delta_- : P_+(M)^- \to L_+(M)$ the canonical projections.

Lemma 2.1.1. *The stabilizer of $e_{I,J}$ under $G_+ \times G_+$ is the subgroup of $P_+(M) \times P_+(M)_-$ consisting of pairs (g, g_-) such that*

$$\delta(g) \equiv \delta(g_-) \mod L_+(J^c)_{\mathrm{der}} T_{I,J}$$

where $T_{I,J}$ is a subtorus of T_+.

2.2 Adjoint quotient

We keep the assumption that G is semisimple simply-connected. The adjoint action of G on the Vinberg monoid Vin_G is the restriction of left and right multiplication by $G \times G$ along the diagonal. In other words, for any $g \in G$ and $\gamma \in \mathrm{Vin}_G$, the adjoint action is defined by $\mathrm{Ad}(g)(\gamma) := g\gamma g^{-1}$. Note that this action factors through the adjoint group G_{ad}.

For any $\gamma \in \mathrm{Vin}_G$, we let G_γ be the centralizer of γ in G, i.e. the stabilizer of γ under the adjoint action of G. If $\gamma \in G_+$ belongs to the unit group of Vin_G, we know that $\dim G_\gamma \geq \dim T = r$. By upper-semicontinuity of stabilizer dimension (cf. [ABD+65, VI B.4, Prop. 4.1]), we see that $\dim G_\gamma \geq \dim T$ for all $\gamma \in \mathrm{Vin}_G$.

Definition 2.2.1. An element $\gamma \in \mathrm{Vin}_G$ is *regular* if $\dim G_\gamma = r$ (i.e. smallest possible). Let $\mathrm{Vin}_G^{\mathrm{reg}} \subset \mathrm{Vin}_G$ be the open subset consisting of regular elements.

Definition 2.2.2. The *extended Steinberg base* $\mathfrak{C}_+ := \mathrm{Vin}_G//\mathrm{Ad}(G)$ is defined to be the invariant quotient. Let

$$\chi_+ : \mathrm{Vin}_G \to \mathfrak{C}_+$$

be the canonical quotient map.

The functions α_i^+ define a canonical map $\beta : \mathfrak{C}_+ \to A_G$ so that $\alpha = \beta \circ \chi_+$. The following result is [Bou15a, Proposition 1.7]:

Theorem 2.2.1. *The closed embedding $\overline{T_+} \subset \mathrm{Vin}_G$ induces an isomorphism $\overline{T_+}//W \cong \mathfrak{C}_+$. Moreover, the functions α_+ and $\mathrm{Tr}(\rho_i^+)$ define isomorphism*

$$\mathfrak{C}_+ \cong A_G \times \mathbb{A}^r \cong \mathbb{A}^{2r}.$$

The canonical projection $q : \overline{T_+} \to \mathfrak{C}_+$ is a finite flat, generically Galois étale with Galois group W.

2.2.1 Nilpotent cone

Our exposition in this part follow a suggestion of Xinwen Zhu. Let $\mathcal{N} := \chi_+^{-1}(0)$ be the *nilpotent cone* in the Vinberg monoid Vin_G. Let $\mathcal{N}^0 := \mathcal{N} \cap \mathrm{Vin}_G^0$ and $\mathcal{N}^{\mathrm{reg}} := \mathcal{N} \cap \mathrm{Vin}_G^{\mathrm{reg}}$ be the corresponding open subsets.

For any subset $J \subset \Delta$, denote $J^c := \Delta \setminus J$, then we have

$$O_{\varnothing,J} \cong (G/G_{J^c}U_{J^c} \times G/G_{J^c}U_{J^c}^-)/Z(L_{J^c}).$$

where $Z(L_{J^c})$, the center of the Levi L_{J^c} acts diagonally on the product. There is a canonical map

$$\pi_{\varnothing,J} : O_{\varnothing,J} \to G/P_{J^c} \times G/P_{J^c}^-$$

The diagonal G-orbits on the product $G/P_{J^c} \times G/P_{J^c}^-$ corresponds bijectively to ${}^{J^c}W^{J^c}$. The element $w \in {}^{J^c}W^{J^c}$ corresponds to the G-orbit of $(\dot{w}, 1)$ for any representative $\dot{w}$ of w

in G. We denote this G-orbit by $Y_{\varnothing,J,w}$ and let $X_{\varnothing,J,w}$ be its inverse image under $\pi_{\varnothing,J}$. Then we have

$$X_{\varnothing,J,w} = \mathrm{Ad}(G)(Z(L_{J^c})\dot{w}e_{\varnothing,J}). \tag{2.2.1}$$

The G-orbit $Y_{\varnothing,J,w}$ has codimension $l(w)$ in $G/P_{J^c} \times G/P_{J^c}^-$. Hence we have

$$\begin{aligned} \dim X_{\varnothing,J,w} &= 2\dim(G/P_{J^c}) - l(w) + \dim Z(L_{J^c}) \\ &= \dim G - \dim L_{J^c} - l(w) + |J|. \end{aligned} \tag{2.2.2}$$

Let $S = \{s_1, \ldots, s_r\}$ be the set of simple reflections in W corresponding to our choice of simple roots Δ. Let $l : W \to \mathbb{N}$ be the length function determined by S. For each $w \in W$, let $\mathrm{Supp}(w) \subset S$ be the subset consisting of those simple reflections which occurs in one (and hence every) reduced word expression of w.

Definition 2.2.3. An element $w \in W$ is called an *S-Coxeter element* if it can be written as products of simple reflections in S, each occuring precisely once. In particular, $l(w) = r$ and $\mathrm{Supp}(w) = S$. Denote by $\mathrm{Cox}(W, S)$ the set of S-Coxeter elements in W.

In general, an element $w \in W$ is called a *Coxeter element* if it is conjugate to an S-Coxeter element in W.

Lemma 2.2.2. *$X_{\varnothing,J,w} \subset \mathcal{N}$ if and only if $J \subset \mathrm{Supp}(w)$.*

Proof. First suppose $X_{\varnothing,J,w} \subset \mathcal{N}$. Then in particular $\dot{w}e_{\varnothing,J} \in \mathcal{N}$. Recall that the idempotent $e_{\varnothing,J}$ acts as projector to highest weight space in the representation V_{ω_i} if $i \in J$ and acts by 0 if $i \notin J$. If there exists $j \in J$ but $j \notin \mathrm{Supp}(w)$, then $\rho_{\omega_j}(\dot{w})$ preserves the highest weight space in V_{ω_j} and hence $\mathrm{Tr}(\rho_{\omega_j}(\dot{w}e_{\varnothing,J})) \neq 0$, contradiction the assumption that $\dot{w}e_{\varnothing,J} \in \mathcal{N}$.

Conversely suppose that $J \subset \mathrm{Supp}(w)$. Let $x = t\dot{w}e_{\varnothing,J}$ where $t \in Z(L_{J^c}) \subset T$. Then $\rho_{\omega_i}(x) = 0$ if $i \notin J$. If $i \in J$, so $i \in \mathrm{Supp}(w)$, then by a standard result in root system we have $w(\omega_i) \neq \omega_i$ (see, for example [HT06, Lemma 3.5]). Thus we have $\mathrm{Tr}(\rho_{\omega_i}(x)) = 0$ as

$t \in T$ preserve the weight spaces and $\dot{w}$ maps the highest weight space into the weight space with weight $w(\omega_i)$. Thus $x \in \mathcal{N}$. $\square$

Corollary 2.2.3. *(a) There is a stratification of $\mathcal{N}$ into $\mathrm{Ad}(G)$-stable pieces*

$$\mathcal{N} = \bigsqcup_{J \subset \Delta} \bigsqcup_{\substack{w \in {}^{J^c}W^{J^c} \\ \mathrm{Supp}(w) \supset J}} X_{\varnothing, J, w}.$$

(b) $\mathcal{N}^0 = \bigsqcup_{w \in W} X_{\varnothing, \Delta, w}$.

(c) For each $w \in \mathrm{Cox}(W, S)$(cf. Definition 2.2.3), $X_{\varnothing,\Delta,w}$ is a single $\mathrm{Ad}(G)$-orbit and $\mathcal{N}^{\mathrm{reg}} = \bigsqcup_{w \in \mathrm{Cox}(W,S)} X_{\varnothing,\Delta,w}$. *In particular $\mathcal{N}^{\mathrm{reg}} \subset \mathcal{N}^0$.*

(d) $\dim \mathcal{N} = \dim \mathcal{N}^{\mathrm{reg}} = \dim G - r$ and the dimension of the complement $\mathcal{N} \setminus \mathcal{N}^{\mathrm{reg}}$ is strictly less than $\dim \mathcal{N}$.

Proof. Part (a) and (b) are immediate from Lemma 2.2.2. For each strata $X_{\varnothing,J,w} \subset \mathcal{N}$ as in Lemma 2.2.2, we have $l(w) \geq |J|$ since $J \subset \mathrm{Supp}(w)$. From (2.2.2) we see that

$$\dim X_{\varnothing,J,w} \geq \dim G - \dim L_{J^c} \geq \dim G - r$$

and equality is reached precisely when $J = \Delta$ and $l(w) = r$. This condition means that $w \in \mathrm{Cox}(W, S)$. Hence part (d) follows from part (c).

It remains to show that for each $w \in \mathrm{Cox}(W, S)$, $X_{\varnothing,\Delta,w}$ is a single $\mathrm{Ad}(G)$-orbit. By (2.2.1), we have

$$X_{\varnothing,\Delta,w} = \mathrm{Ad}(G)(Twe_{\varnothing,\Delta}).$$

So it suffices to show that for each $t \in T$, the elements $t\dot{w}e_{\varnothing,\Delta}$ and $\dot{w}e_{\varnothing,\Delta}$ are conjugate. Since w is a Coxeter element, by [Ste65, Lemma 7.6] there exists $s \in T$ such that $t = s^{-1}\dot{w}s\dot{w}^{-1}$. This implies that $s^{-1}\dot{w}e_{\varnothing,\Delta}s = t\dot{w}e_{\varnothing,\Delta}$ since $s, t \in T$ and hence commute with $e_{\varnothing,\Delta}$. $\square$

Remark 2.2.1. Another way to show that $X_{\varnothing,\Delta,w}$ consists of a single $\mathrm{Ad}(G)$-orbit is to show that the centralizer of $we_{\varnothing,\Delta}$ in G has dimension r, i.e. $we_{\varnothing,\Delta} \in \mathcal{N}^{\mathrm{reg}}$. For then the $\mathrm{Ad}(G)$-orbit of $we_{\varnothing,\Delta}$ is contained in the irreducible set $X_{\varnothing,\Delta,w}$ and has the same dimension, thus equals to $X_{\varnothing,\Delta,w}$.

Corollary 2.2.4. *The morphism $\chi_+ : \mathrm{Vin}_G \to \mathfrak{C}_+$ is flat.*

Proof. There exists a nonempty open subset $U \subset \mathfrak{C}_+$ such that the fibres of χ_+ over U have dimension $\dim \mathrm{Vin}_G - \dim \mathfrak{C}_+ = \dim G - r$. Since χ_+ is Z_+ equivariant, U is Z_+-stable. By Corollary 2.2.3(d) we know that $0 \in U$ and hence we have $U = \mathfrak{C}_+$. By [BK05, 6.2.9], Vin_G is Cohen-Macaulay. Moreover, $\mathfrak{C}_+ \cong \mathbb{A}^{2r}$ is regular and hence χ_+ is flat. □

Corollary 2.2.5. $\mathrm{Vin}_G^{\mathrm{reg}} \subset \mathrm{Vin}_G^0$.

Proof. Let $F := \mathrm{Vin}_G^{\mathrm{reg}} \setminus \mathrm{Vin}_G^0$. By Corollary 2.2.3(c), we have $\mathcal{N}^{\mathrm{reg}} \subset \mathcal{N}^0$ and hence $F \cap \mathcal{N} = \varnothing$. On the other hand, F is a Z_+-stable closed subset of $\mathrm{Vin}_G^{\mathrm{reg}}$, so we must have $F = \varnothing$. □

Proposition 2.2.6. *The nilpotent cone $\mathcal{N}$ is connected and equidimensional. Moreover, there exist bijections*

$$\mathrm{Cox}(W,S) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{N}^{\mathrm{reg}}) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{N}^0) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{N})$$

which send $w \in \mathrm{Cox}(W,S)$ to the irreducible component containing $\dot{w}e_{\varnothing,\Delta}$.

Proof. Since χ_+ is flat, its fibre $\mathcal{N} = \chi_+^{-1}(0)$ is equidimensional. Since χ_+ is the invariant quotient under a reductive group, there is a unique closed orbit in $\mathcal{N}$, namely $0 \in \mathcal{N}$. In particular, $\mathcal{N}$ is connected.

From Corollary 2.2.3, we see that $\mathcal{N}^{\mathrm{reg}}$ is dense in $\mathcal{N}^0$ and $\mathcal{N}$ and $\mathrm{Irr}(\mathcal{N}^{\mathrm{reg}})$ is in bijection with $\mathrm{Cox}(W,S)$. Hence $\mathrm{Irr}(\mathcal{N}^0)$ and $\mathrm{Irr}(\mathcal{N})$ are also in bijection with $\mathrm{Cox}(W,S)$. □

Remark 2.2.2. As a comparison, note that in the Lie algebra case, the fibers of the map $\mathfrak{g}^{\mathrm{reg}} \to \mathfrak{c}$ are irreducible and consist of a single adjoint G-orbit.

2.2.2 Discriminant divisor

Recall that on T we have the discriminant function

$$\mathrm{Disc}(t) := \prod_{\alpha\in\Phi}(1-\alpha(t))$$

which is W-equivariant and descends to a regular function on the Steinberg base $\mathfrak{C} := T//W$. We extend the function Disc to a function Disc_+ on $T_+^{\mathrm{sc}} = (T\times T)/Z_G$ by

$$\mathrm{Disc}_+(t_1,t_2) := 2\rho(t_1)\mathrm{Disc}(t_2).$$

Then Disc_+ extends to a regular function on $\overline{T_+}$, which further descends to a regular function on $\mathfrak{C}_+$. The vanishin loci of Disc_+ is a principal divisor on $\mathfrak{C}_+$ which we call *extended discriminant divisor* and denote by $\mathfrak{D}_+$.

From the definition, we see that Disc_+ is an eigenfunction for the Z_+-action on $\overline{T_+}$ and $\mathfrak{C}_+$, with eigen-value 2ρ. Hence the subschemes $\mathfrak{D}_+$ is Z_+-invariant.

For $t_+ = (t,t^{-1}) \in T_{\mathrm{diag}} \subset T_+$, we have

$$\begin{aligned} D_+(t_+) &= 2\rho(t)\prod_{\alpha\in\Phi_+}(1-\alpha(t))(1-\alpha(t^{-1})) \\ &= (-1)^{|\Phi_+|}\prod_{\alpha\in\Phi_+}(1-\alpha(t))^2 \end{aligned} \tag{2.2.3}$$

For each $\alpha\in\Phi_+$, $D_\alpha := (1-\alpha(t))^2$ extends to a polynomial function on $\overline{T_{\mathrm{diag}}} \cong \mathbb{A}^r$.

2.2.3 Adjoint orbits in extended Steinberg fibre

An element $\gamma\in\mathrm{Vin}_G$ is called *semisimple* if it is G-conjugate to an element in $\overline{T_+}$. Let $\mathrm{Vin}_G^{\mathrm{rs}}$ be the subset of Vin_G consisting of elements that are *both regular and semisimple*.

Lemma 2.2.7. *The centralizer of any semisimple element $\gamma\in\mathrm{Vin}_G$ in G is a Levi subgroup* *of G.*

Proof. We may assume that $\gamma \in \overline{T_+}$ so that $\gamma = te_{I,J}$ for some $t \in T_+$ and idempotent $e_{I,J}$.

For any $g \in G_+$, we have $g\gamma g^{-1} = \gamma$ if and only if

$$t^{-1}gte_{I,J}g^{-1} = e_{I,J}.$$

By the description of the stabilizer of $e_{I,J}$ under the action of $G_+ \times G_+$, we see that $g \in (G_+)_\gamma$ if and only if the following 2 conditions are satisfied:

- $(t^{-1}gt, g) \in P_M \times P_M^-$;
- $\delta(t^{-1}gt)\delta_-(g)^{-1} \in (L_{J^c})_{\mathrm{der}}T_{I,J}$.

Here $M := I \cap J^0 \sqcup J^c$. Since $t \in L_M$, the first condition implies that $g \in L_M$. Since the roots in $I \cap J^0$ and J^c are orthogonal to each other, the second condition implies that $(G_+)_\gamma$ is the subgroup of L_M generated by T_+, L_{J^c} and the centralizer of t in $L_{I\cap J^0}$. This shows that $(G_+)_\gamma$ is a Levi subgroup of G_+ and hence G_γ is a Levi subgroup of G. □

Lemma 2.2.8. *For any closed point $c \in \mathfrak{C}_+$, the fibre $\chi_+^{-1}(c)$ is connected and equidimensional of dimension $\dim G - r$. The open $\mathrm{Ad}(G)$-orbits in $\chi_+^{-1}(c)$ are precisely the regular conjugacy classes in $\chi_+^{-1}(c)$. On the other hand, there is a* unique *closed $\mathrm{Ad}(G)$-orbit in $\chi_+^{-1}(c)$ which is also the unique semisimple conjugacy class in $\chi_+^{-1}(c)$.*

Proof. By Corollary 2.2.4, χ_+ is flat. Hence $\chi_+^{-1}(c)$ is equidimensional of dimension $\dim G - r$. Since χ_+ is the invariant quotient by the reductive group G, there is a unique closed orbit in $\chi_+^{-1}(c)$. This closed orbit is connected since G is connected. Consequently $\chi_+^{-1}(c)$ is also connected.

The regular conjugacy classes in $\chi_+^{-1}(c)$ are locally closed subsets of the same dimension as $\chi_+^{-1}(c)$. Hence they are precisly the open $\mathrm{Ad}(G)$-orbits in $\chi_+^{-1}(c)$.

Finally by [Ren88], closed $\mathrm{Ad}(G)$-orbits are precisely the semisimple conjugacy classes. □

Unlike the group case, there might be more than one regular conjugacy class in an extended Steinberg fibre $\chi_+^{-1}(c)$, as we see in Proposition 2.2.6 for the nilpotent cone $\mathcal{N} = \chi_+^{-1}(0)$. On the other hand, regular semisimple conjugacy classes are the only $\mathrm{Ad}(G)$ orbit in the extended Steinberg fibre they live in. We give another characterization of regular semisimple conjugacy classes using the discriminant function Disc_+. The following is a generalization of [Bou15a, 2.19]

Proposition 2.2.9. *Denote $\overline{T_+}^{\mathrm{reg}} := \overline{T_+} \cap \mathrm{Vin}_G^{\mathrm{reg}}$. For any $\gamma \in \overline{T_+}$, the following are equivalent:*

1. $\gamma \in \overline{T_+}^{\mathrm{reg}}$*;*
2. $\mathrm{Disc}_+(\gamma) \neq 0$*;*
3. *The map $q : \overline{T_+} \to \mathfrak{C}_+$ is étale at γ;*
4. $G_\gamma = T$*.*

Proof. (1)⇒(2): Suppose $\gamma \in \overline{T_+}^{\mathrm{reg}}$. By Corollary 2.2.5, we have $\gamma \in \mathrm{Vin}_G^0 \cap \overline{T_+}$. After conjugation and multiplying by the center Z_+, we may assume that $\gamma \in \overline{T_{\mathrm{diag}}}$. If $\mathrm{Disc}_+(\gamma) = 0$, then there exists $\alpha \in \Phi_+$ such that $D_\alpha(\gamma) = 0$. This implies that γ lies in the closure of the diagonal embedding of $\ker(\alpha)$. Since the centralizers of elements in $\ker(\alpha)$ have dimension at least $r+1$, the same is true for G_γ by upper semicontinuity of centralizer dimension. This contradicts the assumption that γ is regular and we must have $\mathrm{Disc}_+(\gamma) \neq 0$.

(1)⇔(3)⇔(4): Since $\mathfrak{C}_+ = \overline{T_+}//W$, the finite cover $q : \overline{T_+} \to \mathfrak{C}_+$ is étale at γ if and only if the stabilizer of γ in W is trivial, which is equivalent to the fact $G_\gamma = T$ since G_γ is a standard Levi subgroup of G by the proof of Lemma 2.2.7.

(2)⇒(1): Let $V \subset \overline{T_+}$ be the open subset where Disc_+ is nonzero and we need to show that $V = \overline{T_+}^{\mathrm{reg}}$. In the implication "(1)⇒(2)" we proved that $\overline{T_+}^{\mathrm{reg}} \subset V$.

Consider the stratification of $\overline{T_+}$ induced by the T_{ad}-orbits on $A_G = \mathbb{A}^r$. The open strata is T_+, the unit group of $\overline{T_+}$. The codimension 1 stratas are described as follows: for each $1 \leq i \leq r$, let $\mathcal{O}_i$ be the codimension 1 strata consisting of $x \in \overline{T_+}$ such that the i-th coordinate of $\alpha(x)$ vanishes and the other coordinates are nonzero. Consider the complement $F := V \setminus \overline{T_+}^{\mathrm{reg}}$, which is a closed subset of V. It is a classical fact that $F \cap T_+ = \varnothing$. Also, we have $e_{\varnothing,\Delta} \in \overline{T_+}^{\mathrm{reg}}$ by direct calculation of its centralizer. Hence $e_{\varnothing,\Delta}$ lies in the closure $\overline{\mathcal{O}_i}$ for all $1 \leq i \leq r$. This shows that the generic point of $\mathcal{O}_i$ lies in $\overline{T_+}^{\mathrm{reg}}$ for all i, which implies that F has codimension at least 2 in $\overline{T_+}$. But by the equivalence "(1)$\Leftrightarrow$(3)" we just proved and purity of branch locus (see, for example [Sta17, Tag 0BMB]), the complement $\overline{T_+} \setminus \overline{T_+}^{\mathrm{reg}}$ is pure of codimension 1 in $\overline{T_+}$. This forces F, an open subset of $\overline{T_+} \setminus \overline{T_+}^{\mathrm{reg}}$ to be empty and hence $V = \overline{T_+}^{\mathrm{reg}}$. □

Corollary 2.2.10. $\mathrm{Vin}_G^{\mathrm{rs}} = \chi_+^{-1}(\mathfrak{C}_+ \setminus \mathfrak{D}_+)$. *Moreover, G acts transitively on each fibre of χ_+ over $\mathfrak{C}_+ \setminus \mathfrak{D}_+$.*

Proof. By Proposition 2.2.9, we have $\mathrm{Vin}_G^{\mathrm{rs}} \subset \chi_+^{-1}(\mathfrak{C}_+ \setminus \mathfrak{D}_+)$.

Let $c \in \mathfrak{C}_+ \setminus \mathfrak{D}_+$. By Lemma 2.2.8 and Proposition 2.2.9, the unique closed orbit in $\chi_+^{-1}(c)$ is also open. Hence $\chi^{-1}(c)$ is a single $\mathrm{Ad}(G)$-orbit consisting of elements that are both regular and semisimple. This proves the inverse inclusion. □

For this reason, we denote $\mathfrak{C}_+^{\mathrm{rs}} := \mathfrak{C}_+ \setminus \mathfrak{D}_+$ and call it the *regular semisimple* open subset of $\mathfrak{C}_+$.

2.2.4 Extended Steinberg section

For each S-Coxeter element $w \in \mathrm{Cox}(W,S)$ (cf. Definition 2.2.3), each choice of representatives $\dot{s}_i \in N_G(T)$ of the simple roots s_i, Steinberg defines a section $\epsilon^w : \mathfrak{C}_G \to G$ of the adjoint quotient map $\chi_G : G \to \mathfrak{C}_G$. Moreover, it is shown that the equivalence class of ϵ^w depends neither on w nor the choices $\dot{s}_i$, see [Ste65, 7.5 and 7.8]. Here we say that two sections ϵ, ϵ' are *equivalent* if for all $a \in \mathfrak{C}_G$, $\epsilon(a)$ and $\epsilon'(a)$ are conjugate under G.

Following [Bou15a], we extend the Steinberg sections ϵ^w to the Vinberg monoid Vin_G as follows. For each $(b,a) \in \mathfrak{C}_+ \cong \mathbb{A}^{2r}$ where $b \in A_G \cong \mathbb{A}^r$, define a map

$$\epsilon_+^w : \mathfrak{C}_+ \to \mathrm{Vin}_G$$

by $\epsilon_+^w(b,a) := \epsilon^w(a)\mathfrak{s}(b)$ where $\mathfrak{s} : A_G \to \mathrm{Vin}_G$ is the section of the abelianization map α defined in § 2.1.1.

