THE UNIVERSITY OF CHICAGO

SUBADDITIVE THERMODYNAMIC FORMALISM FOR FIBER-BUNCHED COCYCLES

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To my family and friends.

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ABSTRACT

In this thesis, we study subadditive thermodynamic formalism of fiber-bunched cocycles. In particular, we study the norm potentials and their equilibrium states of such cocycles. We present mainly three results, each of which can be viewed as a suitable generalization of the corresponding result for locally constant cocycles.

The first result concerns with an open and dense subclass of fiber-bunched $\mathrm{GL}_d(\mathbb{R})$-cocycles called the typical cocycles. We show that the norm potentials of typical cocycles have unique equilibrium states with the subadditive Gibbs property. The main body of work amounts to establishing a property called quasi-multiplicativity, which implies the stated result. As a corollary, we obtain the continuity of the subadditive pressure and the equilibrium state (with respect to the weak-$*$ distance) on such cocycles.

Second, in joint work with Clark Butler, we study thermodynamic formalism of fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles. Exploiting the low dimensionality of the cocycles, we fully analyze their norm potentials. We show that irreducible fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles have unique equilibrium states. We also provide a criterion for their norm potentials to have multiple ergodic equilibrium states.

Lastly, in joint work with Benjamin Call, we study the ergodic properties on these unique equilibrium states. Under the settings considered in the above two results, we show that if the unique equilibrium state is totally ergodic, then it has the Kolmogorov property. This is done by extending a result of Ledrappier.

CHAPTER 1

INTRODUCTION

Given a finite set of $M_{d\times d}(\mathbb{R})$ matrices $\mathsf{A} = \{A_1, \ldots, A_q\}$ and an infinite word $x^+ = x_0x_1x_2\ldots$ where each $x_j \in \{1, 2, \ldots, q\}$, consider the products

$$A_{x_n} \ldots A_{x_0}, \tag{1.1}$$

for $n = 1, 2, \ldots$. The study of such products naturally arises in many settings and has numerous applications. For instance, suppose each A_i is contracting, and T_i is an affine transformation of $\mathbb{R}^d$ whose linear part is A_i; that is, $T_i(x) = A_ix + r_i$ for some translation vector r_i. Then there exists a unique self-affine attractor $X \subset \mathbb{R}^d$ invariant under $\{T_1, \ldots, T_q\}$, in the sense that $X = \bigcup_{i=1}^{q} T_iX$; see [31]. The local geometry of the attractor X depends on the properties of the composition of the linear contractions (1.1); for example, the Hausdorff dimension of X is intimately related to the growth rate of the product (1.1) over all possible words x^+.

The product of matrices (1.1) can be placed in a broader context. To any dynamical system $f\colon X \to X$ and map $\mathcal{A}\colon X \to M_{d\times d}(\mathbb{R})$, we associate a *linear cocycle* $F_{\mathcal{A}}\colon X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ given by

$$F_{\mathcal{A}}(x, v) = (fx, \mathcal{A}(x)v).$$

We say that $F_{\mathcal{A}}$ is *generated* by f and $\mathcal{A}$. By an abuse of notation, we will often refer to the cocycle $F_{\mathcal{A}}$ simply by $\mathcal{A}$. For $n \in \mathbb{N}$ and $x \in X$, we write $F^n_{\mathcal{A}}(x, v) = (f^nx, \mathcal{A}^n(x)v)$, where

$$\mathcal{A}^n(x) := \mathcal{A}(f^{n-1}x) \ldots \mathcal{A}(fx)\mathcal{A}(x).$$

The definition of linear cocycle F also extends to (not necessarily trivializable) vector bundles $\mathcal{E}$ over X as a family of linear maps $F_x\colon \mathcal{E}_x \to \mathcal{E}_{fx}$ covering a base system (X, f).

When the base system is the left shift operator σ on a one-sided shift $\Sigma_q^+ = \{1, 2, \ldots, q\}^{\mathbb{N}_0}$, then the cocycle generated by a map $\mathcal{A}\colon \Sigma_q^+ \to M_{d\times d}(\mathbb{R})$ defined by $x = (x_i)_{i\in\mathbb{N}_0} \mapsto A_{x_0}$ encodes the products (1.1) in the sense that $\mathcal{A}^n(x) = A_{x_{n-1}} \ldots A_{x_0}$. Such cocycle is an example of a locally constant cocycle.

Another natural class of linear cocycles comes from smooth dynamics. When the base system $f\colon M \to M$ is a smooth map or diffeomorphism of a closed Riemannian manifold M, the *derivative cocycle* Df is a cocycle generated by the map $\mathcal{A}(x) = D_x f\colon T_x M \to T_{fx} M$. More generally, for any Df-invariant sub-bundle $E \subset TM$, the derivative map restricted to E gives rise to a linear cocycle $Df|_E$. If f is uniformly hyperbolic (i.e., expanding or Anosov), then there exists a symbolic coding of f by a subshift of finite type [45, 10]. From such a coding, the derivative cocycle of a uniformly hyperbolic map can effectively be regarded as a linear cocycle over a subshift of finite type.

Our main objects of interest are linear cocycles $\mathcal{A}$ over two-sided subshifts of finite type (Σ_T, σ). Any $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ defines a sequence of continuous functions $\{\varphi_{\mathcal{A},n}\}_{n\in\mathbb{N}}$ on Σ_T given by

$$\varphi_{\mathcal{A},n}(x) = \|\mathcal{A}^n(x)\|,$$

where $\|\cdot\|$ is the operator norm. The submultiplicativity of the norm $\|\cdot\|$ implies that this sequence is *submultiplicative* in the sense that for any $m, n \in \mathbb{N}$,

$$0 \leq \varphi_{\mathcal{A},n+m} \leq \left(\varphi_{\mathcal{A},n} \circ \sigma^m\right) \cdot \varphi_{\mathcal{A},m}. \tag{1.2}$$

Such a submultiplicative sequence gives rise to a *norm potential* $\Phi_{\mathcal{A}} = \{\log \varphi_{\mathcal{A},n}\}_{n\in\mathbb{N}}$.

The norm potentials $\Phi_{\mathcal{A}}$ are example of *subadditive potentials* $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ which can be thought of as generalizations of the Birkhoff sums $S_n\varphi$ for continuous potentials $\varphi \in C(\Sigma_T, \mathbb{R})$. Cao, Feng, and Huang [17] showed that the usual thermodynamical notions of the pressure and the equilibrium states for additive potentials naturally extend to subadditive potentials. In particular, for any subadditive potential Φ they established the existence of the subadditive pressure $\mathsf{P}(\Phi)$ and showed that it satisfies the *subadditive variational principle*:

$$\mathsf{P}(\Phi) = \sup_{\mu \in \mathcal{M}(\sigma)} \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log \varphi_n \, d\mu \right\}$$

where the supremum is taken over the set $\mathcal{M}(\sigma)$ of all σ-invariant measures. Any σ-invariant measure $\mu \in \mathcal{M}(\sigma)$ achieving the supremum is called an *equilibrium state* for Φ. Such equilibrium states have important applications in dimension theory of fractals; see for instance [23, 47, 19, 3, 29, 4] and references therewithin. See also Subsection 2.2.4 and Section 3.4 for a brief discussion on this topic.

The above discussed topic of subadditive thermodynamic formalism has its basis in the classical thermodynamic formalism where additive potentials and their equilibrium states are studied. Even for the classical thermodynamic formalism, it is often difficult to determine the number and the properties of the equilibrium states given arbitrary dynamical systems and potentials. On the other hand, there are suitable assumptions which reveal relevant thermodynamical information. Such information is then used in studying the underlying dynamical systems.