Proposition 2.2.11. *The map* ϵ_+^w *is a section of the adjoint quotient* $\chi_+ : \mathrm{Vin}_G \to \mathfrak{C}_+$*. Moreover, the image of* ϵ_+^w *is contained in* $\mathrm{Vin}_G^{\mathrm{reg}}$.

Proof. The first statement is [Bou15a, Proposition 1.10]. The second statement is Proposition 1.16 in *loc. cit.* □

Remark 2.2.3. For each $w \in \mathrm{Cox}(W,S)$, the equivalence class of the extended section ϵ_+^w is independant of the choice of representatives $\dot{s}_i$ of the simple reflections. However, for two *different* $w, w' \in \mathrm{Cox}(W,S)$, the sections ϵ_+^w and $\epsilon_+^{w'}$ are not equivalent since, as we will see, $\epsilon_+^w(0)$ and $\epsilon_+^{w'}(0)$ are not conjugate.

Next we examine the interaction of the extended Steinberg section ϵ_+^w with the action of the central torus Z_+.

To this end, we drop the semisimple simply connected assumption and allow G to be any connected reductive group. Then the adjoint action of G_{ad} on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ induces an action of G on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ which we also denote by "Ad". Let $\mathfrak{C}_+ = \mathrm{Vin}_{G^{\mathrm{sc}}}//\mathrm{Ad}(G) = \mathrm{Vin}_{G^{\mathrm{sc}}}//\mathrm{Ad}(G^{\mathrm{sc}})$ be the extended Steinberg base for $\mathrm{Vin}_{G^{\mathrm{sc}}}$. The central torus $Z_+^{\mathrm{sc}} = T^{\mathrm{sc}}$ acts naturally on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ and $\mathfrak{C}_+$ such that the morphism $\chi_+ : \mathrm{Vin}_{G^{\mathrm{sc}}} \to \mathfrak{C}_+$ is T^{sc}-equivariant. Hence χ_+ induces a morphism between stacks

$$[\chi_+] : [\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G) \times T^{\mathrm{sc}})] \to [\mathfrak{C}_+/T^{\mathrm{sc}}] \tag{2.2.4}$$

We would like to see if ϵ_+^w induces a section $[\chi_+]$. It turns out that this is not true in general. To remedy it we consider the homomorphism $\psi : T^{\mathrm{sc}} \to G_{\mathrm{ad}}$ defined as the following composition

$$\psi : T^{\mathrm{sc}} \xrightarrow{\omega_\bullet} \mathbb{G}_m^r \xrightarrow{\mathfrak{s}} G_+^{\mathrm{sc}} \to G_{\mathrm{ad}} \tag{2.2.5}$$

where the first arrow is $\omega_\bullet := (\omega_1, \ldots, \omega_r)$, the second arrow is induced by the canonical section of the abelianization α (cf. Equation 2.1.1) and the third arrow is the canonical quotient morphism.

Consider the action of $T^{\mathrm{sc}} \times T^{\mathrm{sc}}$ on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ where the first copy of T^{sc} acts by composing ψ with the adjoint action of G_{ad} and the second copy of T^{sc} acts as central torus. In [Bou15a, Proposition 1.11], by examining the action on weight vectors of fundamental representations, it is shown that for all $a_+ \in \mathfrak{C}_+$ and $z \in T^{\mathrm{sc}}$ we have

$$\epsilon_+^w(z \cdot a_+) = z \cdot \mathfrak{s}(\omega_\bullet(z))\epsilon_+^w(a_+)\mathfrak{s}(\omega_\bullet(z))^{-1}$$

This shows that ϵ_+^w is equivariant with respect to the diagonal embedding $T^{\mathrm{sc}} \to T^{\mathrm{sc}} \times T^{\mathrm{sc}}$ and hence induces a morphism

$$[\mathfrak{C}_+/T^{\mathrm{sc}}] \to [\mathrm{Vin}_{G^{\mathrm{sc}}}/\psi(T^{\mathrm{sc}}) \times T^{\mathrm{sc}}]$$

If $G = G_{\mathrm{ad}}$, then this leads to a section $[\epsilon_+^w]$ of $[\chi_+]$. In general, let $c = |Z(G_{\mathrm{der}})|$ be the order of the center of the derived group G_{der}. Then by extracting c-th roots, we would get a lifting $\psi_{[c]} : T^{\mathrm{sc}} \to G_{\mathrm{der}} \subset G$ of ψ. More precisely, $\psi_{[c]}$ is defined by the following commutative diagram

$$\begin{tikzcd} T^{\mathrm{sc}} \arrow[r, "\psi_{[c]}"] \arrow[d, "c"] & G \arrow[d] \\ T^{\mathrm{sc}} \arrow[r, "\psi"] & G_{\mathrm{ad}} \end{tikzcd}$$

where the left vertical map is raising to c-th power.

The c-th power map $T^{\mathrm{sc}} \to T^{\mathrm{sc}}$ induces a morphism between classifying stacks $\mathbb{B}T^{\mathrm{sc}} \to$

$\mathbb{B}T^{\mathrm{sc}}$. Base changing $[\chi_+]$ along this map, we obtain a Cartesian diagram

$$\begin{tikzcd}
{[\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G)\times T^{\mathrm{sc}})]_{[c]}} \arrow[r] \arrow[d, "{[\chi_+]_{[c]}}"'] & {[\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G)\times T^{\mathrm{sc}})]} \arrow[d, "{[\chi_+]}"] \\
{[\mathfrak{C}_+/T^{\mathrm{sc}}]_{[c]}} \arrow[r] & {[\mathfrak{C}_+/T^{\mathrm{sc}}]}
\end{tikzcd}$$

where on the left, the T^{sc} action is the composition of the c-th power map and the usual action.

Proposition 2.2.12. *The map ϵ_+^w induces a section $\epsilon_{+,[c]}^w : [\mathfrak{C}_+/T^{\mathrm{sc}}]_{[c]} \to [\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G)\times T^{\mathrm{sc}})]_{[c]}$ of $[\chi_+]_{[c]}$ whose image lies in the open substack*

$$[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/(\mathrm{Ad}(G)\times T^{\mathrm{sc}})]_{[c]}$$

Proof. By what we have discussed, ϵ_+^w induces a morphism

$$[\mathfrak{C}_+/T^{\mathrm{sc}}]_{[c]} \to [\mathrm{Vin}_{G^{\mathrm{sc}}}/\psi_{[c]}(T^{\mathrm{sc}})\times T^{\mathrm{sc}}]$$

where on the right, the second copy of T^{sc} acts by composing the c-th power map and the usual action. Since $\psi_{[c]}(T^{\mathrm{sc}})\subset G$, there is a canonical morphism

$$[\mathrm{Vin}_{G^{\mathrm{sc}}}/\psi_{[c]}(T^{\mathrm{sc}})\times T^{\mathrm{sc}}] \to [\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G)\times T^{\mathrm{sc}})]_{[c]}.$$

Composing the two morphisms above we obtain the morphism $\epsilon_{+,[c]}^w$ with the desired property. $\square$

2.3 Regular centralizer for the group

In this section we let (G, G') be a pair of connected reductive groups equipped with an isomorphism of their derived groups $G_{\mathrm{ad}} \cong G'_{\mathrm{ad}}$. Assume moreover that the derived group of G is simply connected. Then there is a natural adjoint action of G' on G and the action factors through $G'_{\mathrm{ad}} \cong G_{\mathrm{ad}}$. Let $\mathfrak{C}_G := G//\mathrm{Ad}(G')$ be the invariant quotient. Then there is a canonical isomorphism $\mathfrak{C}_G \cong T//W$. The natural map $T \to \mathfrak{C}_G$ is finite flat and its restriction to $\mathfrak{C}_G^{\mathrm{rs}}$ is a Galois étale cover with Galois group W.

Consider the centralizer group scheme $I_{G'}$ over G defined by

$$I_{G'} := \{(g,x) \in G' \times G | \mathrm{Ad}(g)x = x\}.$$

In other words, the fiber of $I_{G'}$ over $x \in G$ is the centralizer G'_x of x in G'. Since the derived group of G is simply connected, G'_x is connected for semisimple $x \in G$ If moreover $x \in G^{\mathrm{rs}}$ is regular semisimple, then G'_x is a maximal torus in G'. More generally, the restriction $I_{G'}|_{G^{\mathrm{reg}}}$ to the regular open subscheme G^{reg} is a smooth commutative group scheme of relative dimension $\dim(T)$. The following lemma is the group version of [Ngô10, Lemme 2.1.1]

Lemma 2.3.1. *There exists a unique smooth commutative group scheme $J_{G'}$ over $\mathfrak{C}_G$ such that we have a G'-equivariant isomorphism*

$$(\chi^* J_{G'})_{G^{\mathrm{reg}}} \cong I_{G'}|_{G^{\mathrm{reg}}}.$$

Moreover, this isomorphism extends uniquely to a homomorphism $\chi^* J_{G'} \to I_{G'}$.

Proof. The proof of [Ngô10, Lemme 2.1.1] works in the group case. For the last statement, we use the fact that the complement of G^{reg} in G has codimension at least 2, c.f. [Ste65]. □

Fix a maximal torus $T' \subset G'$. Consider the Weil restriction of the torus $T' \times T$ on T to $\mathfrak{C}_G$:

$$\Pi_G := \Pi_{T/\mathfrak{C}_G}(T' \times T).$$

In other words, for any $\mathfrak{C}$-scheme S, we have

$$\Pi_G(S) = \mathrm{Hom}_T(S \times_C T, T' \times T)$$

The diagonal action of W on $T' \times T$ induces an action of W on Π_G. The fixed point subscheme of Π_G^W is a closed smooth subscheme of Π_G since the characteristic of the base field does not divide the order of W.

Proposition 2.3.2. *There exists a canonical open embedding $J_{G'} \to \Pi_G^W$.*

Proof. We follow the argument for the Lie algebra case in [Ngô10, §2.4]. First we define a morphism $J \to \Pi_G^W$. By adjunction, this is the same as giving a morphism $q^*J \to T \times T$ where $q : T \to \mathfrak{C}$ and we view $T \times T$ as a constant group scheme over T. One construct this morphism by descent along the smooth morphism $\tilde{\chi}^{\mathrm{reg}} : \widetilde{G}^{\mathrm{reg}} \to T$ which sits in the Cartesian diagram

$$\begin{tikzcd} \widetilde{G}^{\mathrm{reg}} \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{\chi}"'] & G^{\mathrm{reg}} \arrow[d, "\chi"] \\ T \arrow[r, "q"] & \mathfrak{C}_G \end{tikzcd}$$

Hence it suffices to construct a G-equivariant morphism $(\tilde{\chi}^{\mathrm{reg}})^* q^* J_G \to T \times \widetilde{G}^{\mathrm{reg}}$. The upshot is that for all $x \in G$ and Borel subgroup $x \in B \subset G$, we have $I_x \subset B$ by the argument of [Ngô10, Lemme 2.4.3]. Hence when composed with the quotient $B \to T$, we obtain a map $I_x \to T$ depending on the choice of Borel B containing x. Thus we get the desired morphism $(\tilde{\chi}^{\mathrm{reg}})^* q^* J_G \cong \tilde{q}^* I_{G^{\mathrm{reg}}} \to T \times \widetilde{G}^{\mathrm{reg}}$ which is G-equivariant by construction.

To show that the morphism $J_G \to \Pi_G^W$ constructed above is an isomorphism, it suffices to show the isomorphism over an open subset of $\mathfrak{C}$ whose complement has codimension at least 2.

For each simple root $\alpha \in \Phi^+$, let T_α be the kernel of α, which is a subscheme of codimension 1 in T. Then the discriminant divisor $\mathfrak{D} \subset \mathfrak{C}$ is the union of $q(T_\alpha)$ for all simple root α. Let $T_\alpha^\circ \subset T_\alpha$ be the open subscheme consisting of points that does not lie in T_β for any $\beta \neq \alpha$. Then

$$\mathfrak{C}^{\mathrm{rs}} \cup \big(\bigcup_{\alpha \in \Phi^+} q(T_\alpha^\circ) \big)$$

is an open subset of $\mathfrak{C}$ whose complement has codimension 2. It follows from construction that it is an isomorphism over $\mathfrak{C}^{\mathrm{rs}}$. Hence it remains to show that $J_G \to \Pi_G^W$ is an isomorphism when restricted to $q(T_\alpha^\circ)$ for each positive root α.

Let $t \in T_\alpha^\circ$ and we will show that $J \to \Pi_G^W$ is an isomorphism in an étale neighbourhood of t. Let G_α be the centralizer of T_α in G and $\mathfrak{C}_{G_\alpha}$ its adjoint quotient. Then the natural morphism $\pi_\alpha : \mathfrak{C}_{G_\alpha} \to \mathfrak{C}$ is étale in a neighbourhood of $q_\alpha(t)$ where $q_\alpha : T \to \mathfrak{C}_{G_\alpha}$ is the natural map. This implies that in an étale neighbourhood of $q_\alpha(t)$ the group schemes $\Pi_G^W \times_{\mathfrak{C}} \mathfrak{C}_{G_\alpha}$ and $\Pi_{G_\alpha}^{s_\alpha}$ are isomorphic.

There is a natural open embedding $G^{\mathrm{reg}} \cap G_\alpha \subset G_\alpha^{\mathrm{reg}}$. Consider the open subset

$$\mathfrak{C}_\alpha^{G-\mathrm{reg}} := \chi_\alpha(G^{\mathrm{reg}} \cap G_\alpha)$$

As $t \in T_\alpha^\circ$, one can choose a unipotent element $u \in G_\alpha$ such that $tu \in G^{\mathrm{reg}} \cap G_\alpha$. In particular, $q_\alpha(t) \in \mathfrak{C}_\alpha^{G-\mathrm{reg}}$. It is clear that

$$I_{G_\alpha}|_{G^{\mathrm{reg}} \cap G_\alpha} \cong I_G|_{G^{\mathrm{reg}} \cap G_\alpha}$$

This implies that $(\pi_\alpha^* J_G)|_{\mathfrak{C}_\alpha^{G-\mathrm{reg}}} \cong (J_{G_\alpha})|_{\mathfrak{C}_\alpha^{G-\mathrm{reg}}}$.

In summary, the base change of J_G and Π_G^W to an étale neighbourhood of $q(t)$ are isomorphic to the corresponding groups defined for the group G_α. Note that by assmption, G_α is of rank 1 and has semisimple derived group, thus isomorphic to the product of a torus with either GL_2 or SL_2. So we are finally reduced to the case of GL_2 and SL_2, on which the isomorphism follows by direct calculation. $\square$

2.4 Regular centralizer for Vinberg monoid

In this section we let G be an arbitrary connected reductive group over k. Let G^{sc} be the simply-connected cover of its derived group. Then there is a natural adjoint action of G on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ and the action factors through G_{ad}.

Consider the centralizer group scheme $\mathcal{I}$ over $\mathrm{Vin}_{G^{\mathrm{sc}}}$ defined by

$$\mathcal{I} = \{(g, \gamma) \in G \times \mathrm{Vin}_{G^{\mathrm{sc}}} | \mathrm{Ad}(g)\gamma = \gamma\}$$

Then $\mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$ is smooth of relative dimension r. By [Ren88], the fibres of $\mathcal{I}$ over $\mathrm{Vin}^{\mathrm{rs}}_{G^{\mathrm{sc}}}$ are maximal tori in G. In particular, $\mathcal{I}|_{\mathrm{Vin}^{\mathrm{rs}}_{G^{\mathrm{sc}}}}$ is commutative. Hence $\mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$ is also commutative.

2.4.1 Open cover of regular locus

For each $w \in \mathrm{Cox}(W, S)$, define $\mathcal{J}^w := (\epsilon^w_+)^*\mathcal{I}$. Then $\mathcal{J}^w$ is a smooth commutative group scheme on $\mathfrak{C}_+$. The morphism

$$\begin{aligned} c_w : G \times \mathfrak{C}_+ &\longrightarrow \mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}} \\ (g, a) &\longmapsto g\epsilon^w_+(a)g^{-1} \end{aligned}$$

factors through $(G \times \mathfrak{C}_+)/\mathcal{J}^w$ and induces a quasi-finite morphism

$$\bar{c}_w : (G \times \mathfrak{C}_+)/\mathcal{J}^w \to \mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$$

Since $\bar{c}_w$ is an isomorphism over $G^{sc,\mathrm{reg}}_+$, it is birational. Since $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$ is normal, $\bar{c}_w$ is an open embedding by Zariski Main Theorem.

Denote by $\mathrm{Vin}^w_{G^{\mathrm{sc}}}$ the image of $\bar{c}_w$, which is an open subscheme of $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$. The union $U := \bigcup_{w \in \mathrm{Cox}(W,S)} \mathrm{Vin}^w_{G^{\mathrm{sc}}}$ is a Z^{sc}_+-stable open subset of $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$. By Proposition 2.2.6, it coincides with $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$ over $0 \in \mathfrak{C}_+$. Hence it equals to $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$. In other words, the sets

$\mathrm{Vin}^{w}_{G^{\mathrm{sc}}}$ form an open cover of $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$:

$$\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}} = \bigcup_{w \in \mathrm{Cox}(W,S)} \mathrm{Vin}^{w}_{G^{\mathrm{sc}}} \tag{2.4.1}$$

We generalize Lemma 2.3.1 to $\mathrm{Vin}_{G^{\mathrm{sc}}}$:

Lemma 2.4.1. *There is a unique smooth commutative group scheme $\mathcal{J}$ over $\mathfrak{C}_+$ such that we have a G-equivariant isomorphism $(\chi_+^{\mathrm{reg}})^*\mathcal{J} \cong \mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$. Moreover, this isomorphism extends uniquely to a homomorphism $\chi_+^*\mathcal{J} \to \mathcal{I}$.*

Proof. By the same argument as Lemma 2.3.1, for each $w \in \mathrm{Cox}(W,S)$, $\mathcal{J}^w$ is the unique commutative smooth group scheme over $\mathfrak{C}_+$ such that

$$(\chi_+^*\mathcal{J}^w)|_{\mathrm{Vin}^{w}_{G^{\mathrm{sc}}}} \cong \mathcal{I}|_{\mathrm{Vin}^{w}_{G^{\mathrm{sc}}}}$$

Next we show that for any $w, w' \in \mathrm{Cox}(W,S)$, the group schemes $\mathcal{J}^w$ and $\mathcal{J}^{w'}$ are canonically isomorphic. It suffices to show that they are canonically isomorphic over certain open subset whose complement has codimension at least 2. From Lemma 2.3.1, we have the isomorphism over the open subset $\mathfrak{C}_{G^{\mathrm{sc}}_+}$. Over $\mathfrak{C}_+^{\mathrm{rs}}$, each fiber of χ_+ consists of a single $\mathrm{Ad}(G)$ orbit by Lemma 2.2.8. In other words, G acts transitively on each fibre of χ_+ over $\mathfrak{C}_+^{\mathrm{rs}}$. Hence $\mathrm{Vin}^{\mathrm{rs}}_{G^{\mathrm{sc}}} \subset \mathrm{Vin}^{w}_{G^{\mathrm{sc}}}$ for all $w \in \mathrm{Cox}(W,S)$. Thus by uniqueness of $\mathcal{J}^w$ we see that $\mathcal{J}^w$ and $\mathcal{J}^{w'}$ are isomorphic over $\mathfrak{C}_+^{\mathrm{rs}}$.

The complement of $\mathfrak{C}_{G^{\mathrm{sc}}_+}$ is the union of the closure of codimension 1 stratas in $\mathfrak{C}_+$. Since the idempotent $e_\varnothing$ is regular semisimple and belongs each of the strata closure we see that on each strata, the regular semisimple locus is nonempty open. Hence the complement of $\mathfrak{C}_{G^{\mathrm{sc}}_+} \cup \mathfrak{C}_+^{\mathrm{rs}} \subset \mathfrak{C}_+$ has codimension at least 2.

Consequently there is a unique commutative smooth group scheme $\mathcal{J}$ over $\mathfrak{C}_+$ which comes with a unique isomorphism $(\chi_+^{\mathrm{reg}})^*\mathcal{J} \cong \mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$. We know from Lemma 2.3.1 that this isomorphism extends uniquely to a homomorphism between $\chi_+^*\mathcal{J}$ and $\mathcal{I}$ over the open subset $G_+^{\mathrm{sc}} \cup \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}$ whose complement has codimension at least 2. Hence it extends further to the whole space $\mathrm{Vin}_{G^{\mathrm{sc}}}$. $\square$

Proposition 2.4.2. *The classifying stack* $\mathbb{B}\mathcal{J}$ *acts naturally on* $[\mathrm{Vin}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$. *The action preserves the open substacks* $[\mathrm{Vin}_{G^{\mathrm{sc}}}^0/\mathrm{Ad}(G)]$, $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/\mathrm{Ad}(G)]$ *and* $[\mathrm{Vin}_{G^{\mathrm{sc}}}^w/\mathrm{Ad}(G)]$ *for each* $w \in \mathrm{Cox}(W,S)$. *Moreover, the morphism*

$$[\chi_+^w] : [\mathrm{Vin}_{G^{\mathrm{sc}}}^w/\mathrm{Ad}(G)] \to \mathfrak{C}_+$$

induced by χ_+ *is a* $\mathbb{B}\mathcal{J}$ *gerbe, neutralized by the extended Steinberg section* ϵ_+^w.

The proof is the same as [Ngô10, Proposition 2.2.1].

Proposition 2.4.3. *The number of irreducible components of the fibers of the map*

$$\chi_+^{\mathrm{reg}} : \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}} \to \mathfrak{C}_+$$

is bounded above by $|\mathrm{Cox}(W,S)|$ *and equality is achieved at* $\mathcal{N}^{\mathrm{reg}} = (\chi_+^{\mathrm{reg}})^{-1}(0)$.

Proof. The first statement follows from (2.4.1). The second statement is in Proposition 2.2.6. $\square$

Remark 2.4.1. Consequently, unless all simple factors of G^{sc} are SL_2, the action of $\mathbb{B}\mathcal{J}$ on $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/\mathrm{Ad}(G)]$ is not transitive. In other words, $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/\mathrm{G}]$ is not a $\mathbb{B}\mathcal{J}$-gerbe, but rather a finite union of $\mathbb{B}\mathcal{J}$ gerbes as in Proposition 2.4.2. This is different from Lie algebra situation, cf [Ngô10, Proposition 2.2.1].

2.4.2 Galois description of universal centralizer

Let $\prod_{\overline{T_+^{\mathrm{sc}}}/\mathfrak{C}_+} (T \times \overline{T_+^{\mathrm{sc}}})$ be the restriction of scalar which associates to any $\mathfrak{C}_+$-scheme S the set

$$\prod_{\overline{T_+}/\mathfrak{C}_+} (T \times \overline{T_+})(S) = \mathrm{Hom}_{\overline{T_+}}(S \times_{\mathfrak{C}_+} \overline{T_+}, T \times \overline{T_+})$$

Then W acts diagonally on $\prod_{\overline{T_+}/\mathfrak{C}_+}(T' \times \overline{T_+})$ and we consider its fixed point subscheme

$$\mathcal{J}^1 := (\prod_{\overline{T_+}/\mathfrak{C}_+} T \times \overline{T_+})^W.$$

The following is proved in [Bou17, Proposition 11].

Proposition 2.4.4. *$\mathcal{J}^1$ is a smooth commutative group scheme over $\mathfrak{C}_+$. Moreover, there exists an open embedding $\mathcal{J} \to \mathcal{J}^1$ whose restriction to $\mathfrak{C}_+^{\mathrm{rs}}$ is an isomorphism.*

2.5 Arc space of Vinberg monoid

In this section, we assume that G is semisimple and simply connected.

There is a stratification of the space of nondegenerate arcs of $A_G \supset T_{\mathrm{ad}}$ by $T_{\mathrm{ad}}(\mathcal{O})$ orbits:

$$A_G(\mathcal{O}) \cap T_{\mathrm{ad}}(F) = \bigsqcup_{\lambda \in X_*(T_{\mathrm{ad}})_+} L^\lambda A_G$$

where $L^\lambda A_G$ is the $T_{\mathrm{ad}}(\mathcal{O})$-orbit of $\varpi^{-w_0(\lambda)}$.

This inducs a stratification of the space of nondegenerate arcs of $\mathrm{Vin}_G \supset G_+$ into $G_+(\mathcal{O})$-stable pieces:

$$\mathrm{Vin}_G(\mathcal{O}) \cap G_+(F) = \bigsqcup_{\lambda \in X_*(T_{\mathrm{ad}})_+} L^\lambda \mathrm{Vin}_G$$

where $L^\lambda \mathrm{Vin}_G$ is the inverse image of $L^\lambda A_G$ under the abelianization map $L^+\alpha : \mathrm{Vin}_G(\mathcal{O}) \to A_G(\mathcal{O})$. Also we denote

$$L^\lambda \mathrm{Vin}_G^0 := L^\lambda \mathrm{Vin}_G \cap \mathrm{Vin}_G^0(\mathcal{O}).$$

Lemma 2.5.1. *For any $g_+ \in G_+(F)$, we have $g_+ \in L^\lambda \mathrm{Vin}_G$ if and only if $\alpha(g_+) \in$*

$\varpi^{-w_0(\lambda)}T_{\mathrm{ad}}(\mathcal{O})$ *and the image of* g_+ *in* $G_{\mathrm{ad}}(F)$ *belongs to*

$$\overline{G_{\mathrm{ad}}(\mathcal{O})\varpi^{\lambda}G_{\mathrm{ad}}(\mathcal{O})} = \bigcup_{\substack{\mu\in X_*(T_{\mathrm{ad}})^+ \\ \mu\leq\lambda}} G_{\mathrm{ad}}(\mathcal{O})\varpi^{\mu}G_{\mathrm{ad}}(\mathcal{O}).$$

Moreover, $g_+ \in L^{\lambda}\mathrm{Vin}^0_G$ *if and only if* $\alpha(g_+) \in \varpi^{-w_0(\lambda)}T_{\mathrm{ad}}(\mathcal{O})$ *and the image of* g_+ *in* $G_{\mathrm{ad}}(F)$ *belongs to the double coset* $G_{\mathrm{ad}}(\mathcal{O})\varpi^{\lambda}G_{\mathrm{ad}}(\mathcal{O})$*.*

Proof. The coweight lattice for T_+ can be expressed as

$$X_+(T_+) = \{(\lambda_1,\lambda_2) \in X_*(T_{\mathrm{ad}}) \times X_*(T_{\mathrm{ad}}) | \lambda_1 + \lambda_2 \in X_*(T)\}$$

For $(\lambda_1,\lambda_2) \in X_*(T_+)$, we have $\varpi^{(\lambda_1,\lambda_2)} \in L^{\lambda}\mathrm{Vin}_G$ if and only if

- $\alpha(\varpi^{(\lambda_1,\lambda_2)}) \in \varpi^{-w_0(\lambda)}T_{\mathrm{ad}}(\mathcal{O})$ and
- The matrix $\rho^+_{\omega_i}(\varpi^{(\lambda_1,\lambda_2)}) \in \mathrm{End}(V_{\omega_i})$ has entries in $\mathcal{O}$ for all $1 \leq i \leq r$.

Since $\alpha(\varpi^{(\lambda_1,\lambda_2)}) = \varpi^{\lambda_1}$, the first condition means that $\lambda_1 = -w_0(\lambda)$. Then the second condition means that

$$\langle(-w_0(\lambda),\lambda_2),\chi_+\rangle \geq 0$$

for all $1 \leq i \leq r$ and all weights χ_+ in the G_+-representation $\rho^+_{\omega_i}$. Since the weights of the representation $\rho^+_{\omega_i}$ lie in the convex hull of the W-orbit of the highest weight (ω_i,ω_i) where W acts on the second factor, the above inequality is equivalent to

$$\langle -w_0(\lambda),\omega_i\rangle + \langle\lambda_2, w(\omega_i)\rangle = \langle(-w_0(\lambda),\lambda_2),(\omega_i,w(\omega_i))\rangle \geq 0$$

for all $w \in W$ and $1 \leq i \leq r$. This can be further reformulated as

$$\langle\lambda - w(\lambda_2),\omega_i\rangle \geq 0$$

for all $w \in W$ and $1 \leq i \leq r$.

By the discussion so far, we have

$$L^\lambda \mathrm{Vin}_G \cap T_+(F) = \bigcup_{\substack{\mu \in X_*(T_{\mathrm{ad}}) \\ \mu_{\mathrm{dom}} \leq \lambda}} \varpi^{(-w_0(\lambda),\mu)} T_+(\mathcal{O})$$

where μ_{dom} denotes the unique dominant coweight in the W-orbit of μ. As $L^\lambda \mathrm{Vin}_G$ is stable under the action of $G_+(\mathcal{O}) \times G_+(\mathcal{O})$, it is a union of $G_+(\mathcal{O})$ double cosets in $G_+(F)$. Thus by Cartan decomposition we get

$$L^\lambda \mathrm{Vin}_G = \bigsqcup_{\substack{\mu \in X_*(T_{\mathrm{ad}})^+ \\ \mu \leq \lambda}} G_+(\mathcal{O}) \varpi^{(-w_0(\lambda),\mu)} G_+(\mathcal{O})$$

Similarly we can get a description of $L^\lambda \mathrm{Vin}_G^0$. The difference is that we require furthermore that $\rho^+_{\omega_i}(\varpi^{(-w_0(\lambda),\lambda_2)})$ have nonzero reduction mod ϖ for all $1 \leq i \leq r$. Hence besides the inequality $\langle \lambda - w(\lambda_2), \omega_i \rangle \geq 0$ for all $w \in W$ and $1 \leq i \leq r$, we require furthermore that for each i, there exists $w \in W$ such that $\langle \lambda - w(\lambda_w), \omega_i \rangle = 0$. This condition means that implies that λ_2 is in the W-orbit of λ and hence

$$L^\lambda \mathrm{Vin}_G^0 = G_+(\mathcal{O}) \varpi^{(-w_0(\lambda),\lambda)} G_+(\mathcal{O}).$$

From these description the lemma follows. □

Lemma 2.5.2. *Suppose* $n \geq b(\lambda) := \max_{1 \leq i \leq r} \langle \lambda, \omega_i - w_0(\omega_i) \rangle$. *Then for all* $\gamma, \gamma' \in L^\lambda \mathrm{Vin}_G$ *having the same image in* $\mathrm{Vin}_G(\mathcal{O}/\varpi^n\mathcal{O})$, *there exists* $g \in G_+(\mathcal{O})$ *such that* $\gamma' = \gamma g$.

Proof. The following argument is due to Zhiwei Yun. Let $i \mapsto i^*$ be the involution on the set $\{1, \ldots, r\}$ such that $\omega_{i^*} = -w_0(\omega_i)$. For each $1 \leq i \leq r$, there exists natural pairing between V_i and V_{i^*} such that for all $x \in G_+$, $v \in V_i$ and $v^* \in V_{i^*}$, we have

$$\langle \rho_i^+(x)v, \rho_{i^*}^+(x)v^* \rangle = (\omega_i + \omega_{i^*})(\alpha(x))\langle v, v^* \rangle.$$

Thus for each $x \in G_+(F)$, under the natural pairing above, the lattice $\rho_i^+(x)V_i(\mathcal{O})$ in $V_i(F)$ is dual to the lattice

$$(\omega_i + \omega_{i^*})(\alpha(x)^{-1})\rho_{i^*}^+(g)V_{i^*}(\mathcal{O}) \subset V_{i^*}(F).$$

For $\gamma \in L^\lambda \mathrm{Vin}_G \subset \mathrm{Vin}_G(\mathcal{O})$, we have $\rho_i^+(\gamma)V_i(\mathcal{O}) \subset V_i(\mathcal{O})$ for all $1 \leq i \leq r$. Taing duals, we get

$$V_{i^*}(\mathcal{O}) \subset \varpi^{-\langle \lambda, \omega_i + \omega_{i^*} \rangle} \rho_{i^*}^+(\gamma)V_{i^*}(\mathcal{O}).$$

In other words, we have shown that for all $1 \leq i \leq r$,

$$\varpi^{\langle \lambda, \omega_i + \omega_{i^*} \rangle} V_i(\mathcal{O}) \subset \rho_i^+(\gamma)V_i(\mathcal{O}) \subset V_i(\mathcal{O}).$$

Thus if γ and γ' have the same image in $\mathrm{Vin}_G(\mathcal{O}/\varpi^n\mathcal{O})$ for $n \geq b(\lambda)$, the lattices $\rho_i^+(\gamma)V_i(\mathcal{O})$ and $\rho_i(\gamma')V_i(\mathcal{O})$ are the same and hence $\gamma' = \gamma g$ for some $g \in G_+(\mathcal{O})$. $\square$

CHAPTER 3

KOTTWITZ-VIEHMANN VARIETIES

We fix a connected reductive group G. Let $T \subset G$ be a maximal torus and $\lambda \in X_*(T)_+$ a dominant coweight. Let $\gamma \in G^{\mathrm{rs}}(F)$ be a regular semisimple element.

We study the following sets associated to the pair (γ, λ), which we both refer to as *Kottwitz-Viehmann varieties*:

$$X_\gamma^\lambda = \{g \in G(F)/G(\mathcal{O}) | \mathrm{Ad}(g)^{-1}(\gamma) \in G(\mathcal{O})\varpi^\lambda G(\mathcal{O})\}$$

$$X_\gamma^{\leq\lambda} = \{g \in G(F)/G(\mathcal{O}) | \mathrm{Ad}(g)^{-1}(\gamma) \in \overline{G(\mathcal{O})\varpi^\lambda G(\mathcal{O})}\}$$

3.1 Nonemptiness

The first immediate question is when the sets $X_\gamma^\lambda, X_\gamma^{\leq\lambda}$ are nonempty. To answer this we need to recall the notion of Newton points and Kottwitz map.

3.1.1 Newton Points

Following [KV12, §4], for each $\gamma \in G(F)^{\mathrm{rs}}$, one associate a rational dominant coweight $\nu_\gamma \in X_*(T)^+_{\mathbb{Q}}$, called the *Newton point of* γ.

Definition 3.1.1. The *discriminant valuation* for $\gamma \in G(F)^{\mathrm{rs}}$ is defined by

$$d(\gamma) := \mathrm{val}\det(\mathrm{Id} - \mathrm{ad}_\gamma : \mathfrak{g}(F)/\mathfrak{g}_\gamma(F) \to \mathfrak{g}(F)/\mathfrak{g}_\gamma(F))$$

where $\mathfrak{g}$ is the Lie algebra of G and γ_γ is the centralizer of γ, i.e. the fixed locus of the adjoint action ad_γ.

Lemma 3.1.1. *Let $\gamma \in G(F)^{\mathrm{rs}}$ and $\nu_\gamma \in \Lambda_{\mathbb{Q}}^{+}$ its Newton point. Let $\bar{\gamma} \in T(\bar{F})^{\mathrm{rs}}$ be a $G(\bar{F})$-conjugate of γ such that $\mathrm{val}(\alpha(\gamma)) \geq 0$ for all positive root α. Then we have*

$$d(\gamma) = 2 \sum_{\alpha \in \Phi^+} \mathrm{val}(\alpha(\bar{\gamma}) - 1) - \langle 2\rho, \nu_\gamma \rangle$$

where we have extended the valuation on F to its separable closure $\bar{F}$.

Proof. From the definition we see that

$$d(\gamma) = \sum_{\alpha \in \Phi} \mathrm{val}(\alpha(\gamma) - 1).$$

Separate the sum over Φ according to whether $\langle \alpha, \nu_\gamma \rangle = 0$ or not, then we get

$$d(\gamma) = \sum_{\substack{\alpha \in \Phi \\ \langle \alpha, \nu_\gamma \rangle = 0}} \mathrm{val}(\alpha(\bar{\gamma}) - 1) + \sum_{\substack{\alpha \in \Phi \\ \langle \alpha, \nu_\gamma \rangle < 0}} \langle \alpha, \nu_\gamma \rangle. \tag{3.1.1}$$

By our assumption that $\mathrm{val}(\alpha(\gamma)) \geq 0$ for $\alpha \in \Phi^+$, the first term in (3.1.1) equals to

$$2 \sum_{\substack{\alpha \in \Phi^+ \\ \langle \alpha, \nu_\gamma \rangle = 0}} \mathrm{val}(\alpha(\gamma) - 1) = 2 \sum_{\alpha \in \Phi^+} \mathrm{val}(\alpha(\gamma) - 1)$$

while the second term of (3.1.1) equals to

$$\sum_{\alpha \in \Phi^-} \langle \alpha, \nu_\gamma \rangle = - \sum_{\alpha \in \Phi^+} \langle \alpha, \nu_\gamma \rangle = -\langle 2\rho, \nu_\gamma \rangle.$$

Hence the lemma follows. □

3.1.2 Kottwitz map

The fundamental group of G is $\pi_1(G) := X_*(T)/X_*(T^{\mathrm{sc}})$, the quotient of coweight lattice by the coroot lattice. Let $p_G : X_*(T) \to \pi_1(G)$ be the canonical projection. Following [KV12], one defines a group homomorphism

$$\kappa_G : G(F) \to \pi_1(G)$$

which we refer to as Kottwitz homomorphism. Note that in *loc. cit.*, this map is denoted by w_G.

Lemma 3.1.2. *Suppose that $\kappa_G(\gamma) = p_G(\lambda)$. Then there exists an element $\gamma_\lambda \in G_+^{\mathrm{sc}}(F)$ such that*

- *the image of γ_λ in $G_{\mathrm{ad}}(F)$ coincides with the image of γ in $G_{\mathrm{ad}}(F)$;*
- *$\alpha(\gamma_\lambda) = \varpi^{-w_0(\lambda_{\mathrm{ad}})} \in T_{\mathrm{ad}}(F) \cap A_{G^{\mathrm{sc}}}(\mathcal{O})$ where $\lambda_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})^+$ is the image of $\lambda \in X_*(T)^+$.*

Moreover, γ_λ is uniquely determined up to multiplication by an element in $Z_{G^{\mathrm{sc}}}(F)$.

Proof. Let $\gamma_{\mathrm{ad}} \in G_{\mathrm{ad}}(F)$ be the image of γ. Choose any $\tilde{\gamma} \in G_+^{\mathrm{sc}}(F)$ that maps to γ_{ad}. Suppose $\alpha(\tilde{\gamma}) \in \varpi^\mu T_{\mathrm{ad}}(\mathcal{O})$ for $\mu \in X_*(T_{\mathrm{ad}})^+$. By the assumption $\kappa_G(\gamma) = p_G(\lambda)$, we have $\lambda_{\mathrm{ad}} - \mu \in X_*(T)$. Let $\gamma_\lambda := \varpi^{\lambda_{\mathrm{ad}}-\mu}\tilde{\gamma}$ where we view $\varpi^{\lambda_{\mathrm{ad}}-\mu} \in T(F) = Z_+(F)$ as a central element in $G_+^{\mathrm{sc}}(F)$. Then we have $\alpha(\gamma_\lambda) = \varpi^{-w_0(\lambda_{\mathrm{ad}})}$ and the image of γ_λ in $G_{\mathrm{ad}}(F)$ equals to γ_{ad}.

Suppose $\gamma_\lambda, \gamma'_\lambda \in G_+^{\mathrm{sc}}(F)$ both satisfy the requirement of the Lemma. Then $\gamma'_\lambda\gamma_\lambda^{-1} \in G^{\mathrm{sc}}(F)$ and its image in $G_{\mathrm{ad}}(F)$ is the identity. Hence $\gamma'_\lambda\gamma_\lambda^{-1} \in Z_{G^{\mathrm{sc}}}(F)$. □

Now we can state the non-emptiness criterions.

Proposition 3.1.3. *The following are equivalent:*

1. *X_γ^λ is nonempty;*
2. *$X_\gamma^{\leq\lambda}$ is nonempty;*

3. *$\kappa_G(\gamma) = p_G(\lambda)$ and $\nu_\gamma \leq_{\mathbb{Q}} \lambda$, i.e. $\lambda - \nu_\gamma$ is a $\mathbb{Q}$-linear combinition of simple coroots with non-negative coefficients;*

4. *$\kappa_G(\gamma) = p_G(\lambda)$ and $\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$, where $\gamma_\lambda \in G_+^{\mathrm{sc}}(F)$ is defined in Lemma 3.1.2.*

Proof. The implication "(1)⇒(2)" is tautological. The implication "(1)⇒(3)" is done in [KV12, Corollary 3.6].

(3)⇒(4): Let F'/F be a finite extension of degree e so that γ (and hence γ_λ) is split in $G(F')$. Let $\varpi' = \varpi^{\frac{1}{e}}$ be a uniformizer of F' and $\mathcal{O}' = k[[\varpi']] \subset F'$ be the ring of integers. Then $e \cdot \nu_\gamma \in X_*(T)_+$ and γ is $G(F')$-conjugate to an element in $(\varpi')^{e\cdot\nu_\gamma}T(\mathcal{O}')$. From (3) we deduce that γ_λ is $G_+^{\mathrm{sc}}(F')$-conjugate to an element in $\mathrm{Vin}_G^{\mathrm{sc}}(\mathcal{O}')$. Therefore

$$\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O}') \cap \mathfrak{C}_+(F) = \mathfrak{C}_+(\mathcal{O}).$$

(2)⇒(4): Let $g \in X_\gamma^{\leq\lambda}$. Then $\mathrm{Ad}(g)^{-1}(\gamma) \in G(\mathcal{O})\varpi^\mu G(\mathcal{O})$ for some $\mu \in X_*(T)_+$ with $\mu \leq \lambda$. Then we have $\omega_G(\gamma) = p_G(\mu) = p_G(\lambda)$. In particular we can define the element $\gamma_\lambda \in G_+^{\mathrm{sc}}(F)$ as in Lemma 3.1.2. Then by Lemma 2.5.1 we have $\mathrm{Ad}(g)^{-1}(\gamma_\lambda) \in L^\lambda \mathrm{Vin}_{G^{\mathrm{sc}}} \subset \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})$. Thus $\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$.

(4)⇒(1): Let $a_+ := \chi_+(\gamma_\lambda)$. So $a \in \mathfrak{C}_+(\mathcal{O})$ by condition (4). Then for any Coxeter element $w \in \mathrm{Cox}(W, S)$ (cf. Definition 2.2.3), we have $\epsilon_+^w(a) \in \mathrm{Vin}^0_{G^{\mathrm{sc}}}(\mathcal{O})$. It remains to show that there exists $h \in G(F)$ such that $\mathrm{Ad}(h)^{-1}(\gamma_\lambda) = \epsilon_\lambda^w(a)$, for then $h \in X_\gamma^\lambda$. To see this, notice that the transporter from γ to $\epsilon_+^w(a)$ in G is a torsor under the torus G_{γ_λ} over F. Any such torsor is trivial since $H^1(F, G_{\gamma_\lambda})$ by a theorem of Steinberg (using the fact that the residue field k is algebraic closed). Thus the transporter has an F-point $h \in G(F)$. □

3.2 Ind-scheme structure

3.2.1 First approach

We will equip the sets X_γ^λ and $X_\gamma^{\leq\lambda}$ with an ind-scheme structure. We present two approaches, one based on the original definition, the other using Vinberg monoid.

Let $\mathrm{Gr}_G := LG/L^+G$ be the affine Grassmanian for G, which are known to be ind-projective ind-scheme over k. The positive loop group L^+G acts by left multiplication on Gr_G. Let $(LG)_\lambda := L^+G\varpi^\lambda L^+G$ (resp. $(LG)_{\leq\lambda}$) be the k-scheme whose set of k-points is $G(\mathcal{O})\varpi^\lambda G(\mathcal{O})$ (resp. $\overline{G(\mathcal{O})\varpi^\lambda G(\mathcal{O})}$).

Definition 3.2.1. Let $\mathcal{X}_\gamma^\lambda$ be the k-functor which associates to any k-algebra R the set

$$\mathcal{X}_\gamma^\lambda(R) = \{g \in \mathrm{Gr}_G(R) | g^{-1}\gamma g \in (LG)_\lambda(R)\}.$$

Also, we define the k-functor $\mathcal{X}_\gamma^{\leq\lambda}$ by replacing $(LG)_\lambda$ with $(LG)_{\leq\lambda}$ in the above definition

By definition, $\mathcal{X}_\gamma^{\leq\lambda}$ is a closed sub-indscheme of Gr_G and $\mathcal{X}_\gamma^\lambda$ is an open sub-indscheme of $\mathcal{X}_\gamma^{\leq\lambda}$. Let X_γ^λ (resp. $X_\gamma^{\leq\lambda}$) be the reduced structure of $\mathcal{X}_\gamma^\lambda$ (resp. $\mathcal{X}_\gamma^{\leq\lambda}$).

3.2.2 Second approach

Now we use Vinberg monoids to define certain analogue of affine Springer fibers, which turns out to be isomorphic to Kottwitz-Viehmann varieties.

Let $\mathfrak{C}_+$ be the extended Steinberg base for the monoid $\mathrm{Vin}_{G^{\mathrm{sc}}}$. Let $a \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_+(F)^{\mathrm{rs}}$ and suppose that

$$\beta(a) \in \varpi^{-w_0(\lambda_{\mathrm{ad}})} T_{\mathrm{ad}}(\mathcal{O}) \subset A_{G^{\mathrm{sc}}}(\mathcal{O}) \cap T_{\mathrm{ad}}(F)$$

where $\lambda_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})$ is the image of $\lambda \in X_*(T)$. Moreover, let $\gamma_+ \in G_+^{\mathrm{sc}}(F)$ be an element such that $\chi_+(\gamma_+) = a$.

Definition 3.2.2. The *generalized affine Springer fibre* $\mathcal{S}p_{G,\gamma_+}$ associates to any k-algebra

R the set of isomorphism classes of pairs (h, ι) where h is the horizontal arrow in the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R[[\varpi]] \arrow[r, "h"] \arrow[dr, "a"'] & {[\mathrm{Vin}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]} \arrow[d] \\ & \mathfrak{C}_+ \end{tikzcd}$$

and ι is an isomorphism between the restriction of h to $\operatorname{Spec} R((\varpi))$ and the composition

$$\operatorname{Spec} R((\varpi)) \xrightarrow{\gamma_+} \mathrm{Vin}_{G^{\mathrm{sc}}} \to [\mathrm{Vin}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)].$$

Also, we define k-functors $\mathcal{S}p^0_{G,\gamma_+}$ (resp. $\mathcal{S}p^{\mathrm{reg}}_{G,\gamma_+}$) by replacing $\mathrm{Vin}_{G^{\mathrm{sc}}}$ with $\mathrm{Vin}^0_{G^{\mathrm{sc}}}$ (resp. $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$).

By definition $\mathcal{S}p_{G,\gamma_+}$ is a closed sub-indscheme of Gr_G and $\mathcal{S}p^{\mathrm{reg}}_{G,\gamma_+} \subset \mathcal{S}p^0_{G,\gamma_+}$ are its open sub-indschemes. We let Sp_{G,γ_+} (resp. $\mathrm{Sp}^0_{G,\gamma_+}$, $\mathrm{Sp}^{\mathrm{reg}}_{G,\gamma_+}$) be the reduced structures of $\mathcal{S}p_{G,\gamma_+}$ (resp. $\mathcal{S}p^0_{G,\gamma_+}$, $\mathcal{S}p^{\mathrm{reg}}_{G,\gamma_+}$).

The isomorphism class of Sp_{G,γ_+} and $\mathrm{Sp}^0_{G,\gamma_+}$ only depends on $a = \chi_+(\gamma_+)$, so we will also denote them by $\mathrm{Sp}_{G,a}$ and $\mathrm{Sp}^0_{G,a}$. We will simplify notation as Sp_{γ_+}, Sp_a etc. if the group G is clear from the context.

Next we relate the two definitions given above. Let (γ, λ) be as in the beginning of this chapter. Suppose that the ind-scheme X^λ_γ is nonempty. Then by Proposition 3.1.3 we have $\kappa_G(\gamma) = p_G(\lambda)$ and $a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$ where $\gamma_\lambda \in G^{\mathrm{sc}}_+(F)$ is defined in Lemma 3.1.2. It is not hard to see that

$$X^\lambda_\gamma \cong \mathrm{Sp}^0_a \quad \text{and} \quad X^{\leq\lambda}_\gamma \cong \mathrm{Sp}_a.$$

Conversely, let $a \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_+(F)^{\mathrm{rs}}$ and suppose that

$$\beta(a) \in \varpi^{-w_0(\lambda)} T_{\mathrm{ad}}(\mathcal{O}) \subset A_{G^{\mathrm{sc}}}(\mathcal{O}) \cap T_{\mathrm{ad}}(F)$$

for some $\lambda \in X_*(T_{\mathrm{ad}})_+$. Let $\gamma^w_a \in G_{\mathrm{ad}}(F)$ be the image of $\epsilon^w_+(a) \in G_+(F) \cap \mathrm{Vin}^0_{G^{\mathrm{sc}}}(\mathcal{O})$

under the natural quotient $G_+(F) \to G_{\mathrm{ad}}(F)$. Then we have

$$\mathrm{Sp}_a \cong X_{\gamma_a^w}^{\leq\lambda} \quad \text{and} \quad \mathrm{Sp}_a^0 \cong X_{\gamma_a^w}^{\lambda}.$$

Note that the isomorphism class of $X_{\gamma_a^w}^{\leq\lambda}$ and $X_{\gamma_a^w}^{\lambda}$ does not depend on the choice of $w \in \mathrm{Cox}(W, S)$.