For example, in his fundamental work on thermodynamic formalism, Bowen [11] showed that Hölder potentials on mixing hyperbolic systems such as (Σ_T, σ) have unique equilibrium states with the Gibbs property; see Proposition 2.18 for the precise statement. This class of unique equilibrium states is now well-studied, and known to possess further ergodic and statistical properties; see [11, 12, 44, 42]. In suitable settings, commonly studied invariant measures such as the measures of maximal entropy or Sinai-Ruelle-Bowen measures belong to this class of equilibrium states.

It is natural to ask if Bowen's result (with suitable generalizations) on the uniqueness of the equilibrium states holds for subadditive potentials such as $\Phi_{\mathcal{A}}$. Unfortunately, the analogue of Bowen's theorem does not necessarily hold for general subadditive potentials: while the existence of at least one equilibrium states is guaranteed from the upper semi-continuity of the maps $\mu \mapsto h_\mu(\sigma)$ and $\mu \mapsto \lim_{n\to\infty} \frac{1}{n}\int \log \varphi_n \, d\mu$, there are examples of subadditive potentials with multiple ergodic equilibrium states; see [28] and also the discussion at the end of Chapter 2.

The main theme of this thesis is to investigate and study sufficient conditions for subadditive potentials arising from matrix cocycles to have unique equilibrium states. We also study ergodic properties of such equilibrium states. In what remains of this introduction, we will outline what each chapter is about and state the main results. For terminologies appearing in the results, we refer the readers to Chapter 2 for their precise definitions.

1.1 Typical cocycles and Quasi-multiplicativity

In Chapter 3, which is the the first main chapter following the chapter on preliminary results, we will study a property called quasi-multiplicativity for subadditive potentials. This property is important because it is a sufficient condition to extend the above mentioned result of Bowen on the uniqueness of the equilibrium states; see Proposition 2.26 for the precise statement.

Denoting the set of all admissible words of Σ_T by $\mathcal{L}$, for any cocycle $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ and $\mathrm{I} \in \mathcal{L}$, we define

$$\|\mathcal{A}(\mathrm{I})\| := \max_{x\in[\mathrm{I}]} \varphi_{\mathcal{A},|\mathrm{I}|}(x) = \max_{x\in[\mathrm{I}]} \|\mathcal{A}^{|\mathrm{I}|}(x)\|. \tag{1.3}$$

We say $\mathcal{A}$ is *quasi-multiplicative* if there exist $c > 0$ and $k \in \mathbb{N}$ such that for any words $\mathrm{I}, \mathrm{J} \in \mathcal{L}$, there exists $\mathrm{K} = \mathrm{K}(\mathrm{I},\mathrm{J}) \in \mathcal{L}$ with $|\mathrm{K}| \leq k$ such that $\mathrm{IKJ} \in \mathcal{L}$ and

$$\|\mathcal{A}(\mathrm{IKJ})\| \geq c\|\mathcal{A}(\mathrm{I})\|\|\mathcal{A}(\mathrm{J})\|. \tag{1.4}$$

Quasi-multiplicativity may be thought of as follows: given any two words $\mathrm{I}, \mathrm{J} \in \mathcal{L}$ of arbitrary length, we can obtain the reverse inequality to the submultiplicativity (1.2) at a cost of inserting a connecting word $\mathrm{K} = \mathrm{K}(\mathrm{I}, \mathrm{J})$ in between I and J of uniformly bounded length. While we have formulated the above quasi-multiplicativity (1.4) for matrix cocycles and their norm potentials, we note that the same notion of quasi-multiplicativity makes sense for any subadditive potentials that are not necessarily given by the norm potentials; see Definition 2.25.

Quasi-multiplicativity is also known to be verifiable in certain cases. For instance, if $\mathcal{A}$ is a locally constant $\mathrm{GL}_d(\mathbb{R})$-cocycle over a full shift generated by an irreducible set of matrices, then $\mathcal{A}$ is quasi-multiplicative; see [26]. For such cocycles, quasi-multiplicativity ensures that their norm potentials have unique equilibrium states; see Proposition 2.27.

In Chapter 3, we address the question of whether quasi-multiplicativity holds for more general cocycles beyond locally constant cocycles. It is not entirely clear what the natural counterpart to irreducibility might be for general cocycles. Even in the settings where a suitable version of irreducibility can be defined, it is not known whether it translates to quasi-multiplicativity in the most general setting. On the other hand, since quasi-multiplicativity is a common feature among locally constant cocycles, it is reasonable to expect that quasi-multiplicativity holds for a more general class of cocycles under suitable assumptions.

Throughout this thesis, we will restrict our attention to Hölder continuous and fiber-bunched cocycles, a class that contains the locally constant cocycles. The fiber-bunching assumption is an open condition which roughly says that the cocycle is nearly conformal. We denote the space of α-Hölder and fiber-bunched functions by $C^\alpha_b(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$, viewed as a subset of $C^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$.

Our main result for Chapter 3 establishes that quasi-multiplicativity holds generically among the fiber-bunched cocycles. More precisely, Bonatti and Viana in [8] introduced the notion of *typical* cocycles among fiber-bunched cocycles (see Definition 2.8, 2.9, and 2.11 for precise formulations). The set

$$\mathcal{U} := \{\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_d(\mathbb{R})) \colon \mathcal{A} \text{ is typical}\}$$

is open in $C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$, and they also proved that $\mathcal{U}$ is dense in $C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ and that its complement has infinite codimension.

Theorem A. Every $\mathcal{A} \in \mathcal{U}$ is quasi-multiplicative. Moreover, the constants c, k in (1.4) can be chosen uniformly in a neighborhood of $\mathcal{A}$ in $\mathcal{U}$.

Since quasi-multiplicativity implies the uniqueness of the equilibrium states for fiber-bunched cocycles (see Proposition 2.26), the above theorem has an immediate corollary:

Corollary 1.1. For $\mathcal{A} \in \mathcal{U}$, the norm potential $\Phi_{\mathcal{A}}$ has a unique equilibrium state $\mu_{\mathcal{A}}$.

As an application of Theorem A, we use the uniformity of the constants c, k to prove the continuity of the subadditive pressure $\mathsf{P}(\Phi_{\mathcal{A}})$ and the equilibrium states $\mu_{\mathcal{A}}$ on $\mathcal{U}$.

Theorem B. The following statements hold true:

1. The map $\mathcal{A} \mapsto \mathsf{P}(\Phi_{\mathcal{A}})$ is continuous on $\mathcal{U}$.

2. The unique equilibrium state $\mu_{\mathcal{A}}$ varies weak-$*$ continuously on $\mathcal{U}$.

Via a different approach, Cao, Pesin, and Zhao [18] recently proved a result that implies Theorem B (1). See comments at the end of Section 3.3 for further remarks on related results.

The paper [41], where Theorem A and B originally appear, contains further applications and generalizations these theorems. However, for coherence of this thesis, we will only briefly comment on them at the end of Chapter 3.

1.2 Fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles

While the results formulated above show that the norm potentials of typical cocycles have unique equilibrium states, it is not known whether the same can be said for cocycles that fail to satisfy the typicality assumption. In the special case of fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles, we exploit the low dimensionality of the cocycles and fully analyze their norm potentials in all cases as described in the next two results.

We say a fiber-bunched cocycle is *reducible* if there exists a proper sub-bundle that is invariant under the cocycle as well as the canonical holonomies. For fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles, such a sub-bundle is necessasrily be a line bundle. We say a fiber-bunched cocycle is *irreducible* if it is not reducible; see Definition 2.6 for the precise definition. The main result of Chapter 4 formulated below establishes that irreducibility implies the uniqueness of the equilibrium states for fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles.

Theorem C. Let $\mathcal{A}$ be a fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycle. If $\mathcal{A}$ is irreducible, then the norm potential $\Phi_{\mathcal{A}}$ has a unique equilibrium state $\mu_{\mathcal{A}}$.

In the proof of this theorem, we introduce a class of weakly typical cocycles that generalizes the notion of typical cocycles, and we use such cocycles to provide a classification of irreducible fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles; see Theorem 4.8.

We will also show that Theorem C may alternatively be obtained by manipulating a result of Bochi and Garibaldi [6]. They introduced and established of a property called spannability that is aimed at different applications. It turns out that the spannability implies quasi-multiplicativity, and hence, the uniqueness of the equilibrium state. This approach is outlined at the end of Chapter 4.

We also study thermodynamic formalism of reducible cocycles. By straightening out the invariant line bundle, a reducible fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycle $\mathcal{A}$ admits a Hölder conjugacy $\mathcal{C}\colon \Sigma_T \to \mathrm{GL}_2(\mathbb{R})$ such that $\mathcal{B}(x) := \mathcal{C}(\sigma x)\mathcal{A}(x)\mathcal{C}(x)^{-1}$ is upper triangular for every $x \in \Sigma_T$. Since $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$ have the same set of equilibrium states (see Remark 2.22) the study of reducible cocycles reduces to the study of Hölder cocycles taking values in upper triangular matrices.

Theorem D. Let $\mathcal{B} \in C^{\alpha}(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ be an α-Hölder cocycle taking values in the group of upper triangular matrices:

$$\mathcal{B}(x) := \begin{pmatrix} a(x) & b(x) \\ 0 & c(x) \end{pmatrix}. \tag{1.5}$$

The norm potential $\Phi_{\mathcal{B}}$ has a unique equilibrium state, unless

1. $\log|a|$ is not cohomologous to $\log|c|$, and

2. $\mathsf{P}(\log|a|) = \mathsf{P}(\log|c|)$.

If these two conditions hold, then $\Phi_{\mathcal{B}}$ has exactly two distinct ergodic equilibrium states.

1.3 Kolmogorov property

In the final chapter this thesis, we study the ergodic properties of the unique equilibrium states for the norm potentials established in the above formulated results. From uniqueness, it is clear that they are ergodic. Due to intimate connections with their defining cocycles, it is also reasonable to expect that suitable assumptions on the structure of the cocycles impose further stronger ergodic properties on such equilibrium states.

In this direction, Morris [37, 38, 39] recently showed in a sequence of papers that if the unique equilibrium states for irreducible locally constant cocycles are totally ergodic, then they are actually Bernoulli, which is the highest (strongest) ergodic property in the hierarchy of various ergodic properties. He does so by providing a characterization for the failure of total ergodicity in terms of certain structures on the cocycle.

The following main result of Chapter 5 is similar in flavor to Morris' result and shows that under suitable assumptions, total ergodicity which sits lower in the hierarchy implies an evidently stronger ergodic property called the Kolmogorov property (K-property). The K-property is stronger than mixing of all orders and weaker than Bernoulli. While the result is not as strong as the result of Morris, our result is applicable to a broader class of subadditive potentials.

Theorem E. Let $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ be a subadditive potential on Σ_T, and suppose it is quasi-multiplicative and has bounded distortion. If the unique equilibrium state μ for Φ is totally ergodic, then μ has the K-property.

We note that the bounded distortion property appearing as one of the assumptions in this theorem is a mild assumption that is satisfied by all subadditive potentials considered in this thesis, including the norm potentials of fiber-bunched cocycles.

The key result used in proving this theorem is a result which shows that for subadditive equilibrium states, weak mixing is equivalent to the K-property under some suitable assumptions, similar to those used in [11]. This result which holds even for non-symbolic systems is formulated in Theorem 5.6, and we expect it to be of independent interest.

The remaining results in the chapter are obtained by applying Theorem E to equilibrium states appearing in the previously stated results. By verifying total ergodicity on the unique equilibrium states for typical cocycles, we obtain the following theorem:

Theorem F. Let $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ be a typical cocycle. Then the unique equilibrium state $\mu_{\mathcal{A}}$ from Corollary 1.1 has the K-property.

Lastly, by considering all cases depending on the number of equilibrium states for fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles as done in Theorem D, we study their K-property. The following theorem may be thought of as follows: for fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles $\mathcal{A}$, the case specified in the theorem is the only case where the ergodic equilibrium states for $\Phi_{\mathcal{A}}$ fail to be K.

Theorem G. Let $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_2(\mathbb{R})$ be a Hölder continuous and fiber-bunched cocycle. Every ergodic equilibrium state of $\Phi_{\mathcal{A}}$ is K up to a period, that is, $(\Sigma_T, \sigma, \mu_{\mathcal{A}})$ is isomorphic to a K-system times a finite rotation. Indeed, the only case when $\mu_{\mathcal{A}}$ is not K is when $\mathcal{A}$

can be conjugated to another cocycle

$$\mathcal{B}(x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}$$

such that $f(x) := \log|a(\sigma x)b(x)|$ and $g(x) := \log|b(\sigma x)a(x)|$ viewed as potentials over (Σ_T, σ^2) have distinct equilibrium states μ_1 and μ_2.

CHAPTER 2

PRELIMINARIES

2.1 Symbolic dynamics

An *adjacency matrix* T is a square (0,1)-matrix. A one-sided *subshift of finite type* defined by a $q \times q$ adjacency matrix T is a dynamical system (Σ_T^+, σ) where

$$\Sigma_T^+ := \{(x_i)_{i\in\mathbb{N}_0} \colon x_i \in \{1, 2, \ldots, q\} \text{ and } T_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}_0\}$$

and σ is the left shift operator. Similarly, we define a two-sided subshift of finite type (Σ_T, σ) using the same notation σ for the shift operator where

$$\Sigma_T := \{(x_i)_{i\in\mathbb{Z}} \colon x_i \in \{1, 2, \ldots, q\} \text{ and } T_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.$$

Then (Σ_T, σ) is the *natural extension* of (Σ_T^+, σ): denoting the projection from Σ_T onto Σ_T^+ by $\pi \colon \Sigma_T \to \Sigma_T^+$, each $x \in \Sigma_T$ corresponds to one possible sequence of preimage of $\pi(x) \in \Sigma_T^+$ under σ.

We will always assume that the adjacency matrix T is *primitive*, meaning that there exists $N > 0$ such that all entries of T^N are positive. The primitivity of T is equivalent to the mixing property of the corresponding subshift of finite type (Σ_T, σ).

Fix $\theta \in (0, 1)$ and endow Σ_T with the metric d defined as follows: for $x = (x_i)_{i\in\mathbb{Z}}, y = (y_i)_{i\in\mathbb{Z}} \in \Sigma_T$, we have

$$d(x, y) = \theta^k,$$

where k is the largest integer such that $x_i = y_i$ for all $|i| < k$. Equipped with such metric, the left shift operator $\sigma \colon \Sigma_T \to \Sigma_T$ becomes a hyperbolic homeomorphism of a compact metric space Σ_T.