3.3 Symmetries

Assume X_γ^λ is nonempty. Then by Proposition 3.1.3 we have $\kappa_G(\gamma) = p_G(\lambda)$ and

$$a = \chi_+(\gamma_\lambda) \in \mathfrak{C}_{G_+^{\mathrm{sc}}}^{\mathrm{rs}}(F) \cap \mathfrak{C}_+(\mathcal{O}).$$

Let J_a be the commutative group scheme over $\operatorname{Spec}\mathcal{O}$ obtained by pulling back $\mathcal{J}$ along $a : \operatorname{Spec}\mathcal{O} \to \mathfrak{C}_+$. Since a is generically regular semisimple, there is a canonical isomorphism $LJ_a \cong LG_\gamma^0$ which allows us to identify the positive loop group L^+J_a as a subgroup of LG_γ^0. Consider the quotient group

$$P_a := LJ_a/L^+J_a \cong LG_\gamma^0/L^+J_a.$$

In other words, $\mathcal{P}_a$ is the affine Grassmanian of J_a classifying isomorphism classes of J_a-torsors on $\operatorname{Spec}\mathcal{O}$ with a trivialization of its restriction to $\operatorname{Spec} F$.

The loop group LG_γ^0 acts naturally on X_γ^λ and this action factors through P_a. Using the isomorphism $X_\gamma^\lambda \cong \mathrm{Sp}_a^0$, the P_a action is induced by the $\mathbb{B}\mathcal{J}$ action on $[V_{G^{\mathrm{sc}}}^0/\mathrm{Ad}(G)]$ in Proposition 2.4.2. Moreover, P_a preserve the open subspaces $\mathrm{Sp}_a^{\mathrm{reg}}$ and Sp_a^w for each $w \in \mathrm{Cox}(W, S)$.

Proposition 3.3.1. *For each $w \in \mathrm{Cox}(W, S)$, Sp_a^w is a torsor under P_a.*

Proof. This is a consequence of 2.4.2. $\square$

Remark 3.3.1. Unlike the Lie algebra case, $\mathrm{Sp}_a^{\mathrm{reg}}$ may not be a P_a-torsor in general. See the discussion in § 3.9.3.

Let R_a be the finite free $\mathcal{O}$-algebra defined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{X}_a := \operatorname{Spec} R_a \arrow[r] \arrow[d] & \overline{T_+} \arrow[d] \\ \operatorname{Spec} \mathcal{O} \arrow[r, "a"] & \mathfrak{C}_+ \end{tikzcd} \tag{3.3.1}$$

Let $R_a^\flat$ be the normalization of R_a and $\widetilde{X}_a^\flat := \operatorname{Spec} R_a^\flat$. Then W acts naturally on the $\mathcal{O}$-algebras R_a and $R_a^\flat$.

Let $J_a^\flat$ be the finite type Neron model of J_a. Hence $J_a^\flat$ is a smooth commutative group scheme over $\mathcal{O}$ such that $J_a^\flat(F) = J_a(F) = G_\gamma^0(F)$ and $J_a^\flat(\mathcal{O})$ is the maximal bounded subgroup of $G_\gamma^0(F)$.

Lemma 3.3.2. *There is a canocical isomorphism*

$$J_a^\flat \cong (\prod_{R_a^\flat/\mathcal{O}} T \times \widetilde{X}_a^\flat)^W$$

Proof. The proof is the same as [Ngô10, Proposition 3.8.2]. $\square$

Corollary 3.3.3. $\mathrm{Lie}(P_a) = (\mathfrak{t} \otimes_k (R_a^\flat/R_a))^W$

Proof. The quotient $L^+J_a^\flat/L^+J_a$ is an open subgroup of P_a. Hence we have isomorphism of $\mathcal{O}$ modules

$$\mathrm{Lie}P_a \cong \mathrm{Lie}(L^+J_a^\flat)/\mathrm{Lie}(L^+J_a).$$

On the other hand, by 2.4.4, we have

$$\mathrm{Lie}L^+J_a = (\mathfrak{t} \otimes_k R_a)^W$$

and by 3.3.2,

$$\mathrm{Lie} L^+ J_a^\flat = (\mathfrak{t} \otimes_k R_a^\flat)^W.$$

Hence the Corollary follows. $\square$

3.4 Admissible subsets of loop spaces

In this section we closely follow [GHKR06, §5].

Let M be a standard Levi subgroup of G and $P = MN$ the standard parabolic subgroup where N is the unipotent radical of P. Let $Z(M)^0$ be the neutral component of the center of M. Then $Z(M)^0$ is a subtorus of T. Let Φ_N be the set of roots of $Z(M)^0$ acting on N and $\Phi_N^\vee$ the corresponding set of coroots. For each $\alpha \in \Phi_N$, let N_α be the corresponding root subgroup. Then each N_α is isomorphic to a product of several copies of $\mathbb{G}_a$ and is preserved by the adjoint action of M. Denote δ_N half sum of elements in $\Delta_N^\vee$.

For each $\alpha \in \Delta_N$, denote $\mathrm{ht}_N(\alpha) := \langle \delta_N, \alpha \rangle$. Let $l = \max_{\alpha \in \Phi_N} \mathrm{ht}_N(\alpha)$. For each $1 \leq i \leq l$, let $N[i]$ be the subgroup of N generated by root groups N_α with $\mathrm{ht}_N(\alpha) \geq i$. Also we denote $N[l+1] = 1$. Then $N[1] = N$ and for each $1 \leq i \leq s+1$, $N[i]$ is a normal subgroup of N and the successive quotients $N\langle i \rangle := N[i]/N[i+1]$ are commutative groups isomorphic to products of some copies of $\mathbb{G}_a$. Let LN and L^+N be the loop space and arc space of N. For each integer $n \geq 0$, let $N_n := \ker(L^+N \to L_n^+ N)$. Then $\{N_n\}_{n \geq 0}$ form a decreasing sequence of compact open subgroups of LN.

For each $\gamma \in M(F) \cap G(F)^{\mathrm{rs}}$, consider the map

$$\begin{aligned} f_\gamma : LN &\longrightarrow LN \\ u &\longmapsto u^{-1}\gamma u \gamma^{-1} \end{aligned} \tag{3.4.1}$$

Then f_γ preserves the root subgroups N_α and hence each normal subgroup $N[i]$. In particular, f_γ induces morphism $f_\gamma[i] : LN[i] \to LN[i]$ and $f_\gamma\langle i \rangle : LN\langle i \rangle \to LN\langle i \rangle$.

For each $1 \leq i \leq l$, denote $r_i := \mathrm{val}\det(f_\gamma\langle i \rangle)$. Note that there is a M-equivariant isomorphism $N\langle i\rangle \cong \mathrm{Lie}N\langle i\rangle$ from which we see that

$$r_i = \mathrm{val}\det(\mathrm{ad}_\gamma : \mathrm{Lie}N\langle i\rangle(F) \to \mathrm{Lie}N\langle i\rangle(F)).$$

Consider the following invariant of γ:

$$r_N(\gamma) := \mathrm{val}\det(\mathrm{ad}_\gamma : \mathrm{Lie}N(F) \to \mathrm{Lie}N(F)) \tag{3.4.2}$$

Then we also have $r_N(\gamma) = \sum_{i=1}^{l} r_i$.

Now assume that $\gamma \in M(F)_+$, we have $f_\gamma(U_n) \subset U_n$ for all $n \geq 0$.

Let $f_0 : L^+N \to L^+N$ be the restriction of f_γ to the arc space L^+N.

Lemma 3.4.1. *For any $1 \leq i \leq l+1$ and any positive integer n such that $n \geq \sum_{j=i}^{l+1} r_j$ we have $N[i]_n \subset f_\gamma(L^+N[i])$.*

Proof. We prove by desending induction on i. The case $i = l+1$ is trivial since $N[l+1] = 1$. Assume the statement is true for $i+1$. Let $x \in N[i]_n$. To show that $x \in f_\gamma(L^+N[i])$ it suffices to find $u \in N[i](\mathcal{O})$ with $x * u = 1$, for then $f_\gamma(u^{-1}) = x$.

Let $x_i \in N\langle i\rangle_n$ be the image of x. Since $\mathrm{val}\det(f_\gamma\langle i\rangle) = r_i$, we have $\varpi^{r_i} N\langle i\rangle(\mathcal{O}) \subset f_\gamma\langle i\rangle(N\langle i\rangle(\mathcal{O}))$. Hence there exists $u_i \in N[i]_{n-r_i}$ such that $x_i * u_i = 1$ in $N\langle i\rangle(\mathcal{O})$ and hence $x * u_i \in N[i+1]_{n-r_i}$. By induction hypothesis, there exists $v \in N[i+1](\mathcal{O})$ such that $(x * u_i) * v = 1$. Then $u = u_i v$ satisfies $x * u = 1$. □

A subset of L^+N is *admissible* if it is the pre-image of a locally closed subset of L_n^+N for some n. A subset Z of LN is *admissible* of it is conjugate under $G(F)$ to an admissible subset of L^+N.

Lemma 3.4.2. *Let V be an admissible subset of L^+N. Let $n \geq r_N(\gamma)$ be a positive integer such that V is right invariant under N_n. Suppose moreover that $V \subset f_0(L^+N)$. Then the set $f_0^{-1}(V)$ is admissible and right invariant under N_n. Moreover, f_0 induces a smooth* *surjective map*

$$f_0^{-1}(V)/N_n \to V/N_n$$

whose fibers are isomorphic to $\mathbb{A}^{r_N(\gamma)}$.

Proof. Let $\bar{f}_0 : L_n^+ N \to L_n^+ N$ be the map induced by f_0. Since V is right invariant under N_n, a straightforward calculation shows that $f_0^{-1}(V)$ is also right invariant under N_n. Denote $\overline{V} := V/U_n$. Then we have $f_0^{-1}(V)/U_n = \bar{f}_0^{-1}(\overline{V})$, a locally closed subset of $L_n^+ N$. In particular, $f_0^{-1}(V)$ is admissible. Since $V \subset f_0(L^+ N)$, the induced map $\bar{f}_0^{-1}(\overline{V}) \to \overline{V}$ is surjective and it remains to show that it is smooth with fibers isomorphic to $\mathbb{A}^{r(\gamma)}$.

Denote $H := L_n^+ N$, $H[i] := L_n^+(N[i])$ and $H\langle i\rangle := L_n^+(N\langle i\rangle)$. Then for each $1 \leq i \leq l+1$, $H[i]$ is a normal subgroup of H and $H[i]/H[i+1] \cong H\langle i\rangle$. For each $1 \leq j \leq n$, we define a normal subgroup $H_j := \ker(H \to L_j^+ N)$ of H; and similarly we define normal subgroups $H[i]_j$ (resp. $H\langle i\rangle_j = \varpi^j H\langle i\rangle$) of $H[i]$ (resp. $H\langle i\rangle$).

Consider the right action of H on itself defined by $v * u := u^{-1} v \gamma u \gamma^{-1}$ for $u, v \in H(k) = N(\mathcal{O}/\varpi^n \mathcal{O})$. Then $\bar{f}_0(u) = 1 * u$ and hence $\bar{f}_0$ is the orbit map at 1 of the H-action. In particular, all fibres of $\bar{f}_0$ are isomorphic to the stabilizer $S := \bar{f}_0^{-1}(1)$.

Now we take a closer look at the structure of the stabilizer S. First note that the action $*$ induces actions of $H[i]$ and $H\langle i\rangle$ on themselves. Let $S[i]$ (resp. $S\langle i\rangle$) be the stabilizer of 1 under the $H[i]$ (resp. $H\langle i\rangle$) action.

We claim that for all i, the canonical homomorphism $S[i] \to S\langle i\rangle$ is surjective. Let $s \in S\langle i\rangle$ and choose a representative $h \in H[i]$ of s. Since

$$S\langle i\rangle = \ker(\bar{f}_0\langle i\rangle) \subset \varpi^{n-r_i} H\langle i\rangle$$

we have $h \in H[i]_{n-r_i}$ and $1 * h \in H[i]_{n-r_i} \cap H[i+1] = H[i+1]_{n-r_i}$. By assumption $n - r_i \geq \sum_{j=i+1}^{l+1} r_j$, then we can apply Lemma 3.4.1 to obtain an element $h' \in H[i+1]$ such that $1 * (hh') = 1$. Thus $hh' \in S[i]$ maps to $s \in S\langle i\rangle$ and the claim follows.

The kernal of the surjective homomorphism $S[i] \to S\langle i\rangle$ is $S[i] \cap H[i+1] = S[i+1]$.

Moreover, we have

$$S\langle i\rangle \cong (f_0\langle i\rangle)^{-1}(\varpi^n N\langle i\rangle)/\varpi^n N\langle i\rangle \cong \mathbb{A}^{r_i}$$

From this we see that $S \cong \mathbb{A}^{r_N(\gamma)}$ as a scheme. $\square$

The proof of the following lemma is inspired by [KV12, Lemma 3.8].

Lemma 3.4.3. *For any $n \geq r_N(\gamma)$, we have $f_\gamma^{-1}(N_n) \subset N_{n-r_N(\gamma)}$.*

Proof. Let $u \in N(F)$ with $f_\gamma(u) \in N_n$. We will show by induction that

$$u \in N[i](F) \cdot N_{n-\sum_{j<i} r_j}.$$

The case $i = 1$ says $u \in N[1](F) = N(F)$ which is clear and the case $i = s+1$ gives the lemma since $\sum_{i=1}^{s} r_i = r_N(\gamma)$ and $N[s+1] = 1$.

It remains to finish the induction step. By induction hypothesis we have $u = u_i v$ with $u_i \in N[i](F)$ and $v \in N_{n-\sum_{j<i} r_j}$. By assumption,

$$f_\gamma(u) = f_\gamma(u_i v) = v^{-1} \cdot u_i^{-1}\gamma u_i \gamma^{-1} \cdot \gamma v \gamma^{-1} \in N_n$$

from which it follows that

$$u_i^{-1}\gamma u_i\gamma^{-1} \in N[i](F) \cap v \cdot N_n \cdot (\gamma v^{-1}\gamma^{-1}) \subset N[i]_{n-\sum_{j<i} r_j}$$

Let $\bar{u}_i \in N\langle i\rangle$ be the image of u_i. Then we have

$$f_\gamma\langle i\rangle(\bar{u}_i) \in N\langle i\rangle_{n-\sum_{j<i} r_j}$$

Since $\operatorname{val}\det(f_\gamma\langle i\rangle) = r_i$, we get that $\bar{u}_i \in N\langle i\rangle_{n-\sum_{j<i+1} r_j}$ and hence

$$u = u_i v \in N[i+1](F) \cdot N_{n-\sum_{j<i+1} r_j}$$

This finishes the induction step. $\square$

Proposition 3.4.4. *Let Z be an admissible subset of the loop space LN. Then $f_\gamma^{-1}(Z)$ is admissible and there exists a positive integer m such that for all $n \geq m$, $f_\gamma^{-1}(Z)$ and Z are right invariant under the group N_n and the map*

$$f_\gamma^{-1}(Z)/N_n \to Z/N_n$$

induced by f_γ is smooth surjective whose geometric fibers are irreducible of dimension $r_N(\gamma)$.

Proof. Let $n_0 \geq r(\gamma)$ be a positive integer. Choose a coweight $\mu_0 \in X_*(Z(M)^0)$ such that

$$Z^{\mu_0} := \mathrm{Ad}(\varpi^{\mu_0})(Z) \subset N_{n_0}.$$

Then by Lemma 3.4.3 we have

$$f_\gamma^{-1}(Z^{\mu_0}) \subset f_\gamma^{-1}(N_{n_0}) \subset N_{n_0 - r(\gamma)} \subset L^+N$$

Hence in particular

$$\mathrm{Ad}(\varpi^{\mu_0})(f_\gamma^{-1}(Z)) = f_\gamma^{-1}(Z^{\mu_0}) = f_0^{-1}(Z^{\mu_0})$$

Moreover, since Z^{μ_0} is an admissible subset of L^+N, $f_0^{-1}(Z^{\mu_0})$ is an admissible subset of L^+N by Lemma 3.4.2. This shows that $f_\gamma^{-1}(Z)$ is admissible.

Let $n_1 > n_0$ be a positive integer such that Z^{μ_0} and $f_\gamma^{-1}(Z^{\mu_0})$ are invariant under right multiplication by N_{n_1}. For all $n \geq n_1$, since the map f_γ commutes with conjugation by ϖ^{μ_0}, Z and $f_\gamma^{-1}(Z)$ are right invariant under the group $N_n^{-\mu_0} := \varpi^{-\mu_0} N_n \varpi^{\mu_0}$. Then we get the following commutative diagram

$$\begin{tikzcd}
f_\gamma^{-1}(Z)/N_n^{-\mu_0} \arrow[r] \arrow[d, "\simeq"] & Z/N_n^{-\mu_0} \arrow[d, "\simeq"] \\
f_\gamma^{-1}(Z^{\mu_0})/N_n \arrow[r] & Z^{\mu_0}/N_n
\end{tikzcd}$$

where the horizontal arrows are induced by f_γ and the vertical arrows are isomorphisms induced by $\mathrm{Ad}(\varpi^{\mu_0})$.

By Lemma 3.4.1, $Z^{\mu_0} \subset N_{n_0} \subset f_\gamma(L^+N)$. Therefore we can apply Lemma 3.4.2 to conclude that the lower horizontal map is surjective smooth whose fibers are isomorphic to $\mathbb{A}^{r_N(\gamma)}$. Hence the same is true for the upper horizontal map.

Let m be a positie integer such that for all $n \geq m$, $N_n \supset N_{n'}^{-\mu_0}$ for some $n' \geq n_1$. Consider the following diagram

$$\begin{tikzcd} f_\gamma^{-1}(Z)/N_{n'}^{-\mu_0} \arrow[r] \arrow[d] & Z/N_{n'}^{-\mu_0} \arrow[d] \\ f_\gamma^{-1}(Z)/N_n \arrow[r] & Z/N_n \end{tikzcd}$$

The two vertical maps are smooth surjective with fibers isomorphic to the irreducible scheme $U_n/U_{n'}^{-\mu_0}$ and the upper horizontal map is smooth surjective with fibers isomorphic to $\mathbb{A}^{r_N(\gamma)}$ as we have just seen. Hence the lower horizontal map is smooth surjective with irreducible fibers of dimension $r_N(\gamma)$. □

3.5 The case of unramified conjugacy class

In this section we assume that $\gamma \in G(F)^{\mathrm{rs}}$ is an *unramified* regular semisimple element. Since the residue field k is algebraically closed, after conjugation we may assume that $\gamma \in \varpi^\mu T(\mathcal{O}) \cap G^{\mathrm{rs}}(F)$, where $\mu = \nu_\gamma \in X_*(T)_+$ is the Newton points of γ. In this case, we have $G_\gamma^0 = T$. By Lemma 3.1.1 the discriminant valuation for γ is

$$d(\gamma) = 2 \sum_{\alpha \in \Phi^+} \mathrm{val}(\alpha(\gamma) - 1) - \langle 2\rho, \mu \rangle.$$

We will apply the results in previous section to the case $N = U$ is a maximal unipotent subgroup. In this case, the corresponding invariant for γ is

$$r(\gamma) := r_U(\gamma) = \sum_{\alpha \in \Phi^+} \mathrm{val}(\alpha(\gamma) - 1) = \frac{1}{2} d(\gamma) + \langle \rho, \mu \rangle. \tag{3.5.1}$$

Fix a dominant coweight $\lambda \in \Lambda_+$ such that $\mu \leq \lambda$. By Proposition 3.1.3, this implies that X_γ^λ is nonempty.

3.5.1 Relation with MV-cycles

Let Y_γ^λ be the locally closed sub-indscheme of X_γ^λ whose set of k-points is

$$Y_\gamma^\lambda(k) = \{u \in U(F)/U(\mathcal{O}) | \mathrm{Ad}(u)^{-1}\gamma \in G(\mathcal{O})\varpi^\lambda G(\mathcal{O})\}$$

To understand the structure of Y_γ^λ, we use the map $f_\gamma : LU \to LU$ (cf. (3.4.1)). In the following, we denote $K := L^+G$. Then we have

$$Y_\gamma^\lambda = (f_\gamma^{-1}(K\varpi^\lambda K\varpi^{-\mu} \cap LU)/L^+U$$

Recall the Mirkovic-Vilonen cycles in the affine Grassmanian:

$$S_\mu \cap \mathrm{Gr}_\lambda = (LU\varpi^\mu K \cap K\varpi^\lambda K)/K$$

From this description we get an isomorphism

$$\begin{array}{ccc} (LU \cap K\varpi^\lambda K\varpi^{-\mu})/\varpi^\mu L^+U\varpi^{-\mu} & \longrightarrow & S_\mu \cap \mathrm{Gr}_\lambda \\ u & \longmapsto & u\varpi^\mu \end{array} \tag{3.5.2}$$

In summary, we have the following diagram

$$\begin{tikzcd} f_\gamma^{-1}(K\varpi^\lambda K\varpi^{-\mu} \cap LU) \arrow[r, "f_\gamma"] \arrow[d] & K\varpi^\lambda K\varpi^{-\mu} \cap LU \arrow[d] \\ Y_\gamma^\lambda & S_\mu \cap \mathrm{Gr}_\lambda \end{tikzcd}$$

where the left vertical arrow is an L^+U-torsor and the right vertical arrow is a torsor under the group $\varpi^\mu L^+U\varpi^{-\mu}$.

Theorem 3.5.1. *Y_γ^λ is an equi-dimensional quasi-projective variety of dimension $\langle \rho, \lambda\rangle + \frac{1}{2}d(\gamma)$, where $d(\gamma)$ is the discriminant valuation, cf. Definition 3.1.1. Moreover, the number of irreducible components of Y_γ^λ equals to $m_{\lambda\mu}$, the dimension of μ-weight space in the irreducible representation V_λ of $\hat{G}$ with highest weight λ.*

Proof. Apply Proposition 3.4.4 to the admissible subset $Z = K\varpi^\lambda K\varpi^{-\mu} \cap LU$ of LU, we see that there exists a large enough positive integer n such that in the following diagram

$$\begin{tikzcd} f_\gamma^{-1}(K\varpi^\lambda K\varpi^{-\mu} \cap LU)/U_n \arrow[r, "\bar{f}_\gamma"] \arrow[d] & (K\varpi^\lambda K\varpi^{-\mu} \cap LU)/U_n \arrow[d] \\ Y_\gamma^\lambda & S_\mu \cap \mathrm{Gr}_\lambda \end{tikzcd}$$

1. All schemes are of finite type;

2. The map $\bar{f}_\gamma$ induced by f_γ is smooth surjective whose geometric fibers are irreducible of dimension $r(\gamma)$, where we recall that $r(\gamma)$ is defined in (3.5.1);

3. U_n is contained in $\varpi^\mu L^+U\varpi^{-\mu}$, hence also L^+U;

4. The left vertical map is smooth surjective with fibers isomorphic to the irreducible scheme L^+U/U_n;

5. The right vertical map is smooth with fibers isomorphic to the irreducible scheme $\varpi^\mu L^+U\varpi^{-\mu}/U_n$.

Since Y_γ^λ is of finite type, it is a locally closed subscheme of a closed Schubert variety. In particular, Y_γ^λ is quasi-projective since closed Schubert varieties are projective.