An *admissible word of length* n is a word $i_0 \dots i_{n-1}$ with $i_j \in \{1, \dots, q\}$ such that $T_{i_j, i_{j+1}} = 1$ for all $0 \le j \le n-2$. Let $\mathcal{L}$ be the collection of all admissible words. For $\mathrm{I} \in \mathcal{L}$, we denote its length by $|\mathrm{I}|$. For each $n \in \mathbb{N}$, let $\mathcal{L}(n) \subset \mathcal{L}$ be the set of all admissible words of length n. For any $\mathrm{I} = i_0 \dots i_{n-1} \in \mathcal{L}(n)$, we define the associated *cylinder* by

$$[\mathrm{I}] = [i_0 \dots i_{n-1}] := \{y \in \Sigma_T \colon y_j = i_j \text{ for all } 0 \le j \le n-1\}.$$

For $x \in \Sigma_T$ and $n \in \mathbb{N}$, we similarly define

$$[x]_n := \{y \in \Sigma_T \colon y_i = x_i \text{ for all } 0 \le i \le n-1\}.$$

We define the *local stable set* $\mathcal{W}^s_{\text{loc}}(x)$ *of* $x \in \Sigma_T$ by

$$\mathcal{W}^s_{\text{loc}}(x) := \{y \in \Sigma_T \colon x_i = y_i \text{ for all } i \ge 0\}.$$

In other words, $y \in \Sigma_T$ belongs to $\mathcal{W}^s_{\text{loc}}(x)$ if the forward orbit of y exponentially shadows the forward orbit of x, meaning that $d(\sigma^n x, \sigma^n y) \le \theta^{n+1}$ for all $n \ge 0$. We then define the *stable set* $\mathcal{W}^s(x)$ *of* $x \in \Sigma_T$ by

$$\mathcal{W}^s(x) := \{y \in \Sigma_T \colon \sigma^n y \in \mathcal{W}^s_{\text{loc}}(\sigma^n x) \text{ for some } n \ge 0\}.$$

Analogously, the (local) stable set of σ^{-1} is called the *(local) unstable set* $\mathcal{W}^u_{(loc)}$ *of* σ. Explicitly, they are defined as

$$\mathcal{W}^u_{\text{loc}}(x) := \{y \in \Sigma_T \colon x_i = y_i \text{ for all } i \le 0\}$$

and

$$\mathcal{W}^u(x) := \{y \in \Sigma_T \colon \sigma^n y \in \mathcal{W}^u_{\text{loc}}(\sigma^n x) \text{ for some } n \le 0\}.$$

For any $x, y \in \Sigma_T$ with $x_0 = y_0$, we say y is in the *local neighborhood* of x. For such x and y, the following bracket operation

$$[x, y] := \mathcal{W}^u_{\text{loc}}(x) \cap \mathcal{W}^s_{\text{loc}}(y) \in \Sigma_T \tag{2.1}$$

is well-defined. From the definition, $[x, y]$ is the unique point in the local neighborhood of x and y whose orbit exponentially shadows the orbit of x in the past and the orbit of y in the future.

Recall from the introduction that to any dynamical system (X, f) and $M_{d\times d}(\mathbb{R})$-valued function $\mathcal{A}$ on X, we associate a linear cocycle $\mathcal{A}$. It is clear from the definition of $\mathcal{A}^n(\cdot)$ from the introduction that the following *cocycle equation* holds:

$$\mathcal{A}^{n+m}(x) := \mathcal{A}^n(f^m x)\mathcal{A}^m(x) \text{ for all } n, m \in \mathbb{N}.$$

If the base system (X, f) is invertible and the image of $\mathcal{A}$ is a subset of $\text{GL}_d(\mathbb{R})$, then we define $\mathcal{A}^0(\cdot) \equiv I$ and $\mathcal{A}^{-n}(x) := \big(\mathcal{A}^n(f^{-n}x)\big)^{-1}$ for $n \in \mathbb{N}$ so that the cocycle equation holds for all $n, m \in \mathbb{Z}$.

If $\mathcal{A}\colon \Sigma_T \to M_{d\times d}(\mathbb{R})$ is locally constant, then from the compactness of Σ_T, there exists $k \in \mathbb{N}$ such that $\mathcal{A}(x)$ depends only on the word $x_{-k} \dots x_k \in \mathcal{L}(2k+1)$ for every $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma_T$. A *locally constant cocycle* $\mathcal{A}$ is a cocycle whose generator $\mathcal{A}$ is locally constant.

Remark 2.1. For any locally constant $\text{GL}_d(\mathbb{R})$-cocycle $\mathcal{A}$ on Σ_T, there exists a re-coding of Σ_T to another subshift of finite type $\Sigma_{\widetilde{T}}$ such that $\mathcal{A}$ is becomes to a $\text{GL}_d(\mathbb{R})$-cocycle on $\Sigma_{\widetilde{T}}$ depending only on the 0-th entry x_0 of $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma_{\widetilde{T}}$. For simplicity, we assume that all locally constant cocycles considered in this paper are cocycles that depend only on the 0-th entry.

Irreducibility defined below is a commonly imposed assumption on locally constant cocycles over full shifts.

Definition 2.2. A finite set of matrices $\mathsf{A} = \{A_1, \dots, A_q\}$ is *reducible* if there exists a non-trivial proper subspace $\mathbb{V} \subset \mathbb{R}^d$ such that $A_i\mathbb{V} = \mathbb{V}$ for all $1 \le i \le q$. We say A is *irreducible* if it is not reducible.

We say a locally constant cocycle $\mathcal{A}\colon \Sigma_q \to \mathrm{GL}_d(\mathbb{R})$ over a full shift generated by A is *irreducible* if A is irreducible.

2.1.1 Fiber-bunched cocycles and canonical holonomies

In this thesis, we will mostly work with a class of cocycles that generalize locally constant cocycles. Such a class of cocycles consist of fiber-bunched cocycles which we introduce now. Let $\mathcal{A}$ be an α-Hölder $\mathrm{GL}_d(\mathbb{R})$-cocycle on Σ_T, meaning that there exists $C > 0$ such that for all $x, y \in \Sigma_T$,

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \le Cd(x, y)^\alpha,$$

where $\|\cdot\|$ is the standard operator norm.

Definition 2.3. A *local stable holonomy* for the cocycle $\mathcal{A}$ is a family of matrices $H^s_{x,y} \in \mathrm{GL}_d(\mathbb{R})$ defined for any $x, y \in \Sigma_T$ with $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$ such that

1. $H^s_{x,x} = I$ and $H^s_{y,z} \circ H^s_{x,y} = H^s_{x,z}$ for any $y, z \in \mathcal{W}^s_{\mathrm{loc}}(x)$,
2. $\mathcal{A}(x) = H^s_{\sigma y, \sigma x} \circ \mathcal{A}(y) \circ H^s_{x,y}$,
3. $H^s\colon (x, y) \mapsto H^s_{x,y}$ is continuous.

A *local unstable holonomy* $H^u_{x,y}$ is likewise defined for $y \in \mathcal{W}^u_{\mathrm{loc}}(x)$ satisfying the analogous properties above.

Definition 2.4. An α-Hölder $\mathrm{GL}_d(\mathbb{R})$-cocycle $\mathcal{A}$ is *fiber-bunched* if for all $x \in \Sigma_T$

$$\|\mathcal{A}(x)\|\|\mathcal{A}(x)^{-1}\|\theta^\alpha < 1,$$

where θ is the hyperbolicity constant defining the metric on the base Σ_T. We denote the space of α-Hölder and fiber-bunched $\mathrm{GL}_d(\mathbb{R})$-cocycles by $C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$.

It is clear that any conformal cocycles and their perturbations are fiber-bunched. In fact, fiber-bunched cocycles may be thought of as nearly conformal cocycles.

By projectivizing the action on the fibers, any cocycle $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ gives rise to the *projective cocycle* $\mathbb{P}\mathcal{A}\colon \Sigma_T \times \mathbb{P}^{d-1} \to \Sigma_T \times \mathbb{P}^{d-1}$. Then the fiber-bunching condition is equivalent to the condition that the rate of expansion (respectively, contraction) of the projective cocycle $\mathbb{P}\mathcal{A}$ at every point $x \in \Sigma_T$ is bounded above by $1/\theta^\alpha$ (respectively, below by θ^α). In particular, the Hölder continuity and the fiber-bunching assumption on $\mathcal{A} \in C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ together ensure the convergence of the *canonical stable/unstable holonomy* $H^{s/u}_{x,y}$: for any $y \in \mathcal{W}^{s/u}_{\mathrm{loc}}(x)$,

$$H^s_{x,y} := \lim_{n\to\infty} \mathcal{A}^n(y)^{-1}\mathcal{A}^n(x) \quad \text{and} \quad H^u_{x,y} := \lim_{n\to-\infty} \mathcal{A}^n(y)^{-1}\mathcal{A}^n(x). \tag{2.2}$$

See [32] for details. It is clear that once canonical holonomies $H^{s/u}$ converge, then they satisfy the listed properties in Definition 2.3.

A cocycle may admit multiple holonomies. However, when the cocycle is fiber-bunched, the canonical holonomies are unique in the sense that they are the only holonomies varying Hölder continuously in the base points [32] with the same Hölder exponent α: there exists $C > 0$ such that

$$\|H^{s/u}_{x,y} - \mathrm{id}\| \leq C \cdot d(x,y)^\alpha, \tag{2.3}$$

for any $y \in \mathcal{W}^{s/u}_{\mathrm{loc}}(x)$. In particular, the canonical holonomies are uniformly continuous, and for this reason, we will always work with the canonical holonomies for fiber-bunched cocycles.

Remark 2.5. It is worth noting that for locally constant $\mathrm{GL}_d(\mathbb{R})$-cocycle, the canonical holonomies from (2.2) converge trivially to the identity and satisfy the properties listed in Definition 2.3.

Using the second property of Definition 2.3, we can extend the definition of the local stable holonomy to the *global stable holonomy* $H^s_{x,y}$ for $y \in \mathcal{W}^s(x)$ not necessarily in the local stable set of x:

$$H^s_{x,y} := \mathcal{A}^n(y)^{-1} H^s_{\sigma^n x, \sigma^n y} \mathcal{A}^n(x),$$

for some large enough $n \in \mathbb{N}$ so that $\sigma^n y \in \mathcal{W}^s_{\text{loc}}(\sigma^n x)$. We can likewise define the global unstable holonomy.

2.1.2 Irreducibility and typicality

We now formulate two assumptions on fiber-bunched cocycles. First, the following definition of irreducibility has previously appeared in the literature, such as in [6].

Definition 2.6. A cocycle $\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ is *reducible* if there exists an $\mathcal{A}$-invariant and $H^{s/u}$-invariant subbundle over Σ_T. We say $\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ is *irreducible* if $\mathcal{A}$ is not reducible.

Remark 2.7. We remark that for fiber-bunched cocycles that are also locally constant, the irreducibility condition as fiber-bunched cocycles (Definition 2.6) coincides with the irreducibility condition as locally constant cocycles (Definition 2.2).

Indeed, consider a fiber-bunched and locally constant $\mathrm{GL}_d(\mathbb{R})$-cocycle $\mathcal{A}$ over Σ_q which is reducible as in Definition 2.6. Let $\mathbb{V}$ be the corresponding $\mathcal{A}$-invariant and bi-holonomy invariant subbundle. As $\mathcal{A}$ is locally constant, we have $H^{s/u} \equiv \mathrm{id}$, and we may suppose that $\mathcal{A}$ is generated by $\mathsf{A} = \{A_1, \ldots, A_q\}$. From the bi-holonomy invariance, the bundle $\mathbb{V}$ must be constant over each cylinder $[i]$ and consist of q subspaces $\{V_i\}_{i=1}^q$ invariant under the action of $\mathcal{A}$. As Σ_q is a full shift, we must have $A_i V_j = V_i$ for all $1 \leq i, j \leq q$. Fixing an i and varying over all j shows that all V_i's are equal and hence $\mathbb{V}$ is a constant bundle. We have just shown that the generating set A preserves a common subspace $\mathbb{V}$, and hence, is reducible. The other direction of the equivalence also follows from a similar reasoning.

We say a point $z \in \Sigma_T$ is a *homoclinic point* of a periodic point p if $z \in \mathcal{W}^s(p) \cap \mathcal{W}^u(p) \setminus \{p\}$. The homoclinic points of p are characterized as the points other than p whose orbits synchronously approach the orbit of p in both forward and backward time. For a hyperbolic system such as (Σ_T, σ), the set of homoclinic points of any periodic point is dense in Σ_T.

Consider any periodic point p and a homoclinic point z. We define the *holonomy loop* ψ_p^z as the composition of the unstable holonomy from p to z and the stable holonomy from z to p:

$$\psi_p^z := H^s_{z,p} \circ H^u_{p,z}. \tag{2.4}$$

The following definition is a slight weakening of the assumptions of [8, Theorem 1]; see the discussion below Definition 2.11.

Definition 2.8. Let $\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ and $H^{s/u}$ be the canonical holonomies for the cocycle $\mathcal{A}$. We say that $\mathcal{A}$ is *1-typical* if it satisfies the following two extra conditions:

(A0) there exists a periodic point p such that $P := \mathcal{A}^{\mathrm{per}(p)}(p)$ has simple real eigenvalues of distinct norms. Let $\{v_i\}_{1 \le i \le d}$ be the eigenvectors of P.

(B0) there exists a homoclinic point z of p such that ψ_p^z twists the eigendirections of P into general position: for any $1 \le i, j \le d$, the image $\psi_p^z(v_i)$ does not lie in any hyperplane $\mathbb{W}_j$ spanned by all eigenvectors of P other than v_j. Equivalently, the coefficients $c_{i,j}$ in

$$\psi_p^z(v_i) = \sum_{1 \le j \le d} c_{i,j} v_j,$$

are nonzero for all $1 \le i, j \le d$.

We will often refer to (A0) and (B0) by *pinching* and *twisting* conditions, respectively. If z is a homoclinic point of a periodic point p of period n, then $\sigma^{nr} z$ is also a homoclinic point of p for any $r \in \mathbb{Z}$. Their holonomy loops (2.4) are conjugated by P^r:

$$P^r \psi_p^z = \psi_p^{\sigma^{nr} z} P^r.$$

It then follows that if the twisting condition (B0) holds at z, then it also holds at $\sigma^{nr} z$. We also establish the following identity to be used later: assuming that p is a fixed point of σ and that $z \in \mathcal{W}^u_{\text{loc}}(p)$ and $\sigma^\ell z \in \mathcal{W}^s_{\text{loc}}(p)$ for some $\ell \in \mathbb{N}$, ψ_p^z, then

$$\psi_p^z = P^{-\ell} \circ H^s_{\sigma^\ell z, p} \circ \mathcal{A}^\ell(z) \circ H^u_{p,z}. \tag{2.5}$$

For each $1 \leq t \leq d$, we denote by $\mathcal{A}^{\wedge t}$ the action of $\mathcal{A}$ on the exterior product $(\mathbb{R}^d)^{\wedge t}$. For a fiber-bunched cocycle $\mathcal{A}$, its exterior product cocycles $\mathcal{A}^{\wedge t}$, $t \in \{1, \dots, d\}$, also admit stable and unstable holonomies, namely $(H^{s/u})^{\wedge t}$. So, for a 1-typical cocycle $\mathcal{A}$, we consider similar conditions appearing in Definition 2.8 on $\mathcal{A}^{\wedge t}$.