Recall that the MV-cycle $S_\mu \cap \mathrm{Gr}_\lambda$ is equidimensional of dimension $\langle \rho, \lambda + \mu \rangle$. Hence by (2)-(5) we see that Y_γ^λ is equidimensional of dimension

$$\begin{aligned} \dim Y_\gamma^\lambda &= \dim(S_\mu \cap \mathrm{Gr}_\lambda) + \dim \varpi^\mu U(\mathcal{O}) \varpi^{-\mu} / U_n^{-\lambda_0} + r(\gamma) - \dim U(\mathcal{O})/U_n^{-\lambda_0} \\ &= \langle \rho, \lambda + \mu \rangle - \langle 2\rho, \mu \rangle + r(\gamma) = \langle \rho, \lambda \rangle + \frac{1}{2} d(\gamma) \end{aligned} \tag{3.5.3}$$

Moreover, by [Sta17, Tag 037A] the 3 maps in the diagram above induces a canonical bijections between set of irreducible components

$$\mathrm{Irr}(Y_\gamma^\lambda) \xrightarrow{\sim} \mathrm{Irr}(S_\mu \cap \mathrm{Gr}_\lambda).$$

Hence the number of irreducible components of Y_γ^λ equals to the number of irreducible components of the MV-cycle $S_\mu \cap \mathrm{Gr}_\lambda$, which is known to be $m_{\lambda\mu}$. $\square$

Corollary 3.5.2. *Suppose $\gamma \in G(F)^{\mathrm{rs}}$ is unramified (or split) and $\nu_\gamma = \mu \in X_*(T)_+$, then X_γ^λ is a scheme locally of finite type, equidimensional of dimension*

$$\dim X_\gamma^\lambda = \langle \rho, \lambda \rangle + \frac{1}{2} d(\gamma).$$

Moreover, the number of $G_\gamma^0(F)$-orbits on its set of irreducible component $\mathrm{Irr}(X_\gamma^\lambda)$ equals to $m_{\lambda\mu}$.

Proof. There is a natural morphism

$$\begin{aligned} Y_\gamma^\lambda \times X_*(T) &\longrightarrow X_\gamma^\lambda \\ (u, \nu) &\longmapsto u\varpi^\nu \end{aligned}$$

which induces bijection on k-points and a stratification of X_γ^λ such that each strata is isomorphic to Y_γ^λ. Thus X_γ^λ is a scheme locally of finite type and the assertions about equidimensionality and dimension formula follows from the corresponding statements for Y_γ^λ.

The LG_γ^0 action on the set $\mathrm{Irr}(X_\gamma^\lambda)$ factors through $\pi_0(LG_\gamma^0) = X_*(T)$ and hence LG_γ-orbits on $\mathrm{Irr}(X_\gamma^\lambda)$ corresponds bijectively to the set $\mathrm{Irr}(Y_\gamma^\lambda)$. Thus the number of orbits equals to the weight multiplicity $m_{\lambda\mu}$. □

3.6 Finiteness of Kottwitz-Viehmann varieties

In this section we let $\gamma \in G(F)^{\mathrm{rs}}$ be any regular semisimple element and $\lambda \in \Lambda^+$. Assume without loss of generality that X_γ^λ is nonempty and $\det(\gamma) = \det(\varpi^\lambda)$. Then we get an element $\gamma_\lambda \in \mathrm{Vin}_G^\lambda(F)$ as in Lemma 3.1.2. Moreover, the Newton point of γ satisfies $\nu_\gamma \leq_{\mathbb{Q}} \lambda$ and $\chi(\gamma) \in \mathfrak{C}_{\leq\lambda}$ by Proposition 3.1.3.

We show in this section that X_γ^λ, a'priori an ind-scheme, is actually a scheme locally of finite type. This has already been proved for unramified conjugacy classes in Corollary 3.5.2. It remains to reduce the general case to the unramified case. This reduction step is completely analogous to the Lie algebra case. For the reader's convenience, we include the details, following the exposition in [Yun15, §2.5]. See also [Bou15a].

Let F'/F be a finite extension of degree e so that γ splits over F'. Let $\varpi' = \varpi^{1/e} \in F'$ be a uniformizer and $\mathcal{O}' = k[[\varpi']]$ the ring of integers in F'. Let σ be a generator of the cyclic group $\mathrm{Gal}(F'/F)$

Choose $h \in G(F')$ such that $\mathrm{Ad}(h)G_\gamma^0 = T$. Then $h\sigma(h)^{-1} \in N_G(T)(F')$ and we let $w \in W$ be its image.

Consider the embedding

$$\begin{aligned} \iota_\gamma : \Lambda := X_*(T) &\longrightarrow G_\gamma(F') \\ \mu &\longmapsto \mathrm{Ad}(h)^{-1}\varpi^\mu \end{aligned}$$

Let $\Lambda_\gamma := \iota_\gamma^{-1}(G_\gamma(F))$. It follows immediately that $\Lambda_\gamma \subset \Lambda^w$ where Λ^w is the fixed point set of w on Λ. Moreover, Λ_γ can be identified with the coweight lattice of the maximal F-split subtorus of G_γ. In particular, $(\Lambda_\gamma)_{\mathbb{Q}} = (\Lambda^w)_{\mathbb{Q}}$ so that $\Lambda_\gamma \subset \Lambda^w$ is a subgroup of finite index.

Proposition 3.6.1. *There exists a closed subscheme $Z \subset X_\gamma^\lambda$ which is projective over k such that $X_\gamma^\lambda = \cup_{\ell \in \Lambda_\gamma} \ell \cdot Z$. Here $\ell \in \Lambda_\gamma$ acts on X_γ^λ via the embedding ι_γ.*

Proof. We reprase the argument in [Yun15, §2.5.7]. Let $\widetilde{X}_\gamma^{e\lambda}$ be the generalized affine Springer fiber of coweight $e\lambda$ for γ in $\mathrm{Gr}_{G_{F'}}$, the affine Grassmanian of $G_{F'}$. Then σ acts naturally on $\widetilde{X}_\gamma^{e\lambda}$ and the fixed points sub-indscheme $(\widetilde{X}_\gamma^{e\lambda})^\sigma$ contains X_γ^λ (but they are not equal in general). Let $\gamma' = h\gamma h^{-1} \in T(F')$ and $\widetilde{X}_{\gamma'}^{e\lambda}$ the corresponding generalized affine Springer fiber in $\mathrm{Gr}_{G_{F'}}$. Then

$$\widetilde{X}_{\gamma'}^{e\lambda} = h \cdot \widetilde{X}_\gamma^{e\lambda}$$

By Theorem 3.5.1, there is a locally closed subscheme $\widetilde{Y}_{\gamma'}^{e\lambda}$ of $\widetilde{X}_{\gamma'}^{e\lambda}$ such that

$$\widetilde{X}_{\gamma'}^{e\lambda} = \cup_{\ell \in \Lambda} \ell \cdot Y_{\gamma'}^{e\lambda}.$$

Let $\widetilde{Z}$ be the closure of $h^{-1}\widetilde{Y}_{\gamma'}^{e\lambda}$ in $\widetilde{X}_\gamma^{e\lambda}$. Then $\widetilde{Z}$ is projective over k and $\widetilde{X}_\gamma^{e\lambda} = \cup_{\ell \in \Lambda} \ell \cdot \widetilde{Z}$.

Recall that $w \in W$ is represented by $h\sigma(h)^{-1}$. One can check that $\sigma(\widetilde{Z}) = \widetilde{Z}$ and more generally $\sigma(\ell \cdot \widetilde{Z}) = w(\ell) \cdot \widetilde{Z}$ for all $\ell \in \Lambda$.Consequently,

$$(\widetilde{X}_\gamma^{e\lambda})^\sigma = \cup_{\ell \in \Lambda^w} \ell \cdot \widetilde{Z} = \cup_{\ell \in \Lambda_\gamma} \ell \cdot (C \cdot \widetilde{Z})$$

where $C \subset \Lambda^w$ is a finite set of representatives of the quotient Λ^w/Λ_γ. Hence, $C \cdot \widetilde{Z}$ is a finite type scheme.

Finally let $Z := (C \cdot \widetilde{Z}) \cap X_\gamma^\lambda$. Then Z is a finite type subscheme of X_γ^λ. Hence Z is projective over k and $X_\gamma^\lambda = \cup_{\ell \in \Lambda_\gamma} \ell \cdot Z$. $\square$

As a consequence, we immediately get:

Theorem 3.6.2. *The ind-scheme X_γ^λ is a finite dimensional k-scheme, locally of finite type. Moreover, the lattice Λ_γ acts freely on X_γ^λ and the quotient $X_\gamma^\lambda/\Lambda_\gamma$ is representable by a proper algebraic space over k.*

3.7 Dimension of the regular locus

Recall that the regular locus $X_\gamma^{\lambda,\mathrm{reg}}$ is an open subscheme of X_γ^λ on which the action of $P_a = LG_\gamma^0/L^+J_a$ is free (but not necessarily transitive).

Theorem 3.7.1.

$$\dim P_a = \dim X_\gamma^{\lambda,\mathrm{reg}} = \langle\rho,\lambda\rangle + \frac{d(\gamma)-c(\gamma)}{2}$$

where

- $d(\gamma) := \mathrm{val}(\det(\mathrm{Id} - \mathrm{ad}(\gamma) : \mathfrak{g}(F)/\mathrm{g}_\gamma(F) \to \mathfrak{g}(F)/\mathrm{g}_\gamma(F)))$.
- $c(\gamma) := \mathrm{rank}(G) - \mathrm{rank}_F G_\gamma$, *where* $\mathrm{rank}_F G_\gamma$ *is the dimension of the maximal F-split subtorus of G_γ.*

Moreover, $X_\gamma^{\lambda,\mathrm{reg}}$ is equidimensional.

Proof. The first equality follows from the fact that the P_a-orbits in $X_\gamma^{\lambda,\mathrm{reg}}$ are open and the action is free.

When γ is unramified (hence split as k is algebraically closed), the second equality follows from Corollary 3.5.2. It remains to reduce to this case. The argument is similar to that of Bezrukavnikov's in Lie algebra case, cf. [Bez96], which we reformulate using the Galois description of universal centralizer.

Let A be the finite free $\mathcal{O}$-algebra defined by the Cartesian diagram (3.3.1) and $A^\flat$ the normalization of A. Then W acts naturally on the $\mathcal{O}$-algebras A and $A^\flat$ and by 3.3.3, we get

$$\dim \mathcal{P}_a = \dim_k(\mathfrak{t}\otimes_k (A^\flat/A))^W.$$

Let $\tilde{F}/F$ be a ramified extension of degree e, with ring of integers $\tilde{\mathcal{O}} = k[[\varpi^{\frac{1}{e}}]]$, such that γ is split over $\tilde{F}$. Let σ be a generater of the cyclic group $\Gamma := \mathrm{Gal}(F'/F)$. Let $\tilde{A} := A\otimes_{\mathcal{O}}\tilde{\mathcal{O}}$ and $\tilde{A}^\flat$ its normalization. We remark that $\tilde{A}^\flat$ is not the same as $A^\flat\otimes_{\mathcal{O}}\tilde{\mathcal{O}}$ in general. Let

$\tilde{\mathcal{P}}_a = LG_{\gamma,F'}/L^+J_{a,F'}$. Then by the dimension formula in split case, we have

$$\dim_k(\mathfrak{t}\otimes_k \tilde{A}^\flat/\tilde{A})^W = \dim\tilde{\mathcal{P}}_a = \langle\rho, e\lambda\rangle + \frac{1}{2}e\cdot d(\gamma).$$

As γ split over $\tilde{\mathcal{O}}$, we have

$$\tilde{A}^\flat \cong \tilde{\mathcal{O}}[W] := \tilde{\mathcal{O}}\otimes_k k[W]$$

as W-module. Here W acts on $\tilde{\mathcal{O}}[W]$ via right regular representation. Moreover, there exists an element $w_\gamma \in W$ of order e such that under the above isomorphism, the natural action of $\sigma\in\Gamma$ on $\tilde{A}^\flat$ becomes $\sigma\otimes l_{w_\gamma}$ where l_{w_γ} denotes the left regular action of w_γ on $k[W]$. In particular, the action of W and Γ commutes with each other. With these considerations, we obtain an isomorphism

$$(\mathfrak{t}\otimes_k \tilde{A}^\flat)^W \cong \mathfrak{t}\otimes_k\tilde{\mathcal{O}}$$

which intertwines the action of $\sigma\in\Gamma$ on the left hand side with the action of $w\otimes\sigma$ on the right hand side.

Moreover, we have an equality

$$(\mathfrak{t}\otimes_k \tilde{A}^\flat)^\Gamma = \mathfrak{t}\otimes_k A^\flat$$

which remains true after taking W-invariants since the Γ action commutes with W action. In particular, we have

$$M := (\mathfrak{t}\otimes_k\tilde{\mathcal{O}})^\Gamma = (\mathfrak{t}\otimes_k A^\flat)^W$$

Moreover, it is clear that from the definition of W action that

$$(\mathfrak{t}\otimes_k\tilde{A})^W = (\mathfrak{t}\otimes_k A)^W\otimes_{\mathcal{O}}\tilde{\mathcal{O}}.$$

Thus we get

$$
\begin{aligned}
\dim \mathcal{P}_a &= \dim_k(\mathfrak{t} \otimes_k A^\flat/A)^W = \frac{1}{e}\dim_k(\mathfrak{t} \otimes_k (A^\flat/A) \otimes_{\mathcal{O}} \tilde{\mathcal{O}})^W = \frac{1}{e}\dim_k\left(\frac{M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}}{(\mathfrak{t} \otimes_k \tilde{A})^W}\right) \\
&= \langle \rho, \lambda \rangle + \frac{1}{2}d(\gamma) - \frac{1}{e}\dim_k\left(\frac{\mathfrak{t} \otimes_k \tilde{\mathcal{O}}}{M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}}\right)
\end{aligned}
\tag{3.7.1}
$$

Since the element $w_\gamma \in W$ has order e, its eigenvalues are e-th roots of unit. Let ζ be a primitive e-th root of unit and $\mathfrak{t}(i)$ the subspace of $\mathfrak{t}$ on which w_γ acts via the scalar ζ^i. In particular, $\mathfrak{t}(0) = \mathfrak{t}^{w_\gamma}$ is the w_γ invariant subspace. Then we have

$$
M := (\mathfrak{t} \otimes_k \tilde{\mathcal{O}})^\Gamma = \bigoplus_{i=0}^{e-1} \mathfrak{t}(i) \otimes_k \varpi^{\frac{e-i}{e}}
$$

The existence of a W-invariant nondegenerate symmetric bilinear form on $\mathfrak{t}$ gaurantees that $\dim_k \mathfrak{t}(i) = \dim_k \mathfrak{t}(e-i)$, from this we obtain that

$$
\dim_k\left(\frac{\mathfrak{t} \otimes_k \tilde{\mathcal{O}}}{M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}}\right) = e(\dim_k \mathfrak{t} - \dim_k \mathfrak{t}^{w_\gamma}) = e \cdot c(\gamma)
$$

Combined with (3.7.1), we obtain

$$
\dim \mathcal{P}_a = \langle \rho, \lambda \rangle + \frac{1}{2}(d(\gamma) - c(\gamma)).
$$

Finally, $X_\gamma^{\lambda,\mathrm{reg}}$ is equidimensional since it is a finite union of $\mathcal{P}_a$-torsors. $\square$

3.7.1 Some 0-dimensional generalized affine Springer fibers

Suppose X_γ^λ is nonempty. Then there exists $\gamma_\lambda \in G_+^{\mathrm{sc}}$ satisfying the conclusion of Lemma 3.1.2. Let $a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_{G_+^{\mathrm{sc}}}^{\mathrm{rs}}(F)$. Recall the extended discriminant divisor $\mathfrak{D}_+ \subset \mathfrak{C}_+$

defined in § 2.2.2. We define the *extended discriminant valuation* to be

$$d_+(a) := \mathrm{val}(a^*\mathfrak{D}_+) \in \mathbb{Z}$$

From equation (3.1.1) we get

$$\begin{aligned} d_+(a) &= 2\cdot \mathrm{val}(\rho(\alpha(\gamma_\lambda))) + d(\gamma) \\ &= \langle 2\rho, \lambda\rangle + d(\gamma) \\ &= \sum_{\substack{\alpha\in\Phi \\ \langle \alpha,\nu_\gamma\rangle = 0}} \mathrm{val}(\alpha(\gamma)-1) + \langle 2\rho, \lambda - \nu_\gamma\rangle \end{aligned} \tag{3.7.2}$$

Proposition 3.7.2. *Suppose* $d_+(a) = 0$*. Then* γ *is split and* $\dim X_\gamma^\lambda = 0$*. Moreover,* $X_\gamma^\lambda = X_\gamma^{\lambda,\mathrm{reg}}$ *and it is a torsor under* P_a*.*

Proof. The assumption $d_+(a) = 0$ implies that $a \in \mathfrak{C}_+^{\mathrm{rs}}(\mathcal{O})$. Let $\widetilde{X_a} = \operatorname{Spec} R_a$ be defined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{X_a} \arrow[r] \arrow[d] & \overline{T_+} \arrow[d] \\ \operatorname{Spec}\mathcal{O} \arrow[r] & \mathfrak{C}_+ \end{tikzcd}$$

By Proposition 2.2.9, $\widetilde{X_a}$ is an étale cover $\operatorname{Spec}\mathcal{O}$ which must be trivial since the residue field k is algebraically closed. Then we see that $\gamma_\lambda \in T_+(F)$ is split and hence $\gamma \in T(F)$ is split.

Since X_γ^λ is nonempty, we have $\nu_\gamma \leq_{\mathbb{Q}} \lambda$ and hence the terms on the right hand side of (3.7.2) are non-negative. In particular, $d_+(a) = 0$ implies that $\nu_\gamma = \lambda$. Thus the proposition follows from Corollary 3.5.2. $\square$

3.8 The case of central coweight

In this section we deal with the case where $\lambda \in X_*(T)_+$ is a central coweight, i.e. $\langle \lambda, \alpha \rangle = 0$ for all roots α. Then $\lambda \in X_*(Z^0)$ where Z^0 is the maximal torus in the center of G. Consequently we have $X_\gamma^\lambda \cong X^0_{\varpi^{-\lambda}\gamma}$. Hence the essential case is when $\lambda = 0$ and the corresponding Kottwitz-Viehmann variety becomes

$$X_\gamma := \{g \in G(F)/G(\mathcal{O}) | \mathrm{Ad}(g)^{-1}\gamma \in G(\mathcal{O})\}.$$

We first do some routine reductions. Let $P = MN$ be a standard parabolic subgroup with standard Levi M and unipotent radical N. For $\gamma \in M(\mathcal{O}) \cap G^{\mathrm{rs}}(F) \subset M(\mathcal{O}) \cap M^{\mathrm{rs}}(F)$, we consider the Kottwitz-Viehmann variety X_γ (resp. X_γ^M) defined for the groups G (resp. M). We have the discriminant valuation $d(\gamma)$ (resp. $d_M(\gamma)$) defined for G (resp. M). The two discriminant valuations are related by

$$d(\gamma) = d_M(\gamma) + 2r_N(\gamma) \tag{3.8.1}$$

where $r_N(\gamma)$ is defined in (3.4.2).

Proposition 3.8.1. *With notation as above, we have*

$$\dim X_\gamma = \dim X_\gamma^M + \frac{d_G(\gamma) - d_M(\gamma)}{2}$$

Proof. Let $P = MN$ be the standard parabolic subgroup with Levi factor is M and unipotent radical N. The connected components of Gr_M and Gr_P both corresponds bijectively to $\pi_1(M)$, the quotient of $X_*(T)$ by the coroot lattice of M. The canonical map $\mathrm{Gr}_P \to \mathrm{Gr}_G$ induces bijection on k-points by generalized Iwasawa decomposition. For each $\lambda \in \pi_1(M)$, let $X_{\gamma,\lambda}$ be the intersection of X_γ and the connected component of Gr_P corresponding to λ. Similarly, let $X_{\gamma,\lambda}^M$ be the intersection of X_γ^M with the connected component of Gr_M

corresponding to λ. Then there is a canonical morphism

$$p_\gamma^M : X_{\gamma,\lambda} \to X_{\gamma,\lambda}^M$$

It suffices to show that the fibres of this map have dimension $r_N(\gamma)$.

Let $h \in X_{\gamma,\lambda}^M$. Then $\gamma_h := h^{-1}\gamma h \in M(\mathcal{O})$ and we consider the fibre $Y_h := (p_\gamma^M)^{-1}(h)$. Its set of k points is

$$Y_h(k) = \{u \in N(F)/N(\mathcal{O}) | u^{-1}\gamma_h u \in G(\mathcal{O})\}$$

In other words, we have

$$Y_h = f_{\gamma_h}^{-1}(N(\mathcal{O}))/N(\mathcal{O})$$

where $f_{\gamma_h} : N(F) \to N(F)$ is defined by $f_\gamma(u) = u^{-1}\gamma_h u \gamma_h^{-1}$. Apply Proposition 3.4.4 to the admissible set $Z = N(\mathcal{O})$ we see that Y_h is an irreducible affine space of dimension

$$\dim Y_h = r_N(\gamma_h) = r_N(\gamma)$$

and hence we conclude by (3.8.1). $\square$

Corollary 3.8.2. *Let $\lambda \in X_*(T)$ be a central coweight and $\gamma \in G(F)^{\mathrm{rs}}$. Then*

$$\dim X_\gamma^\lambda = \frac{1}{2}(d_\gamma - c_\gamma)$$

Proof. We first assume that $\gamma \in G(\mathcal{O})$ is topologically unipotent mod center. In other words, the reduction mod ϖ of γ is unipotent mod center. After multiplying by an element in $Z(\mathcal{O})$, we may assume that $\gamma \in G^{\mathrm{sc}}(\mathcal{O})$ is topologically unipotent. Then when $G = G^{\mathrm{sc}}$ the argument of [KL88, §4] and [Ngô10, Proposition 3.7.1] generalize verbatim to our situation and proves $\dim X_\gamma^{\mathrm{reg}} = \dim X_\gamma$ and hence the dimension formula in this situation. More generally, we argue as in [Tsa16, Lemma 4.1] to reduce to the case $G = G^{\mathrm{sc}}$.

It remains to reduce to the case where γ is topologicallly unipotent mod center. After multiplying $\gamma \in G(\mathcal{O})$ by an element in $Z(\mathcal{O})$ we may assume that $\gamma \in G^{\mathrm{sc}}(\mathcal{O})$. Then G_γ is a maximal torus in G and $\gamma \in G_\gamma(F) \cap G(\mathcal{O})$. Let S be the maximal split subtorus in the centralizer G_γ. After conjugation we may assume that $S \subset T$. Let $M = C_G(S)$ be the centralizer of S in G. Then M is a standard Levi subgroup of G and $\gamma \in M(\mathcal{O})$. Let $a_M := \chi_M(\gamma) \in \mathfrak{C}_M(\mathcal{O})$. Then the pullback of T along $a_M : \operatorname{Spec} \mathcal{O} \to \mathfrak{C}_M$ is a totally ramified cover of $\operatorname{Spec} \mathcal{O}$ and we deduce that γ is topologically unipotent mod center in $M(\mathcal{O})$. Thus the result follows from the case already proved and Proposition 3.8.1. $\square$

3.9 Irreducible components

3.9.1 Stratification on dominant coweight cone

Let $\Lambda := X_*(T)$ and $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathsf{D} \subset \Lambda_{\mathbb{Q}}$ be the positive coroot cone. In other words, D consists of $\mathbb{Q}$-linear combinations of simple coroots with non-negative coefficients.

For $\lambda \in \Lambda^+$, we define the *dominant coweight polytope* to be:

$$\mathsf{P}_\lambda := \Lambda^+_{\mathbb{Q}} \cap \operatorname{Conv}(W \cdot \lambda) = \Lambda^+_{\mathbb{Q}} \cap (\lambda - \mathsf{D}).$$

where $\operatorname{Conv}(W \cdot \lambda)$ denotes the convex hull of the W-orbit of λ.

Lemma 3.9.1. *For each $\lambda_1, \lambda_2 \in \Lambda^+$ with $\kappa_G(\lambda_1) = \kappa_G(\lambda_2)$, there exists $\mu \in \Lambda^+$ such that $\mu \leq \lambda_1, \mu \leq \lambda_2$ and*

$$(\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D}) = \mu - \mathsf{D}.$$

In particular, we have $\mathsf{P}_{\lambda_1} \cap \mathsf{P}_{\lambda_2} = \mathsf{P}_\mu$.

Proof. Since $\kappa_G(\lambda_1) = \kappa_G(\lambda_2)$, the difference $\lambda_1 - \lambda_2$ lies in the coroot lattice. There exists a partition of the set of simple coroots $\Delta^\vee = \Delta^\vee_1 \sqcup \Delta^\vee_2$ such that

$$\lambda_1 - \lambda_2 = \beta_1 - \beta_2$$

where β_i is a non-negative integral linear combinations of simple coroots in $\Delta_i^\vee$ for $i \in \{1, 2\}$. Let $\Delta = \Delta_1 \sqcup \Delta_2$ be the corresponding partion of the set of simple roots. Consider the coweight $\mu := \lambda_1 - \beta_1 = \lambda_2 - \beta_2$. Then clearly $\mu \leq \lambda_1$ and $\mu \leq \lambda_2$.