Definition 2.9. Let $\mathcal{A}$ be 1-typical. For $2 \leq t \leq d-1$, we say $\mathcal{A}$ is *t-typical* if the same points $p, z \in \Sigma_T$ from Definition 2.8 satisfy

(A1) all the products of t distinct eigenvalues of P are distinct;

(B1) the induced map $(\psi_p^z)^{\wedge t}$ on $(\mathbb{R}^d)^{\wedge t}$ satisfies the analogous statement to (B0) from Definition 2.8 with respect to the eigenvectors $\{v_{i_1} \wedge \ldots \wedge v_{i_t}\}_{1 \leq i_1 < \ldots < i_t \leq d}$ of $P^{\wedge t}$.

Remark 2.10. Denoting by M the matrix of ψ_p^z in a basis of eigenvectors $\mathcal{A}^{\text{per}(p)}(p)$, the twisting assumption (B1) is equivalent to the condition that all $t \times t$ minors of M are non-zero.

Also notice that the definition of t-typicality only asks for (A1) and (B1); that is, the definition does not require that $\mathcal{A}^{\wedge t}$ also be fiber-bunched. On the other hand, we will use the fact that the stable and unstable holonomies $(H^{s/u})^{\wedge t}$ are uniformly continuous; see (2.3).

Definition 2.11. We say $\mathcal{A}$ is *typical* if $\mathcal{A}$ is t-typical for all $1 \leq t \leq d-1$. We denote $\mathcal{U} \subset C_b^{\alpha}(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ to be the set of all typical cocycles.

A few comments regarding the assumptions are in order. Typicality assumption is first introduced in Bonatti and Viana [8] as a sufficient condition to establish the simplicity of the Lyapunov exponents of $\mathcal{A}$ for any ergodic σ-invariant measure with continuous local product structure. Their result generalizes the corresponding result on the simplicity of Lyapunov exponents in the setting of i.i.d. product of random matrices. Indeed, the pinching and twisting conditions of the typicality assumption are coined precisely to replicate the effect of proximality and strong irreducibility of Furstenberg [30]; see also [9].

For their 1-typicality assumption, the pinching assumption is the same as in Definition 2.8 above, but their twisting assumption is formulated as follows: any P-invariant subspaces $E, F \subset \mathbb{R}^d$ whose dimensions add upto a number less than or equal to d satisfy $\psi_p^z(E) \cap F = \{0\}$. Interpreting this formulation in regards to the minors of M where M is as in Remark 2.10, it can be seen that such formulation is equivalent to (B0) together with (B1) for all t. In particular, 1-typical cocycles of [8] satisfy all assumptions of typical cocycles defined in Definition 2.11 except (A1).

Moreover, they also show that $\mathcal{U}$ is open and dense in $C_b^{\alpha}(\Sigma_T, \mathrm{SL}_d(\mathbb{R}))$. While their setting is for fiber-bunched $\mathrm{SL}_d(\mathbb{R})$-cocycles, we remark that the difference in the settings ($\mathrm{SL}_d(\mathbb{R})$ for [8] and $\mathrm{GL}_d(\mathbb{R})$ for this paper) does not cause any issues in translating the relevant statements and results from [8] to this paper.

Avila and Viana in [1] then improved the result by weakening the assumptions: only assuming 1-typicality (as opposed to assuming t-typicality for all $1 \leq t \leq d-1$), they allowed the number of symbols of Σ_T to be countably infinite and proved analogous results to [8]. They call 1-typical cocycles of [8] by simple cocycles, and applied this result to Kontsevich-Zorich cocycles [2].

2.2 Ergodic theory and thermodynamic formalism

Throughout this section, X is a compact metric space, $f : X \to X$ is a homeomorphism, and $\mathcal{M}(f)$ denotes the set of all f-invariant probability measures.

2.2.1 Mixing properties of invariant measures

Although the main ergodic property we establish in Theorem E and Theorem F is the K-property, we will make use of various mixing properties along the way. We give a brief introduction to the ones that we will use, in increasing order of strength. In many cases, there are many equivalent formulations of these definitions where we refer the readers to [43] for a more comprehensive discussion on various mixing properties.

Definition 2.12. A measure-preserving transformation (X, f, μ) is *totally ergodic* if (X, f^n, μ) is ergodic for all $n \in \mathbb{N}$.

Definition 2.13. We say $\mu \in \mathcal{M}(f)$ is *weakly mixing* if for all measurable subsets $A, B \subseteq X$, there exists $E \subset \mathbb{N}$ with upper density $\bar{d}(E) = 0$ such that for $n \notin E$,

$$\lim_{n\to\infty} \mu(f^n A \cap B) = \mu(A)\mu(B).$$

Observe from this definition that if μ is weak mixing, then it is totally ergodic. The following is a now classical result that we will also make use of.

Proposition 2.14. (X, f, μ) is weak mixing if and only if $(X \times X, f \times f, \mu \times \mu)$ is ergodic.

Finally, we introduce the K-property. There are a myriad number of equivalent formulations, for details of which we refer to [21].

Definition 2.15. Let $(X, \mathcal{B}, \mu)$ be a probability space, and let f be a measure-preserving invertible transformation of $(X, \mathcal{B}, \mu)$ The system has the *Kolmogorov property*, or simply the *K-property*, if there exists a sub-σ-algebra $\mathcal{K} \subset \mathcal{B}$ satisfying $\mathcal{K} \subset f\mathcal{K}$, $\bigvee_{i=0}^{\infty} f^i\mathcal{K} = \mathcal{B}$, and $\bigcap_{i=0}^{\infty} f^{-i}\mathcal{K} = \{\emptyset, X\}$.

An equivalent definition of independent interest is that of *completely positive entropy.*

Proposition 2.16. A measure $\mu \in \mathcal{M}(f)$ has the K-property if and only if it has completely positive entropy. Equivalently, the maximal zero entropy factor, called the *Pinsker factor*, is trivial.

Remark 2.17. In particular, any K-system has positive measure-theoretic entropy.

We say a system (X, f, μ) is *Bernoulli* if it is measurably isomorphic to a Bernoulli shift. Without too much difficulty, one can use the above definition to show that every Bernoulli system is K.

2.2.2 Additive thermodynamic formalism

A *potential* on X is a continuous function $\varphi\colon X \to \mathbb{R}$. A subset $E \subset X$ is (n, ε)*-separated* if every pair of distinct $x, y \in E$ satisfies

$$d_n(x, y) := \max_{0 \leq i \leq n-1} d(f^i x, f^i y) \geq \varepsilon.$$

Using (n, ε)-separated subsets, we can define a thermodynamical object called the *pressure* $\mathsf{P}(\varphi)$ of φ as follows:

$$\mathsf{P}(\varphi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \Big\{ \sum_{x \in E} e^{S_n\varphi(x)} \colon E \text{ is an } (n, \varepsilon)\text{-separated subset of } X \Big\},$$

where $S_n\varphi := \varphi + \psi \circ f + \ldots + \varphi \circ f^{n-1}$ is the n-th *Birkhoff sum* of φ. When $\varphi \equiv 0$, the pressure $\mathsf{P}(0)$ is equal to the *topological entropy* $h(f)$, which measures the complexity of the system (X, f).

The pressure satisfies the *variational principle*:

$$\mathsf{P}(\varphi) = \sup\Big\{h_\mu(f) + \int \varphi\, d\mu \colon \mu \in \mathcal{M}(f)\Big\},$$

where $h_\mu(f)$ is the measure-theoretic entropy. See [46].

Any invariant measure $\mu \in \mathcal{M}(f)$ achieving the supremum in the variational principle is called an *equilibrium state* of φ. If the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then any potential has an equilibrium state. However, the existence, the finiteness, or the uniqueness of the equilibrium state for a given potential is a subtle question that depends on the system (X, f) as well as the potential φ.

On the other hand, there are specific settings where such questions have an affirmative answer. When (X, f) is a mixing hyperbolic system such as (Σ_T, σ), and the potential φ is Hölder continuous, then the result of Bowen [11] states that there exists a unique equilibrium state μ_φ, which has the Gibbs property. Bernoullicity of such equilibrium states can be found in [12, 44].

Proposition 2.18. [11, 12] Let (Σ_T, σ) be a mixing subshift of finite type, and φ be Hölder continuous. Then there exists a unique equilibrium state μ_φ for φ, characterized as the unique σ-invariant measure satisfying the Gibbs property: there exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $\mathrm{I} \in \mathcal{L}(n)$, we have

$$C^{-1} \leq \frac{\mu_\varphi([\mathrm{I}])}{e^{-n\mathsf{P}(\varphi)+S_n\varphi(x)}} \leq C \tag{2.6}$$

for any $x \in [\mathrm{I}]$. Moreover, μ_φ is Bernoulli.

Remark 2.19. One necessary condition for the Gibbs property (2.6) to hold is that the variation within each cylinder should be uniformly bounded: there exists a constant $C \geq 0$

such that for every $n \in \mathbb{N}$ and $\mathrm{I} \in \mathcal{L}(n)$,

$$|S_n\varphi(x) - S_n\varphi(y)| \leq C$$

for every $x, y \in [\mathrm{I}]$. We call this property by *bounded distortion.*

In the setting of Bowen's theorem, the hyperbolicity of the system and the Hölder regularity of the potential guarantee the bounded distortion property.

We end this subsection by commenting on a sufficient condition on the potentials which guarantees the coincidence of their equilibrium states.

Definition 2.20. Two continuous functions $\varphi, \psi \in C(X)$ with $\mathsf{P}(\varphi) = \mathsf{P}(\psi)$ are *cohomologous* if there exists a continuous function h such that $\varphi - \psi = h \circ \sigma - h$, and we denote it by $\varphi \sim \psi$.

It is clear from the variational principle that if $\varphi \sim \psi$, then their set of equilibrium states are the same. If the base dynamic (Σ_T, σ) is uniformly hyperbolic, this is an if and only if statement when restricted to the class of Hölder potentials; that is, two Hölder potentials φ and ψ are cohomologous if and only if their unique equilibrium states coincide [12, Theorem 1.28].

2.2.3 Subadditive thermodynamic formalism

The additive theory of thermodynamic formalism extends to the subadditive theory with suitable generalizations. A sequence of continuous functions $\{\varphi_n\}_{n\in\mathbb{N}}$ on Σ_T is *submultiplicative* if each φ_n is a non-negative function on Σ_T satisfying

$$0 \leq \varphi_{m+n} \leq \varphi_n \cdot \varphi_m \circ \sigma^n, \text{ for all } m, n \in \mathbb{N}.$$

If we set $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$, then Φ becomes a *subadditive* sequence of functions on Σ_T. We will consider such Φ obtained from a submultiplicative sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ as a *subadditive potential* on Σ_T. A natural example of a subadditive potential is the *norm potential* $\Phi_{\mathcal{A}}$ of a $\mathrm{GL}_d(\mathbb{R})$-cocycle $\mathcal{A}$ defined as in the introduction.

We define the *subadditive pressure* of a subadditive potential $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ as

$$\mathsf{P}(\Phi) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup\Big\{\sum_{x\in E}\varphi_n(x)\colon E \text{ is an } (n,\varepsilon)\text{-separated subset of } \Sigma_T\Big\}, \quad (2.7)$$

where the existence of the limit is guaranteed from the subadditivity of Φ.

Remark 2.21. When the base dynamical system is a subshift of finite type (Σ_T,σ), due to the expansivity the pressure $\mathsf{P}(\Phi)$ may be expressed as follows:

$$\mathsf{P}(\Phi) = \limsup_{n\to\infty}\frac{1}{n}\ log\sup\Big\{\sum_{x\in E}\varphi_n(x)\colon E \text{ is an } (n,1)\text{-separated subset of } \Sigma_T\Big\}; \quad (2.8)$$

that is, we can compute the pressure by looking at $(n,1)$-separated sets, and drop the limit in ε from the definition of the pressure (2.7). See Section 4 of [34].

There are a few different generalizations of the additive notion of the pressure to the subadditive setting: Barreira [5] defines the subadditive pressure using open covers while Cao, Feng, and Huang [17] define it using Bowen balls. Our definition of the subadditive pressure (2.7) is based on [17]. See also [23]. It is not known whether two definitions of the subadditive pressure are equal in general, but there are known settings in which two definitions agree. In particular, it is shown in [17] that two notions are equivalent when the entropy map $\mu\mapsto h_\mu(f)$ is upper semi-continuous, which includes our setting of mixing subshifts of finite type (Σ_T,σ).

Cao, Feng, and Huang [17] also establish the *subadditive variational principle*:

$$\mathsf{P}(\Phi) = \sup\Big\{h_\mu(f) + \mathcal{F}(\Phi,\mu)\colon \mu\in\mathcal{M}(f),\ \mathcal{F}(\Phi,\mu)\neq-\infty\Big\}, \quad (2.9)$$

where

$$\mathcal{F}(\Phi, \mu) := \lim_{n\to\infty} \frac{1}{n} \int \log \varphi_n \, d\mu = \inf_{n\to\infty} \frac{1}{n} \int \log \varphi_n \, d\mu,$$

whose limit is again guaranteed from the subadditivity of Φ. In the case where Φ is given by a norm potential $\mathcal{A}$ of a matrix cocycle $\mathcal{A}$, we note that $\mathcal{F}(\Phi, \mu)$ is the *top Lyapunov exponent of* $\mathcal{A}$ *with respect to* μ, and we will often denote it by $\lambda_+(\mathcal{A}, \mu)$.

Similar to the additive setting, any invariant measure $\mu \in \mathcal{M}(f)$ achieving the supremum in (2.9) is called an *equilibrium state* of Φ. Also, at least one equilibrium state necessarily exists for any subadditive potential Φ if the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous [27]. See also [35].

Remark 2.22. Similar to cohomologous additive potentials, if a cocycle $\mathcal{A} \in C(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ is continuously conjugated to another cocycle $\mathcal{B} \in C(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$, then from the subadditive variational principle (2.9) the pressures and the set of equilibrium states for the norm potentials $\Phi_\mathcal{A}$ and $\Phi_\mathcal{B}$ are the same. This follows because $\mathcal{F}(\Phi_\mathcal{A}, \mu) = \mathcal{F}(\Phi_\mathcal{B}, \mu)$ for all $\mu \in \mathcal{M}(\sigma)$ as the norm $\|\mathcal{C}(x)\|$ of the continuous conjugacy $\mathcal{C} \colon \Sigma_T \to \mathrm{GL}_2(\mathbb{R})$ is uniformly bounded from the compactness of Σ_T. However, it is not necessarily true that two cocycles $\mathcal{A}$ and $\mathcal{B}$ are conjugated to each other just because their equilibrium states coincide.

2.2.4 Singular value potentials

We have mostly been discussing the norm potentials of matrix cocycles, but there is another important class of subadditive potentials arising from matrix cocycles called the singular value potentials. While most results in this thesis are formulated for the norm potentials for coherency, some of those results readily extend also to the singular value potentials and we will comment on them whenever relevant. Hence, we briefly define here the singular value potentials and describe their applications.