We claim that $\mu \in \Lambda^+$. Take any simple root $\alpha \in \Delta_1$. Since β_2 is positive linear combination of coroots in Δ_2, we have $\langle \alpha, \beta_2 \rangle \leq 0$ and hence $\langle \mu, \alpha \rangle = \langle \lambda_2 - \beta_2, \alpha \rangle \geq 0$. Similarly, using $\mu = \lambda_1 - \beta_1$, we see that for all $\alpha \in \Delta_2$, $\langle \mu, \alpha \rangle \geq 0$. Thus we conclude that $\mu \in \Lambda_+$.

It is clear that $\mu - \mathsf{D} \subset (\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D})$. Now we prove the reverse inclusion. Let $\nu \in (\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D})$. Then for $i \in \{1, 2\}$, $\lambda_i - \nu \in \mathsf{D}$ is a non-negative $\mathbb{Q}$-linear combination of simple coroots and we need to show that $\mu - \nu \in \mathsf{D}$. For any fundamental weight ω, there exists $i \in \{1, 2\}$ so that ω is orthogonal to all coroots in $\Delta_i^\vee$. Without loss of generality assume $i = 1$, then we have

$$\langle \mu - \nu, \omega \rangle = \langle \lambda_1 - \beta_1 - \nu, \omega \rangle = \langle \lambda_1 - \nu, \omega \rangle \geq 0.$$

This means that $\nu \leq_{\mathbb{Q}} \mu$, or $\nu \in (\mu - \mathsf{D})$. Therefore we have shown that $\mu - \mathsf{D} = (\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D})$.

Finally, taking intersection with $\Lambda_{\mathbb{Q}}^+$, we get $\mathsf{P}_{\lambda_1} \cap \mathsf{P}_{\lambda_2} = \mathsf{P}_\mu$. $\square$

For each $\lambda \in \Lambda^+$, define

$$\mathsf{P}_\lambda^\circ := \mathsf{P}_\lambda - \bigcup_{\substack{\mu \in \Lambda^+, \\ \mu < \lambda}} \mathsf{P}_\mu. \tag{3.9.1}$$

Corollary 3.9.2. *For any $\lambda_1, \lambda_2 \in \Lambda_+$ with $\lambda_1 \neq \lambda_2$, we have $\mathsf{P}_{\lambda_1}^\circ \cap \mathsf{P}_{\lambda_2}^\circ = \varnothing$. In particular, we get a well-defined stratification*

$$\{\nu \in \Lambda_{\mathbb{Q}}^+ | p_{G,\mathbb{Q}}(\nu) \in X_*(G_{\mathrm{ab}}) \subset \pi_1(G)_{\mathbb{Q}}\} = \bigsqcup_{\lambda \in \Lambda^+} \mathsf{P}_\lambda^\circ.$$

Proof. If $\det(\varpi^{\lambda_1}) \neq \det(\varpi^{\lambda_2})$, it is clear that P_{λ_1} and P_{λ_2} are disjoint. Suppose $\det(\varpi^{\lambda_1}) \neq$

$\det(\varpi^{\lambda_2})$. Then by Lemma 3.9.1, there exists $\mu \in \Lambda^+$ such that $\mu \leq \lambda_1, \mu \leq \lambda_2$ and

$$\mathsf{P}^\circ_{\lambda_1} \cap \mathsf{P}^\circ_{\lambda_2} \subset \mathsf{P}_{\lambda_1} \cap \mathsf{P}_{\lambda_2} = \mathsf{P}_\mu.$$

But by (3.9.1), we have $\mathsf{P}_\mu \cap \mathsf{P}^\circ_{\lambda_i} = \varnothing$ since $\mu \leq \lambda_i$ for $i \in \{1, 2\}$. Therefore $\mathsf{P}^\circ_{\lambda_1} \cap \mathsf{P}^\circ_{\lambda_2} = \varnothing$. $\square$

3.9.2 Stratification on extended Steinberg base

To get a conceptually simpler formulation of the conjecture on irreducible components, we introduce a stratification on $\mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_{G^{\mathrm{sc}}_+}$.

Recall that $\mathfrak{C}_+ \cong A_{G^{\mathrm{sc}}} \times \mathbb{A}^r$. Consider the strata

$$\mathfrak{C}^\lambda_+ := \varpi^{-w_0(\lambda_{\mathrm{ad}})} T_{\mathrm{ad}}(\mathcal{O}) \times \mathcal{O}^r \subset \mathfrak{C}_+(\mathcal{O})$$

where $\lambda_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})_+$ is the image of λ.

For each $\mu \in \Lambda^+$ such that $\mu \leq \lambda$, we have an embedding

$$i_{\mu\lambda} : \mathfrak{C}^\mu_+ \hookrightarrow \mathfrak{C}^\lambda_+$$

defined by the formula

$$\begin{aligned} &i_{\mu\lambda}(a_1, \ldots, a_r, b_1, \ldots, b_r) = \\ &\quad (\varpi^{\langle -w_0(\lambda-\mu), \alpha_1 \rangle} a_1, \ldots, \varpi^{\langle -w_0(\lambda-\mu), \alpha_r \rangle} a_r, \varpi^{\langle -w_0(\lambda-\mu), \omega_1 \rangle}) b_1, \ldots, \varpi^{\langle -w_0(\lambda-\mu), \omega_r \rangle} b_r) \end{aligned} \tag{3.9.2}$$

Note that we need to choose a uniformiser to define the embedding $i_{\mu\lambda}$ but its image does not depend on this choice.

Proposition 3.9.3. *For any $\lambda, \mu_1, \mu_2 \in \Lambda^+$ with $\mu_1 \leq \lambda$ and $\mu_2 \leq \lambda$, there exists $\mu_3 \in \Lambda^+$*

such that $\mu_3 \leq \mu_1, \mu_3 \leq \mu_2$ *and*

$$i_{\mu_1\lambda}(\mathfrak{C}_+^{\mu_1}) \cap i_{\mu_2\lambda}(\mathfrak{C}_+^{\mu_2}) = i_{\mu_3\lambda}(\mathfrak{C}_+^{\mu_3}).$$

Proof. By Lemma 3.9.1, there exists $\mu_3 \in \Lambda^+$ such that

$$(\mu_1 - \mathsf{D}) \cap (\mu_2 - \mathsf{D}) = \mu_3 - \mathsf{D} \tag{3.9.3}$$

To prove the proposition, it suffices to show that

$$i_{\mu_1\lambda}(\mathfrak{C}_+^{\mu_1}) \cap i_{\mu_2\lambda}(\mathfrak{C}_+^{\mu_2}) \subset i_{\mu_3\lambda}(\mathfrak{C}_+^{\mu_3})$$

Let ι be the involution on the set $\{1, \dots, r\}$ such that $\omega_{\iota(i)} = -w_0(\omega_i)$ for all $1 \leq i \leq r$. For each $c = (c_1, \dots, c_r) \in \mathcal{O}^r$, let $a_i := \mathrm{val}(c_{\iota(i)})$.

Suppose that $\varpi^{(-w_0(\lambda_{\mathrm{ad}})}, c) \in i_{\mu_1\lambda}(\mathfrak{C}_+^{\mu_1}) \cap i_{\mu_2\lambda}(\mathfrak{C}_+^{\mu_2})$, then we get

$$a_i \geq \langle \lambda - \mu_1, \omega_i \rangle \text{ and } a_i \geq \langle \lambda - \mu_2, \omega_i \rangle \quad \text{for all } 1 \leq i \leq r \tag{3.9.4}$$

and we need to show that $a_i \geq \langle \lambda - \mu_3, \omega_i \rangle$ for all $1 \leq i \leq r$.

Let $\mu_1' := \sum_{i=1}^r \langle \mu_1, \omega_i \rangle \alpha_i^\vee$ and define μ_2', μ_3'. Then we have $\mu_1', \mu_2', \mu_3' \in \Lambda_0^+$. Consider the coweight $\nu := \sum_{i=1}^r (\langle \lambda, \omega_i \rangle - a_i)\alpha_i^\vee \in \Lambda_0$. By (3.9.3) and (3.9.4) we have

$$\nu \in (\mu_1' - \mathsf{D}) \cap (\mu_2' - \mathsf{D}) = \mu_3' - \mathsf{D}.$$

This implies that

$$\langle \lambda, \omega_i \rangle - a_i = \langle \nu, \omega_i \rangle \leq \langle \mu_3', \omega_i \rangle = \langle \mu_3, \omega_i \rangle$$

which is what we want. $\square$

For any $\lambda, \mu \in \Lambda^+$ with $\mu \leq \lambda$, define

$$\mathfrak{C}_+^{\lambda\mu} := i_{\mu\lambda}(\mathfrak{C}_+^{\mu}) - \bigcup_{\substack{\nu \in \Lambda_+ \\ \nu < \mu}} i_{\nu\lambda}(\mathfrak{C}_+^{\nu}). \tag{3.9.5}$$

Corollary 3.9.4. *For any $\lambda, \mu_1, \mu_2 \in \Lambda^+$ with $\mu_1 \neq \lambda$ and $\mu_2 \leq \lambda$, we have $\mathfrak{C}_+^{\lambda\mu_1} \cap \mathfrak{C}_+ \lambda\mu_2 = \varnothing$. In particular, we get well-defined stratifications*

$$\mathfrak{C}_+^{\lambda} = \bigsqcup_{\substack{\mu \in \Lambda_+ \\ \mu \leq \lambda}} \mathfrak{C}_+^{\lambda\mu}, \quad \mathfrak{C}_{G_+^{\mathrm{sc}}}(F) \cap \mathfrak{C}_+(\mathcal{O}) = \bigsqcup_{\substack{\lambda, \mu \in \Lambda_+ \\ \mu \leq \lambda}} \mathfrak{C}_+^{\lambda\mu}$$

Proof. The argument is similar to the proof of Corollary 3.9.2, using Proposition 3.9.3 instead of Lemma 3.9.1. □

The following lemma relates the stratas (3.9.5) the stratas (3.9.1).

Lemma 3.9.5. *For any $\lambda \in \Lambda^+$ and $\gamma \in G(F)^{\mathrm{rs}}$ with $\nu_\gamma \leq_{\mathbb{Q}} \lambda$, there exists a* unique *dominant integral coweight $\mu \in \Lambda_+$ with $\mu \leq \lambda$ that satisfies any (hence all) of the following equivalent conditions:*

1. *$\mu \in \Lambda_+$ is a minimal dominant integral coweight such that $\nu_\gamma \leq_{\mathbb{Q}} \mu$;*

2. *$\nu_\gamma \in \mathsf{P}_\mu^\circ$, cf.* (3.9.1)*;*

3. *$\chi_+(\gamma_\lambda) \in \mathfrak{C}_+^{\lambda\mu}$, cf.* (3.9.5).

Proof. The equivalence between (1) and (2) folows from the definition of P_μ. The equivalence of (1) and (3) follows from Proposition 3.1.3.

Finally, the uniqueness of μ follows from Lemma 3.9.1 or Proposition 3.9.3. □

Now we state our conjecture on irreducible components of X_γ^λ:

Conjecture 3.9.1. Let $\lambda \in \Lambda^+$ and $\gamma \in G(F)^{\mathrm{rs}}$ with $\nu_\gamma \leq_{\mathbb{Q}} \lambda$. Let $\mu \in \Lambda_+$ be the "best integral approximation" of ν_γ, i.e. the unique dominant coweight that satisfies the equivalent conditions in Lemma 3.9.5. Then the number of $G^0_\gamma(F)$-orbits on $\mathrm{Irr}(X^\lambda_\gamma)$ equals to the weight multiplicity $m_{\lambda\mu}$.

By Corollary 3.5.2, this conjecture is true when γ is an unramified conjugacy class.

Remark 3.9.1. For irreducible components of affine Deligne-Lusztig varieties, there is a similar conjecture made by Chen-Zhu, see the discussion in [HV17] and [XZ17]. In their setting, they also approximate Newton points of twisted conjugacy classes by integral coweight. However, the "best integral approximation" as defined in [HV17] is the largest integral coweight dominated by the Newton point. Whereas in the formulation of Conjecture 3.9.1, we use the smallest integral coweight dominating the Newton point. Simple examples suggest that these two integral approximations are very likely in the same Weyl group orbit, so we expect the two weight multiplicities to be the same.

3.9.3 Components of the regular locus

The $G^0_\gamma(F)$-orbits on $\mathrm{Irr}(X^{\lambda,\mathrm{reg}}_\gamma)$ corresponds bijectively to $G^0_\gamma(F)$ orbits on $X^{\lambda,\mathrm{reg}}_\gamma$, which are precisely the P_a-orbits of maximal dimension on $\mathrm{Sp}^0_a \cong X^\lambda_\gamma$. We know from Proposition 3.3.1 that these are the varieties $X^{\lambda,w}_\gamma = \mathrm{Sp}^w_a$ for $w \in \mathrm{Cox(W,S)}$.

However, for two different $w, w' \in \mathrm{Cox}(W,S)$, $X^{\lambda,w}_\gamma$ and $X^{\lambda,w'}_\gamma$ might coincide. For example, in the case $\lambda = 0$ and $\gamma \in G(\mathcal{O})$, all $X^{\lambda,w}_\gamma$ coincide (hence equal to $X^{\lambda,\mathrm{reg}}_\gamma$). So in this particular case $X^{\lambda,\mathrm{reg}}_\gamma$ is the unique $\mathcal{P}_a$-orbit of maximal dimension. In general, we know from (2.4.1) that the number of $G^0_\gamma(F)$ orbits in $X^{\lambda,\mathrm{reg}}_\gamma$ is bounded above by the Cardinality of $\mathrm{Cox}(W,S)$. We will see that in many situations, this upper bound can be achieved (in other words $X^{\lambda,w}_\gamma$ are mutually disjoint).

Theorem 3.9.6. *Let $\lambda \in \Lambda^+$ and $\gamma \in G(F)^{\mathrm{rs}}$ with $\nu_\gamma \leq_{\mathbb{Q}} \lambda$. Let $\mu \in \Lambda^+$ be the "best integral approximation" of the Newton point ν_γ as in Lemma 3.9.5. Then we have an inequality*

$$|\{G^0_\gamma(F) \text{ orbits on } X^{\lambda,\mathrm{reg}}_\gamma\}| \leq |\mathrm{Cox}(W,S)|$$

where $\mathrm{Cox}(W,S)$ *is the set of* S*-Coxeter elements defined in Definition 2.2.3. Moreover, when* λ *lies in the interior of the Weyl chamber and* $\lambda-\mu$ *lies in the interior of the positive coroot cone, the equality is achieved.*

Proof. It remains to show the last statement. Suppose λ lies in the interior of the Weyl chamber and $\lambda-\mu$ lies in the interior of the dominant coroot cone. Consider the following Cartesian diagram

$$\begin{tikzcd} \chi_+^{-1}(a) \arrow[r] \arrow[d] & \mathrm{Vin}_{G^{\mathrm{sc}}} \arrow[d, "\chi_+"] \\ \operatorname{Spec}\mathcal{O} \arrow[r, "a"] & \mathfrak{C}_+ \end{tikzcd}$$

For $g\in G(F)$ such that $gG(\mathcal{O})\in X_\gamma^{\lambda,\mathrm{reg}}$, let $\overline{\mathrm{Ad}(g)^{-1}\gamma}$ be the reduction mod ϖ of $\mathrm{Ad}(g)^{-1}\gamma\in\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}(\mathcal{O})$. The condition that λ lies in the interior of the Weyl chamber means that $\langle\lambda,\alpha_i\rangle>0$ for all simple roots α_i. Hence the special fiber of $\chi_+^{-1}(a)$ lies in the asymptotic semigroup $\mathrm{As}(G^{\mathrm{sc}}):=\alpha^{-1}(0)$ and in particular $\overline{\mathrm{Ad}(g)^{-1}\gamma}\in\mathrm{As}(G^{\mathrm{sc}})\cap\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}$.

Furthermore, the assumption that $\lambda-\mu$ lies in the interior of the positive coroot cone implies that $\langle\lambda-\mu,\omega_i\rangle>0$ for all fundamental weight ω_i. Therefore the reduction mod ϖ of a equals to 0 and the special fiber of $\chi_+^{-1}(a)$ is the nilpotent cone $\mathcal{N}$. In particular, we get $\overline{\mathrm{Ad}(g)^{-1}\gamma}\in\mathcal{N}^{\mathrm{reg}}$.

Consequently there is a bijection between $G_\gamma^0(F)$ orbits on $X_\gamma^{\lambda,\mathrm{reg}}$ and G orbits on $\mathcal{N}^{\mathrm{reg}}$, the latter of which corresponds bijectively to $\mathrm{Cox}(W,S)$ by Proposition 2.2.6. □

As an immediate consequence, we mention the following purely combinatorial result, which might be of independant interest:

Corollary 3.9.7. *Let* $\lambda\geq\mu$ *be dominant weights of a complex reductive group* G*. Suppose that* λ *lies in the interior of the Weyl chamber and* $\lambda-\mu$ *lies in the interior of the positive root cone (the "wide cone"). Then we have the following lower bound for the weight multiplicity*

$$m_{\lambda\mu}\geq|\mathrm{Cox}(W,S)|$$

where the set $\mathrm{Cox}(W,S)$ *is defined in §2.2.3.*

Proof. We consider the dual group $G^\vee$ of G over k. Then $\lambda \geq \mu$ are dominant coweights for $G^\vee$. Let $T^\vee \subset G^\vee$ be a maximal torus and $\gamma \in \varpi^\mu T^\vee(\mathcal{O}) \cap G^\vee(F)^{\mathrm{rs}}$. Then the generalized affine Springer fibre X_γ^λ is nonempty and by Corollary 3.5.2, the number of $G_\gamma^{\vee,0}(F)$-orbits on $\mathrm{Irr}(X_\gamma^\lambda)$ equals to $m_{\lambda\mu}$. On the other hand, by Theorem 3.9.6, the number of $G_\gamma^{\vee,0}(F)$-orbits on $\mathrm{Irr}(X_\gamma^{\lambda,\mathrm{reg}})$ equals to $|\mathrm{Cox}(W,S)|$, hence the inequality. □

Remark 3.9.2. If G_{ad} is simple of rank r, then $|\mathrm{Cox}(W,S)| = 2^{r-1}$. In general, if the simple factors of G_{ad} has rank $r_1, \ldots, r_m$, then

$$|\mathrm{Cox}(W,S)| = \prod_{i=1}^{m} 2^{r_i-1}.$$

We expect that there should be a more straightforward proof of Corollary 3.9.7.

Remark 3.9.3. In general, the weight multiplicity $m_{\lambda\mu}$ will increase with λ while the right hand side in Corollary 3.9.7 is a fixed constant independant of λ, μ. Thus in general there will be much more irreducible components in X_γ^λ than the regular open subvariety $X_\gamma^{\lambda,\mathrm{reg}}$.

CHAPTER 4

THE HITCHIN-FRENKEL-NGÔ FIBRATION

In this chapter we study global analogue of Kottwitz-Viehmann varieties, the Hitchin-Frenkel-Ngô fibration. These are certain group analogue of Hitchin fibrations, first introduced in [FN11] and later studied in more detail in [Bou17] and [Bou15b].

Throughout this chapter we let X be a projective smooth curve of genus g over k and G a connected reductive group over k.

4.1 First definitions

Let $\mathcal{L}$ be a $Z_+^{\mathrm{sc}} = T^{\mathrm{sc}}$ torsor on X. Then we can twist the schemes $\mathrm{Vin}_{G^{\mathrm{sc}}}$ (resp. $\mathfrak{C}_+$, $A_{G^{\mathrm{sc}}}$) by $\mathcal{L}$ to form corresponding affine spaces $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$ (resp. $\mathfrak{C}_+^{\mathcal{L}}$, $A_{G^{\mathrm{sc}}}^{\mathcal{L}}$) over X.

Definition 4.1.1. The *Hitchin-Frenkel-Ngô moduli stack* associated to the T^{sc}-torsor $\mathcal{L}$ is the mapping stack

$$\mathcal{M}_{\mathcal{L}} := \mathrm{Hom}(X, [\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}/\mathrm{Ad}(G)])$$

In other words, $\mathcal{M}_{\mathcal{L}}$ classifies pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a G-torsor on X and φ is a section of $\mathcal{E} \wedge^G \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$ where G acts on $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$ by adjoint action, and the action factors through G_{ad}.

Replacing $\mathrm{Vin}_{G^{\mathrm{sc}}}$ by $\mathrm{Vin}_{G^{\mathrm{sc}}}^0$ (resp. $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}$) in the definition of $\mathcal{M}_{\mathcal{L}}$, we define open substacks $\mathcal{M}_{\mathcal{L}}^0$ (resp. $\mathcal{M}_{\mathcal{L}}^{\mathrm{reg}} \subset \mathcal{M}_{\mathcal{L}}$). Also we define

$$\mathcal{A}_{\mathcal{L}} := \mathrm{Hom}_X(X, \mathfrak{C}_+^{\mathcal{L}}), \quad \mathcal{B}_{\mathcal{L}} := \mathrm{Hom}_X(X, A_{G^{\mathrm{sc}}}^{\mathcal{L}})$$

as the space of sections of the affine space $\mathfrak{C}_+^{\mathcal{L}}$ (resp. $A_{G^{\mathrm{sc}}}^{\mathcal{L}}$) over X. More concretely, we can describe $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{B}_{\mathcal{L}}$ as follows.

For each $\omega \in X^*(T)$, let $\omega(\mathcal{L})$ be the invertible sheaf on X defined by pushing $\mathcal{L}$ along the morphism $\omega : T \to \mathbb{G}_m$. Then we have

$$\mathcal{B}_{\mathcal{L}} = H^0(X, A_{G^{\mathrm{sc}}}^{\mathcal{L}}) = \bigoplus_{i=1}^{r} H^0(X, \alpha_i(\mathcal{L}))$$

and

$$\mathcal{A}_{\mathcal{L}} = \mathcal{B}_{\mathcal{L}} \oplus \bigoplus_{i=1}^{r} H^0(X, \omega_i(\mathcal{L})).$$

Let $\beta_{\mathcal{L}} : \mathcal{A}_{\mathcal{L}} \to \mathcal{B}_{\mathcal{L}}$ be the natural projection.

Definition 4.1.2. The *Hitchin-Frenkel-Ngô* fibration is the morphism

$$h_{\mathcal{L}} : \mathcal{M}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$$

induced by $\chi_+ : \mathrm{Vin}_{G^{\mathrm{sc}}} \to \mathfrak{C}_+$.

Also, let $\alpha_{\mathcal{L}} : \mathcal{M}_{\mathcal{L}} \to \mathcal{B}_{\mathcal{L}}$ be the map induced by $\alpha : \mathrm{Vin}_{G^{\mathrm{sc}}} \to A_{G^{\mathrm{sc}}}$. In particular, we have $\alpha_{\mathcal{L}} = \beta_{\mathcal{L}} \circ h_{\mathcal{L}}$.

Each point $b \in \mathcal{B}_{\mathcal{L}}$ can be written as $b = (b_1, \ldots, b_r)$ where $b_i \in H^0(X, \alpha_i(\mathcal{L}))$. Denote by $\mathcal{B}_{\mathcal{L}}^{\circ} \subset \mathcal{B}_{\mathcal{L}}$ the open subset consisting of those b such that b_i is nonzero for all i.

Definition 4.1.3. The *generically regular semisimple locus* $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$ is the open subset of $\mathcal{A}_{\mathcal{L}}$ consisting of sections $a : X \to \mathfrak{C}_+^{\mathcal{L}}$ such that $\beta_{\mathcal{L}}(a) \in \mathcal{B}_{\mathcal{L}}^{\circ}$ and $a(X)$ generically lies in the open subset $\mathfrak{C}_+^{\mathrm{rs},\mathcal{L}} = \mathfrak{C}_+^{\mathcal{L}} - \mathfrak{D}_+^{\mathcal{L}}$.

4.1.1 Global Steinberg section

Let $c = |Z(G_{\mathrm{der}})|$ be the order of the center of the derived group of G. Suppose there exists a T^{sc}-tosor $\mathcal{L}'$ such that $\mathcal{L} \cong (\mathcal{L}')^{\otimes c}$. By definition, there is a canonical map $[\mathrm{ev}]_{\mathcal{L}} : \mathcal{A}_{\mathcal{L}} \times X \to$

$[\mathfrak{C}_+/T^{\mathrm{sc}}]$ making the following diagram commutative:

$$\begin{tikzcd} \mathcal{A}_{\mathcal{L}} \times X \arrow[r, "{[\mathrm{ev}]_{\mathcal{L}}}"] \arrow[d] & {[\mathfrak{C}_+/T^{\mathrm{sc}}]} \arrow[d] \\ X \arrow[r, "\mathcal{L}"] & \mathbb{B}T^{\mathrm{sc}} \end{tikzcd}$$

Here the left arrow is projection to X and the bottom arrow corresponds to the T^{sc}-torsor $\mathcal{L}$.

The choice of c-th root $\mathcal{L}'$ of $\mathcal{L}$ defines a morphism $[\mathrm{ev}]_{\mathcal{L}'} : \mathcal{A}_{\mathcal{L}} \times X \to [\mathfrak{C}_+/T^{\mathrm{sc}}]$ lifting $[\mathrm{ev}]_{\mathcal{L}}$. Then for each $w \in \mathrm{Cox}(W, S)$ (cf. Definition 2.2.3), the composition of $[\mathrm{ev}]_{\mathcal{L}'}$ and the section $\epsilon^w_{+,[c]}$ of $[\chi_+]_{[c]}$ (cf. Proposition 2.2.12) induces a section of $h_{\mathcal{L}}$:

$$\epsilon^w_{\mathcal{L}'} : \mathcal{A}_{\mathcal{L}} \to \mathcal{M}^{\mathrm{reg}}_{\mathcal{L}} \subset \mathcal{M}_{\mathcal{L}}.$$

We refer to $\epsilon^w_{\mathcal{L}}$ as the *global Steinberg section.*

4.2 Symmetries of Hitchin-Frenkel-Ngô fibration

Definition 4.2.1. Let $\mathcal{P}_{\mathcal{L}}$ be the Picard stack over $\mathcal{A}_{\mathcal{L}}$ that associates to any S-point $a \in \mathcal{A}_{\mathcal{L}}(S)$ the Picard groupoid $\mathcal{P}_a$ of $\mathcal{J}_a$ torsors on $X \times S$. Here $\mathcal{J}_a$ is the pull back of the universal centralizer $\mathcal{J}_{\mathcal{L}}$ on $\mathfrak{C}^{\mathcal{L}}_+$ along the map $a : X \times S \to \mathfrak{C}^{\mathcal{L}}_+$.

Proposition 4.2.1. *$\mathcal{P}_{\mathcal{L}}$ is a smooth Picard stack over $\mathcal{A}_{\mathcal{L}}$.*

Proof. The argument of [Ngô10, Proposition 4.3.5] generalize *verbatim* to our situation. $\square$

The action of $\mathbb{B}\mathcal{J}$ on $[\mathrm{Vin}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$ (resp. $[\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$) induces action of $\mathcal{P}_{\mathcal{L}}$ on $\mathcal{M}_{\mathcal{L}}$ (resp. $\mathcal{M}^{\mathrm{reg}}_{\mathcal{L}}$).

To understand the connected components of the fibres of $\mathcal{P}_{\mathcal{L}}$, we utilize cameral covers.

Definition 4.2.2. The *cameral cover* associated to each $a \in \mathcal{A}_{\mathcal{L}}(k)$ is the finite flat cover

$\pi_a : \widetilde{X}_a \to X$ defined by the following Cartesian diagram

$$\begin{tikzcd} \widetilde{X}_a \arrow[r] \arrow[d, "\pi_a"'] & \overline{T_+^{\mathrm{sc}}}^{\mathcal{L}} \arrow[d] \\ X \arrow[r, "a"] & \mathfrak{C}_+^{\mathcal{L}} \end{tikzcd}$$

For any closed point $a \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$, we define the *discriminant divisor* for a to be the effective divisor

$$\Delta_a := a^{-1}(\mathfrak{D}_+^{\mathcal{L}})$$

Over the nonempty open subset $U_a := X - \Delta_a$, the cameral cover π_a is Galois étale with Galois group W. Choosing a point $\tilde{u} \in \widetilde{X}_a$ with $u := \pi_a(\tilde{u}) \in U_a$, we get a homomorphism

$$\rho_a : \pi_1(U_a, u) \to W$$

whose image is a subgroup $W_a \subset W$. Note that the conjugacy class of W_a in W is independant of the choice of base point $\tilde{u}$.

Let $\mathcal{J}_a^0 \subset \mathcal{J}_a$ be the fibrewise neutral component and consider the Picard stack $\mathcal{P}'_a := \mathrm{Bun}_{\mathcal{J}_a^0}$ of $\mathcal{J}_a^0$-torsors on X. Then there is a natural homomorphism of Picard stacks $\mathcal{P}'_a \to \mathcal{P}_a$. The following Lemma is parallel to [Ngô10, Lemme 4.10.2] with exactly the same proof.

Lemma 4.2.2. *The homomorphism $\mathcal{P}'_a \to \mathcal{P}_a$ is surjective with finite kernel. Same is true for the induced homomorphism $\pi_0(\mathcal{P}'_a) \to \pi_0(\mathcal{P}_a)$.*

Corollary 4.2.3. *$\pi_0(\mathcal{P}_a)$ is finite if and only if T^{W_a} is finite.*

Proof. By previous lemma, $\pi_0(\mathcal{P}_a)$ is finite if and only if $\pi_0(\mathcal{P}'_a)$ is finite. By [Ngô06, Corollaire 6.7], $\pi_0(\mathcal{P}'_a) = \hat{T}^{W_a}$. Since the finiteness of T^{W_a} is equivalent to the finiteness of $\hat{T}^{W_a}$, the result follows. $\square$

Definition 4.2.3. The *anisotropic locus* is the subset $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}} \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ consisting of $a \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ such that the component group $\pi_0(\mathcal{P}_a)$ is finite.

For each subset $I \subset \Delta$, we consider the invariant quotient $\overline{T^{\mathrm{sc}}_+}^{W_I}$. Then the natural morphism $\overline{T^{\mathrm{sc}}_+}^{W_I} \to \mathfrak{C}_+$ is finite and $Z^{\mathrm{sc}}_+ = T^{\mathrm{sc}}$ equivariant. Denote $\overline{T^{\mathrm{sc}}_+}^{W_I, \mathcal{L}} := \overline{T^{\mathrm{sc}}_+}^{W_I} \times^{Z^{\mathrm{sc}}_+} \mathcal{L}$.

Let $\mathcal{A}^{W_I}_{\mathcal{L}} := H^0(X, \overline{T^{\mathrm{sc}}_+}^{W_I, \mathcal{L}})$ be the space of sections of the affine scheme $\overline{T^{\mathrm{sc}}_+}^{W_I, \mathcal{L}}$ over X. Consider the map

$$\nu_I : \mathcal{A}^{W_I}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$$

induced by the finite morphism $\overline{T^{\mathrm{sc}}_+}^{W_I, \mathcal{L}} \to \mathfrak{C}^{\mathcal{L}}_+$. Let $\mathcal{A}^{W_I, \heartsuit}_{\mathcal{L}} := \nu_I^{-1}(\mathcal{A}^{\heartsuit}_{\mathcal{L}})$.

Proposition 4.2.4. *Suppose G is semisimple. Then the complement of $\mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$ in $\mathcal{A}^{\heartsuit}_{\mathcal{L}}$ is a finite union*

$$\mathcal{A}^{\heartsuit}_{\mathcal{L}} \setminus \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}} = \bigcup_{I \subsetneqq \Delta} \nu_I(\mathcal{A}^{W_I, \heartsuit}_{\mathcal{L}}).$$

Proof. Let $a \in \mathcal{A}^{\heartsuit}_{\mathcal{L}} - \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$. Then by Corollary 4.2.3, T^{W_a} contains a nontrivial torus S. Since G is semisimple, the centralizer of S is a *proper* Levi subgroup of G whose simple roots form a proper subset $I \subsetneqq W$. Then we have $W_a \subset W_I$.

Consider the following diagram in which both squares are Cartesian:

$$\begin{tikzcd}
\widetilde{X}_a \arrow[r] \arrow[d] \arrow[rr, bend left, "\pi_a"] & Y_a \arrow[r, "\pi^I_a"'] \arrow[d] & X \arrow[d] \\
\overline{T^{\mathrm{sc}}_+}^{\mathcal{L}} \arrow[r] & \overline{T^{\mathrm{sc}}_+}^{W_M, \mathcal{L}} \arrow[r] & \mathfrak{C}^{\mathcal{L}}_+
\end{tikzcd}$$

Let $\widetilde{Y_a} \subset \widetilde{X_a}$ be the union of all irreducible components that contain a point in the W_I-orbit of $\tilde{u}$. Then the image of $\widetilde{Y}_a$ in Y_a is isomorphic to X and hence gives a section of the morphism π^I_a. In other words, there is a section $a_I : X \to \overline{T^{\mathrm{sc}}_+}^{W_M, \mathcal{L}}$ such that $\nu_I(a_I) = a$. This proves that

$$\mathcal{A}^{\heartsuit}_{\mathcal{L}} \setminus \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}} \subset \bigcup_{I \subsetneqq \Delta} \nu_I(\mathcal{A}^{W_I, \heartsuit}_{\mathcal{L}}).$$

Conversely, for any $I \subsetneqq \Delta$ and $a_I \in \mathcal{A}^{W_I, \heartsuit}_{\mathcal{L}}$ with $\nu_I(a_I) = a$, the morphism π^I_a in the diagram above has a section given by a_I. This implies that $W_a \subset W_I$ so that T^{W_a} is not finite. By Corollary 4.2.3 again we see that $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$. $\square$

Corollary 4.2.5. *Suppose G is semisimple. Then $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is an open subset of $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$. Moreover, for any $b \in \mathcal{B}_{\mathcal{L}}^{\circ}$ and any integer N with $N > \max\{2g-2, rg\}$, if $\deg \omega_i(\mathcal{L}) > N$ for all $1 \le i \le r$, then the complement of $\mathcal{A}_{\mathcal{L},b}^{\mathrm{ani}}$ in $\mathcal{A}_{\mathcal{L},b}^{\heartsuit}$ has codimension at least $N - rg$.*

Proof. By valuative criterion and [Ngô06, Lemme 7.3] we see that ν_I is proper. So the images $\nu_I(\mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit})$ are closed subsets of $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$ and their complement $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is open. It remains to calculate the dimension of $\mathcal{A}_{\mathcal{L}}^{W_I}$.

Let $I \subsetneq \Delta$ and L_I a corresponding Levi subgroup of G^{sc}. We label the fundamental weights $\omega_1, \ldots, \omega_r$ of G^{sc} so that $\omega_1, \ldots, \omega_s$ are fundamental weights for L_I where $s = |I| < r$. There is a natural morphism

$$q^I : \overline{T_+^{\mathrm{sc}}}^{W_I} \to A_{G^{\mathrm{sc}}} \times \mathbb{A}^s$$

given by the W_I-invariant functions $(\alpha_i, 0)$ for $1 \le i \le r$ and $(\omega_i, \chi_{\omega_i}^I)$ for $1 \le i \le s$, where $\chi_{\omega_i}^I$ is the character of the irreducible representation of L_I with highest weight ω_i. The map q^I induces a map

$$q_X^I : \mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit} \to \mathcal{B}_{\mathcal{L}}^{\circ} \oplus \bigoplus_{i=1}^{s} H^0(X, \omega_i(\mathcal{L}))$$

The fibres of q^I over the open subset $T_{\mathrm{ad}} \times \mathbb{A}^s \subset A_{G^{\mathrm{sc}}} \times \mathbb{A}^s$ are isomorphic to $\mathbb{G}_m^{r-s}$. This implies that the nonempty fibres of q_X^I are $(k^\times)^{r-s}$. Hence

$$\dim \mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit} \le \dim \mathcal{B}_{\mathcal{L}} + \sum_{i=1}^{s} (\deg(\omega_i(\mathcal{L})) + 1 - g) + r - s.$$

Therefore, the codimension of $\mathcal{A}_{\mathcal{L},b}^{\heartsuit} - \mathcal{A}_{\mathcal{L},b}^{\mathrm{ani}}$ is bounded below by

$$\sum_{i=1}^{r} (\deg(\omega_i(\mathcal{L})) + 1 - g) - [\sum_{i=1}^{s} (\deg(\omega_i(\mathcal{L})) + 1 - g) + r - s] \ge N - rg.$$

$\square$

Denote $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} := h_{\mathcal{L}}^{-1}(\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}})$ the anisotropic open substack. This is nonempty when G is semisimple. Also, let $\mathcal{P}_{\mathcal{L}}^{\mathrm{ani}}$ be the restriction of $\mathcal{P}_{\mathcal{L}}$ to $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$.

Proposition 4.2.6. *$\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ and $\mathcal{P}_{\mathcal{L}}^{\mathrm{ani}}$ are Deligne-Mumford stacks.*

Proof. Let $(\mathcal{E}, \varphi) \in \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}(k)$ and $a = h_{\mathcal{L}}(\mathcal{E}, \varphi)$. Then the k-group $\mathrm{Aut}(\mathcal{E}, \varphi)$ classifies sections of the group scheme $\mathrm{Aut}_G(\mathcal{E})_\varphi$ over X, which is the closed subscheme of centralizer of φ in the group scheme $\mathrm{Aut}_G(\mathcal{E})$.

Choose a geometric point $\bar{\eta}$ over the generic point η of X. Restricting the cameral cover to η along a, we obtain a homomorphism $\rho_a^\eta : \mathrm{Gal}(\bar{\eta}/\eta) \to W$. Let W_a be the image of ρ_a^η. Furthermore, choose a trivialization of $\mathcal{E}$ over the generic point η under which φ maps to a regular semisimple element in $T_+(k(X))$. With these choice we get a closed embeddings $\mathrm{Aut}(\mathcal{E}, \varphi) \subset T^{W_a}$ and $H^0(X, \mathcal{J}_a) \subset T^{W_a}$.

Since $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$, T^{W_a} is finite. Since $\mathrm{char}(k)$ is coprime to the order of W, T^{W_a} is finite unramified k-group. This shows that $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ and $\mathcal{P}_{\mathcal{L}}^{\mathrm{ani}}$ are Deligne-Mumford stacks. □

Theorem 4.2.7. *Assume that the T^{sc}-torsor $\mathcal{L}$ admits a c-th root $\mathcal{L}'$. Then for any $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$, there is a homeomorphism of quotient stacks*

$$[\mathcal{M}_a/\mathcal{P}_a] \cong \prod_{x \in X - U_a} [\mathrm{Sp}_{a_x}/P_{a_x}] \tag{4.2.1}$$

In particular, we have

$$\dim \mathcal{M}_a - \dim \mathcal{P}_a = \sum_{x \in \mathrm{Supp}(\Delta_a)} (\dim \mathrm{Sp}_{a_x} - P_{a_x}).$$

Proof. Choose a Coxeter element $w \in \mathrm{Cox}(W, S)$. The c-th root $\mathcal{L}'$ of $\mathcal{L}$ induces a global Steinberg section $\epsilon_{\mathcal{L}'}^w$, in particular a base point $\epsilon_{\mathcal{L}'}^w(a) \in \mathcal{M}_a^{\mathrm{reg}}$. Using Corollary 2.2.10, we argue as in the proof of [Ngô06, Théorème4.6] to show that there is a morphism as (4.2.1) inducing equivalence of groupoids on k-points. Then the argument of [Ngô10] shows that the map (4.2.1) is a homeomorphism. □

4.3 Properness over the anisotropic locus

Throughout this section, we assume G is semisimple so that $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is nonempty. Our goal is to show that the morphism $h_{\mathcal{L}}^{\mathrm{ani}} : \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} \to \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is proper.

4.3.1 Finiteness properties

We first show that the Hitchin-Frenkel-Ngô fibration is of finite type over the anisotropic locus.

We start with a more general situation. Let $\rho : G \to \mathrm{GL}(V)$ a finite dimensional representation such that $\ker(\rho)$ is contained in the center of G. Fix a torus T and a Borel subgroup B containing T. Let $V^{(1)}, \ldots, V^{(m)}$ be the irreducible consitituents of V (counted with multiplicity) and $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be the corresponding highest weight.

For each $V^{(j)}$, we choose a basis $\{e_i^{(j)}, 1 \le i \le d_j\}$ (where $d_j = \dim V_j$) as follows. Each $e_i^{(j)}$ is a weight vector with weight $\lambda_i^{(j)} \in X^*(T)$. Then we can express $\lambda^{(j)} - \lambda_i^{(j)}$ as a linear combination of positive simple roots with non-negative integer coefficients and we call the sum of coefficients the *height* of $e_i^{(j)}$. The basis elements $e_i^{(j)}$ are indexed so that the height is non-decreasing with respect to i. In particular, $e_1^{(j)}$ is a highest weight vector and $e_{d_j}^{(j)}$ is a lowest weight vector in V_j.

Then under the basis $\{e_i^{(j)}, 1 \le i \le d_j, 1 \le j \le m\}$, $\rho(B)$ consists of upper triangular matrices in $\prod_j \mathrm{End}(V_j)$, which are the stabilizers of the standard flags $0 = L_0^{(j)} \subset L_1^{(j)} \subset \cdots \subset L_{d_j}^{(j)} = V^{(j)}$ where $L_i^{(j)} = \mathrm{Span}(e_1^{(j)}, \ldots, e_i^{(j)})$ for $1 \le i \le d_j$.

Let $I \subset \Delta$ be a subset of simple roots and $P_I \subset G$ the standard parabolic subgroup whose Levi factor has simple roots in I. Then there exists standard parabolic subalgebras $\mathfrak{p}_I^{(j)} \subset \mathrm{End}(V^{(j)})$ such that

$$\rho(P_I) = \rho(G) \cap (\bigoplus_{j=1}^{m} \mathfrak{p}_I^{(j)}).$$

More precisely, $\mathfrak{p}_I^{(j)}$ is the stabilizer of the partial flag in $V^{(j)}$ obtained from the standard flag by replacing $L_i^{(j)}$ with the span of $L_i^{(j)}$ and all basis vectors whose corresponding weight differs from the weight of $e_i^{(j)}$ by a linear combination of simple roots in I.

Fix a divisor D on a smooth projective curve X. Consider the following stack

$$\mathcal{M}_V := \mathrm{Hom}(X, [(\prod_{j=1}^{m} \mathrm{End}(V^{(j)})(D))/G])$$

where the action of G on $\prod_{j=1}^{m} \mathrm{End}(V^{(j)})$ is induced by ρ. More concretely, the moduli stack $\mathcal{M}_V$ classifies tuples $(E, \varphi_j, 1 \le j \le m)$ where E is a G-torsor and $\varphi_j : \rho_j E \to \rho_j E(D)$ is a meromorphic endomorphism of the vector bundle $\rho_j E := E \wedge^{(G,\rho)} V^{(j)}$.

From the definition, we have

$$\mathcal{M}_V = \mathcal{M}_1 \times_{\mathrm{Bun}_G} \mathcal{M}_2 \times_{\mathrm{Bun}_G} \cdots \times_{\mathrm{Bun}_G} \mathcal{M}_m$$

where for each $1 \le j \le m$, we define

$$\mathcal{M}_j = \mathrm{Hom}(X, [(\mathrm{End}(V^{(j)})(D))/G]).$$

By ??? we know that there exists a constant $C > 0$ such that for any G-torsor E on X there exists a Borel reduction E_B of E so that $\deg(E_B)$ belongs to

$$\mathsf{C} := \{H \in \Lambda_{\mathbb{Q}}, \alpha(H) \ge -c \; \forall \alpha \in \Delta\}.$$

Let N be a positive integer which is larger than the sum of coefficients of $\lambda^{(j)} - \lambda_i^{(j)}$ under the basis Δ for all i, j. Let d be an integer such that

$$d > \deg(D) + 2Nc \tag{4.3.1}$$

For each subset $I \subset \Delta$, consider the following cone

$$\mathsf{C}_I := \{H \in \Lambda_{\mathbb{Q}}, \alpha(H) \leq d \, \forall \alpha \in I \text{ and } \alpha(H) \geq d \, \forall \alpha \in \Delta - I\}.$$

Lemma 4.3.1. *Let $(E, \varphi_j) \in \mathcal{M}_V$ and E_B a B-reduction of E. Suppose that $\deg E_B \in \mathsf{C} \cap \mathsf{C}_I$, then we have*

$$\varphi \in \mathfrak{p}(D) \wedge^B E_B$$

where $\mathfrak{p} = \bigoplus_{j=1}^m \mathfrak{p}_I^{(j)}$

Proof. We can treat each factor $\mathcal{M}_j$ separately and assume that V is irreducible. It suffices to prove that under the adjoint action of φ, $E_B \wedge^B \mathfrak{b}$ is sent into $E_B \wedge^B \mathfrak{p}(D)$. Consider a filtration of $\mathrm{End}(V^{(j)})$:

$$(0) = \mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \cdots \subset \mathfrak{b}_r = \mathfrak{b} \subset \mathfrak{p} = \mathfrak{p}_s \subset \mathfrak{p}_{s-1} \subset \cdots \subset \mathfrak{p}_0 = \mathrm{End}(V^{(j)}).$$

stable under adjoint action of B, with one-dimensional successive quotients.

Suppose the image of $E_B \wedge^B \mathfrak{b}$ under $\mathrm{ad}(\varphi)$ is not contained in $E_B \wedge^B \mathfrak{p}(D)$. Then there exists $0 < i \leq r$ and $0 \leq j < s$ such that $\mathrm{ad}(\varphi)$ induces a *non-zero* homomorphism of line bundles

$$E_B \wedge^B (\mathfrak{b}_i/\mathfrak{b}_{i-1}) \to E_B \wedge^B (\mathfrak{p}_j/\mathfrak{p}_{j+1})(D).$$

In particular, the degree of the source is not larger than the degree of the target. More precisely, let γ be the weight of B on $\mathfrak{b}_i/\mathfrak{b}_{i+1}$ and δ the weight of B on $\mathfrak{p}_j/\mathfrak{p}_{j+1}$. Then we have the inequality

$$\langle \deg E_B, \gamma - \delta \rangle \leq \deg D.$$

Note that γ is the difference between the highest weight $\lambda^{(j)}$ and certain weight of the G-representation $V^{(j)}$, hence a non-negative linear combination of simple roots with the sum of coefficients bounded by N. Since $\deg E_B \in \mathsf{C}$, we then have

$$\langle \deg E_B, \gamma \rangle \geq -Nc.$$

On the other hand, by definition of $\mathfrak{p} = \mathfrak{p}_I^{(j)}$, we see that $-\delta$ is a non-negative linear combination of simple roots such that the sum of coefficients is bounded by N and the coefficient of some root in $\Delta - I$ is positive. Hence because $\deg E_B \in \mathsf{C} \cap \mathsf{C}_I$, we have

$$\langle \deg E_B, -\delta \rangle \geq d - Nc.$$

Combining the above two inequalities, we get $d - 2Nc \leq \deg D$ which contradicts (4.3.1) and thus the lemma follows. □

Proposition 4.3.2. *The stack $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ is of finite type.*

Proof. The natural morphism $\mathcal{M}^{\mathrm{ani}} \to \mathrm{Bun}_G$ is of finite type. For each $\nu \in X^*(T)$, the moduli stack Bun_B^{ν} of B-bundles on X with degree ν is of finite type. It suffices to show that there is a finite subset $S \subset X^*(T)$ such that the image of $\mathcal{M}^{\mathrm{ani}}$ in Bun_G is contained in the image of $\cup_{\nu \in S} \mathrm{Bun}_B^{\nu}$ in Bun_G.

Let $m = (\mathcal{E}, \varphi) \in \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}(k)$ and $\mathcal{E}_B$ a B-reduction of $\mathcal{E}$ such that $\deg(\mathcal{E}_B) \in \mathsf{C}$. Let $a = h_{\mathcal{L}}(m) \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$. Suppose that $\deg(\mathcal{E}_B) \in \mathsf{C}_I$ for some *proper* subset $I \subset \Delta$. Then φ maps the generic point of the curve into the proper parabolic subgroup $P_{I,+}$ of G_+^{sc}. This implies that W_a is contained in the Weyl group of the Levi L_I and hence T^{W_a} is not finite, contradicting the fact that $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}(k)$. Consequently, we have $\det \mathcal{E}_B$ lies in the intersection of C and the complement of C_I for any proper subgroup $I \subset \Delta$. This intersection is a bounded subset of $X^*(T)_{\mathbb{R}}$ and hence the set of weights $S \subset X^*(T)$ lying in the intersection is a finite set. □

4.3.2 Valuative criterion

First we have the existence part of the valuative criterion, which is true over the larger open subset $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$.

Proposition 4.3.3. *Let R be a complete discrete valuation ring with algebraically closed residue field containing k. Let K be the fraction field of R. Then for all $a \in \mathcal{A}_G^{\heartsuit}(R)$ and $m_K \in \mathcal{M}_G^{\heartsuit}(K)$ such that $h_G(m_K) = a$, there exists a finite extension K' of K and $m \in \mathcal{M}_G(R')$, where R' is the integral closure of R in K', such that*

1. *The image of m in $\mathcal{M}_G^{\heartsuit}(K')$ is isomorphic to that of m_K;*
2. *$h_G(m) = a$.*

Proof. The argument is the same as [CL10, §8.4]. The key points are: 1. Any G-torsor extends uniquely over a codimension 2 subset; 2. the universal twisted monoid $\mathbb{V}_G$ over $\mathcal{A}_G \times X$ is affine, so that Higgs fields extends over any codimension 2 subset. □

Proposition 4.3.4. *Suppose G is semisimple. Let R be a complete discrete valuation ring with algebraically closed residue field κ containing k. Let $m, m' \in \mathcal{M}^{\mathrm{ani}}(R)$ be two elements and $m_K, m'_K \in \mathcal{M}^{\mathrm{ani}}(K)$ their base change. Suppose that the following two conditions are satisfied:*

1. *$h(m) = h(m')$;*
2. *there exists an isomorphism $\iota_K : m_K \to m'_K$.*

Then there exists a unique isomorphism $\iota : m \to m'$ extending ι_K.

Proof. We follow the argument in [CL10, §9]. Let $m = (\mathcal{E}, \phi)$ and $m' = (\mathcal{E}', \phi')$. Consider the local ring B of the generic point of the special fiber of X_R. Then B is a discrete valuation ring whose residue field is the function field $\kappa(X)$ of X_κ and whose fraction field is the function field F of X_R.

By §9.2 of *loc. cit.*, it suffices to extend ι_K to an isomorphism of G-torsors $\iota : \mathcal{E} \to \mathcal{E}'$ over $\operatorname{Spec} B$. As in §9.3 of *loc. cit.*, it suffices to show that for some finie extension K', the base change $\iota_{K'}$ of ι_K extends to an isomorphism between $\mathcal{E}, \mathcal{E}'$ over $\operatorname{Spec} B'$. Here B' is the integral closure of B in the function field F' of $X_{R'}$ where R' is the integral closure of R in K'.

To achieve this, after taking a finite extension K'/K one can assume that $\mathcal{E}, \mathcal{E}'$ are trivial over $\operatorname{Spec} B$ (since by [DS95, Theorem 2], they will be trivial in a Zariski open neighbourhood of the generic point of the special fibre of X_R after a finite extension of K). Moreover, as in [CL10, Lemme 9.3.1], one can choose trivialization of $\mathcal{E}$ and $\mathcal{E}'$ over $\operatorname{Spec} B$ such that they map the "Higgs fields" ϕ and ϕ' to some element $\gamma \in \mathrm{Vin}_G^{\mathrm{rs}}(B)$. Under these trivializations, the isomorphism ι_K is identified with an element $g \in G(F)$ such that $g^{-1}\gamma g = \gamma$. In other words, $g \in G_\gamma(F)$. Since m, m' lies in the anisotropic open substack and $\gamma \in \mathrm{Vin}_G^{\mathrm{rs}}(B)$, G_γ is an anisotropic torus over $\operatorname{Spec} B$ and hence $G_\gamma(B) = G_\gamma(F)$. Thus in particular, $g \in G(B)$ and the isomorphism ι_K extends. $\square$

Theorem 4.3.5. *The morphism $h_{\mathcal{L}}^{\mathrm{ani}} : \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} \to \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is proper.*

Proof. This follows from what have been proved in this section and the valuative criterion of properness for algebraic stacks. $\square$

CHAPTER 5

FROM GLOBAL TO LOCAL

In this chapter we finish the proof of dimension formula for Kottwitz-Viehmann varieties using a local-global argument.

Let $\lambda \in X_*(T)_+$ and $\gamma \in G(F)^{\mathrm{rs}}$. Suppose that $\kappa_G(\gamma) = p_G(\lambda)$ and $\nu_\gamma \leq_{\mathbb{Q}} \lambda$ so that the generalized affine Springer fibres X_γ^λ and $X_\gamma^{\leq\lambda}$ are nonempty. Let $a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}^{\mathrm{rs}}_{G_+^{\mathrm{sc}}}(F)$ where $\gamma_\lambda \in G_+^{\mathrm{sc,rs}}(F)$ is defined in Lemma 3.1.2. Then we have $X_\gamma^{\leq\lambda} \cong \mathrm{Sp}_a$ and we have a commutative algebraic group P_a acting on Sp_a so that $\mathrm{Sp}_a^{\mathrm{reg}}$ is the union of open orbits.

5.1 Local constancy of generalized affine Springer fibers

Theorem 5.1.1. *There exists an integer N such that for all $a' \in \mathfrak{C}_+(\mathcal{O}_x) \cap \mathfrak{C}^{\mathrm{rs}}_{G_+}(F_x)$ such that*

$$a \equiv a' \mod \varpi^N$$

the generalized affine Springer fiber $\mathrm{Sp}_{a'}$ equipped with the action of $P_{a'}$ is isomorphic to Sp_a equipped with the action of P_a.

We may assume that $G = G_+^{\mathrm{sc}}$. Fix a Coxeter element $w \in \mathrm{Cox}(W, S)$, cf. 2.2.3. Let $\gamma_0 := \epsilon_+^w(a)$ (resp. $\gamma_0' := \epsilon_+^w(a')$) be the extended Steinberg sections for a (resp. a'). Then we have canonical isomorphism between groups schemes over $\mathrm{Spec}\,\mathcal{O}$:

$$J_a \cong I_{\gamma_0}, \quad J_{a'} \cong I_{\gamma_0'}.$$

Lemma 5.1.2. *For any $g \in G(F)$, we have $\mathrm{Ad}(g)^{-1}(\gamma_0) \in \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})$ if and only if*

$$\mathrm{Ad}(g)^{-1}(\gamma_0 I_{\gamma_0}(\mathcal{O})) \subset \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

Proof. Since $\gamma_0 \in \gamma_0 I_{\gamma_0}(\mathcal{O})$, the condition is sufficient. Now assume that $\gamma := \mathrm{Ad}(g)^{-1}(\gamma_0) \in \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})$. Then the centralizer I_γ is a group scheme over $\operatorname{Spec}\mathcal{O}$. By Lemma 2.4.1, the isomorphism of F groups

$$\mathrm{Ad}(g)^{-1} : J_{a,F} = I_{\gamma_0,F} \to I_{\gamma,F}$$

extends to $\operatorname{Spec}\mathcal{O}$. Thus we have

$$\mathrm{Ad}(g)^{-1}(I_{\gamma_0}(\mathcal{O})) \subset I_\gamma(\mathcal{O}) \subset G(\mathcal{O})$$

from which we obtain

$$\mathrm{Ad}(g)^{-1}(\gamma_0 I_{\gamma_0}(\mathcal{O})) = \gamma \mathrm{Ad}(g)^{-1}(I_{\gamma_0}(\mathcal{O})) \subset \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

$\square$

Lemma 5.1.3. *Let $a, a' \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}^{\mathrm{rs}}_{G^{\mathrm{sc}}_+}(F)$ with $a \equiv a' \mod \varpi^N$. Suppose that there exists a W-equivariant isomorphism between the cameral covers $\widetilde{X}_a$ and $\widetilde{X}_{a'}$ lifting the identity modulo ϖ^N. Let $\gamma_0 := \epsilon^w_+(a)$ and $\gamma'_0 = \epsilon^w_+(a')$. Then there exists $g \in G(\mathcal{O})$ such that*

$$\mathrm{Ad}(g)^{-1}(\gamma_0 I_{\gamma_0}(\mathcal{O})) = \gamma'_0 I_{\gamma'_0}(\mathcal{O})$$

Proof. We follow the argument of [Ngô10, Lemme 3.5.4]. Let $\widetilde{X}_a = \operatorname{Spec} R_a$ and $\widetilde{X}_{a'} = \operatorname{Spec} R_{a'}$ where R_a, $R_{a'}$ are finite flat $\mathcal{O}$-algebras. Let $F_a := R_a \otimes_{\mathcal{O}} F$ (resp. $F_{a'} := F_{a'} \otimes_{\mathcal{O}} F$) and $R^\flat_a$ (resp. $R^\flat_{a'}$) be the normalization of R_a (resp. $R_{a'}$) in F_a (resp. $F_{a'}$).

By assumption, we have $R_a/\varpi^N = R_{a'}/\varpi^N$ and there exists a W-equivariant $\mathcal{O}$-isomorphism

$$\iota : R_a \xrightarrow{\sim} R_{a'}$$

that lifts the identity modulo ϖ^N.

By Proposition 2.4.4, the isomorphism $\iota : R_a \cong R_{a'}$ induces an isomorphism $\iota_I : I_{\gamma_0} \to$

$I_{\gamma_0'}$ between group schemes over $\operatorname{Spec}\mathcal{O}$. Since $\gamma_0 \in I_{\gamma_0}(F)$, we have $\iota_I(\gamma_0) \in I_{\gamma_0'}(F)$. We can choose $h \in G(R_a^\flat)$ and $h' \in G(R_{a'}^\flat)$ such that on F-points, the map ι_I is given by the following composition

$$I_{\gamma_0}(F) \xrightarrow{\sim} T(F_a)^W \xrightarrow{\iota} T(F_{a'})^W \xrightarrow{\sim} I_{\gamma_0'}(F). \tag{5.1.1}$$

where the first map is $\operatorname{Ad}(h)$ the and third map is $\operatorname{Ad}(h')^{-1}$. In other words, $\iota_I = \operatorname{Ad}(h'^{-1}\iota(h))$ on F-points. In particular, we have

$$\chi_+(\iota_I(\gamma_0)) = \chi_+(\gamma_0) = a.$$

The assumption that ι is identity modulo ϖ^N implies that $\operatorname{Ad}(h'^{-1}\iota(h)) \equiv \operatorname{Id} \mod \varpi^N$. Thus we get

$$\iota(I_{\gamma_0}(F) \cap \operatorname{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})) \subset I_{\gamma_0'}(F) \cap \operatorname{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

In particular, we have $\iota(\gamma_0) \in I_{\gamma_0'} \cap \operatorname{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})$ and moreover

$$\iota_I(\gamma_0) = \gamma_0 = \gamma_0' \text{ in } \operatorname{Vin}^w_{G^{\mathrm{sc}}}(\mathcal{O}/\varpi^N).$$

Since the map

$$G \times \operatorname{Vin}_G^w \to \operatorname{Vin}_G^w \times_{\mathfrak{C}_+} \operatorname{Vin}_G^w$$

is smooth and surjective, there exists $g \in G(\mathcal{O})$ with $g \equiv 1 \mod \varpi^N$ such that $\operatorname{Ad}(g)^{-1}(\gamma_0) = \iota_I(\gamma_0)$. Therefore

$$\operatorname{Ad}(g)^{-1}(I_{\gamma_0}) = I_{\iota(\gamma_0)} = I_{\gamma_0'}.$$

Finally by Lemma 2.5.2, we have $(\gamma_0')^{-1}\iota_I(\gamma_0) \in G(\mathcal{O}) \cap I_{\gamma_0'}(F) = I_{\gamma'}(\mathcal{O})$ which implies that $\iota_I(\gamma_0) \in \gamma_0' I_{\gamma_0'}(\mathcal{O})$ and hence we are done. $\square$

5.2 Finishing the proof of dimension formula

By Theorem 3.7.1, the dimension formula for $X_\gamma^\lambda \cong \mathrm{Sp}_a$ is reduced to the following

Theorem 5.2.1. $\dim \mathrm{Sp}_a = \dim P_a$.

We use a local-global argument to prove this.

Let X be a projective smooth curve over k and $x \in X$ a closed point. Let $\mathcal{O}_x$ be the completed local ring at x and F_x its fraction field. Choose a uniformiser ϖ_x at x so that we have $\mathcal{O}_x = k[[\varpi_x]]$ and $F_x = k((\varpi_x))$. Also we let $X' = X - \{x\}$ be the open curve.

We view $a \in \mathfrak{C}_+(\mathcal{O}_x)$ as a power series in ϖ_x with coefficients in $\mathfrak{C}_+$. Form the Cartesian diagram

$$\begin{tikzcd} X_a \arrow[r] \arrow[d, "\pi_a"'] & \overline{T_+} \arrow[d, "\pi"] \\ \mathrm{Spec}\,\mathcal{O} \arrow[r, "a"] & \mathfrak{C}_+ \end{tikzcd}$$

where $X_a = \mathrm{Spec}\, R_a$ for a finite flat $\mathcal{O}$ algebra R_a. Moreover, $F_a = R_a \otimes_{\mathcal{O}} F$ is a product of finite totally ramified extension of F of degree e. Then $a(\varpi_x^e) \in \mathfrak{C}_+(\mathcal{O}_x) \cap \mathfrak{C}^{\mathrm{rs}}_{G_+^{\mathrm{sc}}}(F_x)$ will be a split conjugacy class.

For each $s \in k$ we define

$$a_s := a(s\varpi_x + (1-s)\varpi_x^e) \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_{G_+}(F)^{\mathrm{rs}}. \tag{5.2.1}$$

Then $a_1 = a$ and $a_0 = a(\varpi_x^e)$. For each $s \neq 0$, Sp_{a_s} is isomorphic to Sp_a since a_s is obtained from $a = a_1$ by changing uniformizer.

Let $N > 0$ be a positive integer such that both Sp_a and Sp_{a_0} only depends on a (resp. a_0) modulo ϖ_x^N. Then for all $s \in k$, Sp_{a_s} only depends on a_s modulo ϖ_x^N.

Now we choose a T^{sc}-torsor $\mathcal{L}$ on X trivialized on the formal neighbourhood of x such that

1. There exists a T^{sc}-torsor $\mathcal{L}'$ such that $(\mathcal{L}')^{\otimes c} \cong \mathcal{L}$;

2. For all $y \in X' = X - x$, choosing a trivialisation of $\mathcal{L}$ on a formal neighbourhood of y, the induced map

$$\mathrm{ev}_{Nx+2y} : \mathcal{A}_{\mathcal{L}} = H^0(X, \mathfrak{C}_+^{\mathcal{L}}) \to \mathfrak{C}_+(\mathcal{O}_x/\varpi_x^N) \times \mathfrak{C}_+(\mathcal{O}_y/\varpi_y^2) \tag{5.2.2}$$

is surjective.

By Riemann-Roch, condition 2 is satisfied if for all $1 \leq i \leq r$ we have $\deg(\alpha_i(\mathcal{L})) \geq 2g + N$ and $\deg(\omega_i(\mathcal{L})) > 2g + N$.

To each point $b \in \mathcal{B}_{\mathcal{L}}^{\circ}$, we can associate a $X_*(T_{\mathrm{ad}})_+$-valued divisor λ_b on X defined by

$$\lambda_b := \sum_{i=1}^{r} \check{\omega}_i D(b_i)$$

where $D(b_i)$ is the effective divisor on X associated to the global section b_i of the line bundle $\alpha_i(\mathcal{L})$ and $\check{\omega}_i$ is the i-th fundamental coweight. For any $a_+ \in \mathcal{A}_{\mathcal{L}}$ such that $\beta_{\mathcal{L}}(a_+) \in \mathcal{B}_{\mathcal{L}}^{\circ}$, we denote $\lambda_{a_+} := \lambda_{\beta_{\mathcal{L}}(a_+)}$.

Lemma 5.2.2. *Let $\Sigma \subset X$ be a finite subset. The subset $\mathcal{A}_{\mathcal{L}}^{\Sigma} \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ consisting of $a_+ \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ such that*

$$\mathrm{Supp}(\lambda_{a_+}) \cap \mathrm{Supp}(\Delta_{a_+}) \subset \Sigma$$

is constructible.

Proof. For each $1 \leq i \leq r$, consider the closed subscheme $\mathcal{D}_i \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit} \times X$ whose fibre over $a_+ \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ is the effective divisor $D(b_i)$ where b_i is the i-th coordinate of $\beta_{\mathcal{L}}(a_+)$ as above. Similarly, we have the closed subscheme $\Delta \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit} \times X$ whose fibre over a_+ is the discriminant divisor Δ_{a_+}. Let $\mathcal{D}_i^{\Sigma} = \mathcal{D}_i \cap (\mathcal{A}_{\mathcal{L}}^{\heartsuit} \times (X - \Sigma))$ and $\Delta^{\Sigma} := \Delta \cap (\mathcal{A}_{\mathcal{L}}^{\heartsuit} \times (X - \Sigma))$. Then $\mathcal{D}_i^{\Sigma} \cap \Delta^{\Sigma}$ is a locally closed subset of $\mathcal{A}_{\mathcal{L}}^{\heartsuit} \times X$. By construction $\mathcal{A}_{\mathcal{L}}^{\Sigma}$ is the image of $\bigcup_{1 \leq i \leq r} (\mathcal{D}_i^{\Sigma} \cap \Delta^{\Sigma})$ in $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$, hence constructible. $\square$

The one-parameter family (5.2.1) defines a curve C in $\mathfrak{C}_+(\mathcal{O}_x/\varpi_x^N)$. Let $L_C \subset \mathcal{A}_{\mathcal{L}}$ be the closed subset defined as the inverse image of C under the map (5.2.2). For all $s \in k$, let $L_{a_s} \subset \mathcal{A}_{\mathcal{L}}$ be the inverse image of a_s under the map (5.2.2). Since $a_s \in \mathfrak{C}^{\mathrm{rs}}_{G^{\mathrm{sc}}}(F)$ for all $s \in k$, we have $L_C \subset \mathcal{A}^{\heartsuit}_{\mathcal{L}}$.

Definition 5.2.1. Let $Z_C \subset L_C$ be the subset consisting of $a_+ \in L_C$ with $b = \beta_{\mathcal{L}}(a_+)$ such that

- $a_+ \in \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$;
- $\mathrm{Supp}(\lambda_{a_+}) \cap \mathrm{Supp}(\Delta_{a_+}) \subset \{x\}$;
- $a_+(X')$ intersects the discriminant divisor $\mathfrak{D}^{\mathcal{L}}_{+}$ transversally, where $X' = X - \{x\}$.

Lemma 5.2.3. *Z_C is a constructible subset of L_C that is fibrewise dense with respect to the projection $L_C \to C$. In particular, there exists a fibrewise dense open subset U_C of L_C such that $U_C \subset Z_C$.*

Proof. First we show that Z_C is constructible. The first condition in Definition 5.2.1 defines an open subset of L_C. By Lemma 5.2.2, the set $L^x_C := L_C \cap \mathcal{A}^x_{\mathcal{L}}$ determined by the second condition in Definition 5.2.1 is a constructible subset of L_C.

Let $U \subset X' \times L_C$ be the open subset whose fibre over $a_+ \in L_C$ is the open curve $X' - \mathrm{Supp}(\lambda_{a_+})$. The local evaluation maps define a morphism

$$U \to \mathbb{T}\mathfrak{C}^{\mathcal{L}}_{+}$$

where $\mathbb{T}\mathfrak{C}^{\mathcal{L}}_{+}$ is the relative tangent bundle of $\mathfrak{C}^{\mathcal{L}}_{+}$ over X. Let U_1 be the inverse image of

$$\mathbb{T}\mathfrak{D}^{\mathcal{L},\mathrm{sm}}_{+} \cup \mathbb{T}\mathfrak{C}^{\mathcal{L}}_{+} \times_{\mathfrak{C}^{\mathcal{L}}_{+}} \mathfrak{D}^{\mathcal{L},\mathrm{sing}}_{+}.$$

Then the image of U_1 in L_C is a constructible subset that satisfies the third condition in Definition 5.2.1. Hence Z_C is a constructible subset of L_C.

Next we show that Z_C is fibrewise dense with respect to the map $L_C \to C$. We fix a point $a_s \in C$.

For any closed point $y \in X'$, the map

$$L_{a_s} \to \mathbb{T}\mathfrak{C}^{\mathcal{L}}_{+,y} = \mathfrak{C}^{\mathcal{L}}_{+} \otimes_{\mathcal{O}_y} \mathcal{O}_y/\mathfrak{m}_y^2$$

is surjective by our choice of $\mathcal{L}$.

Let $X'' := X' \setminus \mathrm{Supp}(\lambda_b)$. By the same argument as in [Ngô10, Lemme 4.7.2], we know that the subset $Z \subset L_{a_s}$ consisting of $a_+ \in L_{a_s}$ such that $a_+(X'')$ intersects $\mathfrak{D}^{\mathcal{L}}_{+}$ transversally is dense in L_{a_s}.

For each $y \in \mathrm{Supp}(\lambda_b) - \{x\}$, since the map $\mathrm{ev}_y : L_{a_s} \to \mathfrak{C}^{\mathcal{L}}_{+,y}$ is surjective, the subset $\Sigma_y := \mathrm{ev}_y^{-1}(\mathfrak{D}^{\mathcal{L}}_{+}) \subset L_{a_s}$ has codimension 1.

Finally, since L_{a_s} has codimension $2rN$ in $\mathcal{A}^{\heartsuit}_{\mathcal{L}}$ and the complement of $\mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$ in $\mathcal{A}^{\heartsuit}_{\mathcal{L}}$ has codimension strictly larger than $2rN$, we see that

$$Z_{a_s} = (Z - \bigcup_{y \in \mathrm{Supp}(\lambda_\mathrm{b})} \Sigma_y) \cap \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$$

is dense in L_{a_s}. $\square$

Thus we can choose a section σ of the surjective linear map (5.2.2) such that $C' := \sigma(C) \cap U_C$ is nonempty and contains the point $\sigma(a_0)$.

By the product formula 4.2.7, we have

$$\dim \mathcal{M}_{\sigma(a_0)} - \mathcal{P}_{\sigma(a_0)} = \sum_{v \in \mathrm{Supp}(\Delta_a) \cup \{x\}} (\dim \mathrm{Sp}_{\sigma(a_0)_v} - \dim P_{\sigma(a_0)_v})$$

where $\sigma(a_0)_v$ denotes the image of $\sigma(a_0)$ in $\mathfrak{C}_+(\mathcal{O}_v)$.

For summands with $v \neq x$, since $\sigma(a_0) \in Z_C$ we have in particular $\lambda_{\sigma(a_0),v} = 0$ and hence by Corollary 3.8.2 $\dim \mathrm{Sp}_{\sigma(a_0)_v} = \dim P_{\sigma(a_0)_v}$. On the other hand, for the term $v = x$, we know that $\sigma(a_0)_x = a_0$ is split and hence by Corollary 3.5.2 $\dim \mathrm{Sp}_{a_0} = \dim P_{a_0}$. Thus the above equality simplifies to

$$\dim \mathcal{M}_{\sigma(a_0)} - \dim \mathcal{P}_{\sigma(a_0)} = 0.$$

Since $C' \subset \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$, the restriction of the Hitchin-Frenkel-Ngô fibration to C' is proper. Hence by upper semicontinuity of fibre dimension we have for

$$\dim \mathcal{M}_{\sigma(a_s)} \leq \dim \mathcal{M}_{\sigma(a_0)} = \dim \mathcal{P}_{\sigma(a_0)}$$

for all $\sigma(a_s) \in C'$ with $s \neq 0$. Since $\mathcal{P}$ is smooth over $\mathcal{A}_{\mathcal{L}}$ by Proposition 4.2.1, we have $\dim \mathcal{P}_{\sigma(a_s)} = \mathcal{P}_{\sigma(a_0)}$, which forces

$$\dim \mathcal{M}_{\sigma(a_s)} = \dim \mathcal{P}_{\sigma(a_s)}$$

Apply product formula 4.2.7 again we get

$$0 = \dim \mathcal{M}_{\sigma(a_s)} - \dim \mathcal{P}_{\sigma(a_s)} = \sum_{v \in \mathrm{Supp}(\Delta_{a_s}) \cup \{x\}} (\dim \mathrm{Sp}_{\sigma(a_s),v} - \dim P_{\sigma(a_s),v})$$

By similar reasoning as above, all terms in the right hand side where $v \neq x$ are zero; at $v = x$ notice that $\sigma(a_s)_x = a_s$ and then we get

$$\dim \mathrm{Sp}_{a_s} - \dim P_{a_s} = 0.$$

Since $s \neq 0$, we have $\mathrm{Sp}_{a_s} \cong \mathrm{Sp}_a$ and hence

$$\dim \mathrm{Sp}_a = \dim P_a$$

This finishes the proof of Theorem 5.2.1 and hence Theorem 1.2.1.

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