The *singular values* of $A \in M_{d\times d}(\mathbb{R})$ are eigenvalues of $\sqrt{A^*A}$. We define the *singular* *value function* $\varphi^s\colon M_{d\times d}(\mathbb{R}) \to \mathbb{R}$ with parameter $s \geq 0$ as follows:

$$\varphi^s(A) = \begin{cases} \alpha_1(A)\ldots\alpha_{\lfloor s\rfloor}(A)\alpha_{\lceil s\rceil}(A)^{\{s\}} & 0 \leq s \leq d, \\ |\det(A)|^{s/d} & s > d, \end{cases}$$

where $\alpha_1(A) \geq \ldots \geq \alpha_d(A) \geq 0$ are the singular values of A. It is well-known that φ^s is submultiplicative for all s: for any $A, B \in M_{d\times d}(\mathbb{R})$ and $s \in [0, \infty)$,

$$\varphi^s(A)\varphi^s(B) \geq \varphi^s(AB).$$

Moreover, the function $(A, s) \mapsto \varphi^s(A)$ is upper semi-continuous, and has a discontinuity at $s = k \in \mathbb{N}$ only if there is a jump in the singular values of the form $\alpha_{k-1}(A) > \alpha_k(A) = 0$. In particular, if A takes value in $\mathrm{GL}_d(\mathbb{R})$, then $\varphi^s(A)$ is continuous in both A and s.

For any cocycle $\mathcal{A}$ and $s \geq 0$, we define the *s-singular value potential* as

$$\Phi^s_{\mathcal{A}} = \{\log \varphi^s_{\mathcal{A},n}\}_{n\in\mathbb{N}} \text{ where } \varphi^s_{\mathcal{A},n}(x) := \varphi^s(\mathcal{A}^n(x)).$$

Since the norm $\|A\|$ is the 1-singular value $\alpha_1(A)$, it is clear that the singular value potentials are generalizations of the norm potentials.

As mentioned in the introduction, the singular value potentials have applications in the dimension theory of the fractals. For instance, given a mixing repeller $\Lambda \subset M$ with respect to a diffeomorphism on a closed manifold M, we may associate to it a sequence of subadditive potentials Φ^s_Λ defined similar to the singular value potentials above. Under certain conditions, the Hausdorff dimension of the repeller is bounded above by the unique zero of the function $s \mapsto \mathsf{P}(\Phi^s_\Lambda)$. What is more, the unique zero of such a function is often equal to the Hausdorff dimension of the repeller; see [24].

This is a generalization of Bowen's result [13] where he computes the Hausdorff dimension of quasi-circles arising from quasi-Fuchsian groups. Bowen uses the classical thermodynamic formalism involving additive potentials and shows that the unique zero of the so-called *Bowen's equation* computes to the Hausdorff dimension of the quasi-circle.

2.2.5 Quasi-multiplicativity

We can extend the notion of bounded distortion for additive potentials introduced in Remark 2.19 to subadditive potentials. We say a subadditive potential $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ has *bounded distortion* if there exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $\mathrm{I} \in \mathcal{L}(n)$, we have

$$C^{-1} \leq \frac{\varphi_n(x)}{\varphi_n(y)} \leq C \tag{2.10}$$

for any $x, y \in [\mathrm{I}]$. The following lemma shows that the norm potentials of fiber-bunched cocycles have bounded distortion.

Lemma 2.23. Let $\mathcal{A}$ be a Hölder continuous and fiber-bunched $\mathrm{GL}_d(\mathbb{R})$-cocycle over Σ_T. Then its norm potential $\Phi_{\mathcal{A}}$ has bounded distortion (2.10).

Proof. From the uniform continuity of the canonical holonomies $H^{s/u}$, we can fix $c > 1$ such that the norm $\|H^{s/u}_{x,y}\|$ is bounded above by c whenever $y \in \mathcal{W}^{s/u}_{\mathrm{loc}}(x)$.

Consider any $n \in \mathbb{N}$, $\mathrm{I} \in \mathcal{L}(n)$, and $x, y \in [\mathrm{I}]$. Then, setting $z := [y, x]$ and using the second property of the holonomies from Definition 2.3 as well as the submultiplicativity (1.2) of the operator norm, we have

$$c^{-2}\|\mathcal{A}^n(x)\| \leq \|\mathcal{A}^n(z)\| = \|H^s_{\sigma^n x, \sigma^n z} \circ \mathcal{A}^n(x) \circ H^s_{z,x}\| \leq c^2\|\mathcal{A}^n(x)\|.$$

Using the canonical unstable holonomy instead, we have $c^{-2} \leq \|\mathcal{A}^n(y)\|/\|A^n(z)\| \leq c^2$. Then, the statement follows by setting the constant C equal to c^4. □

Remark 2.24. The canonical holonomies $H^{s/u}$ vary continuously in $\mathcal{A}$. Hence, by increasing the constant C from Lemma 2.23 if necessary, the bounded distortion holds on $\Phi_{\mathcal{B}}$ for all $\mathcal{B} \in C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ sufficiently close to $\mathcal{A}$ with a uniform constant C.

Definition 2.25. Given a subadditive potential $\Phi = \{\log \varphi_n\}$ on Σ_T with bounded distortion, consider a subadditive potential $\log \varphi$ on $\mathcal{L}$ given defined by $\varphi(\mathrm{I}) := \max\limits_{x \in [\mathrm{I}]} \varphi_n(x)$. We say Φ is *quasi-multiplicative* if there exist constants $c > 0$ and $k \in \mathbb{N}$ such that for any words $\mathrm{I}, \mathrm{J} \in \mathcal{L}$, there exists $\mathrm{K} = \mathrm{K}(\mathrm{I}, \mathrm{J}) \in \mathcal{L}$ with $|\mathrm{K}| \leq k$ such that $\mathrm{IKJ} \in \mathcal{L}$ and

$$\varphi(\mathrm{IKJ}) \geq c\varphi(\mathrm{I})\varphi(\mathrm{J}).$$

In the same vein, we say $\mathcal{A} \colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ is *quasi-multiplicative* if its norm potential $\Phi_{\mathcal{A}}$ is quasi-multiplicative. This agrees with the definition of quasi-multiplicativity (1.4) of $\mathcal{A}$ from the introduction.

The main usage of quasi-multiplicativity is due to the work of Feng [27] who showed that it is a sufficient condition to guarantee the uniqueness of the equilibrium state.

Proposition 2.26. [27, Theorem 5.5] Let $\Phi = \{\log \varphi_n\}_{n \in \mathbb{N}}$ be a subadditive potential on Σ_T with bounded distortion. If Φ is quasi-multiplicative, then there exists a unique equilibrium state $\mu \in \mathcal{M}(\sigma)$ satisfying the *subadditive Gibbs property*: there exists $C \geq 1$ such that for any $x \in \Sigma_T$ and $n \in \mathbb{N}$, we have

$$C^{-1} \leq \frac{\mu([x]_n)}{e^{-n\mathsf{P}(\Phi)}\varphi_n([x]_n)} \leq C.$$

This result is a partial generalization of Proposition 2.18 to subadditive potentials. The differences, however, are that we need an extra assumption of quasi-multiplicativity on the subadditive potential and that μ is not necessarily Bernoulli. Regarding such differences, we note that the proposition is no longer true without assuming quasi-multiplicativity, and examples of subadditive potentials with multiple ergodic equilibrium states may be found in [28]. The lack of Bernoullicity is partially addressed in Chapter 5 where we establish the K-property on μ under suitable assumptions.

We also note that this proposition is established for subadditive potentials over one-sided subshifts of finite type in [27]. However, the same result readily extends to two-sided subshifts of finite type; see [41].

It is clear from Proposition 2.26 that quasi-multiplicativity is an important property one hopes to verify on subadditive potentials. In the case where subadditive potentials are norm potentials of locally constant cocycles, the following result of Feng [26] shows that irreducibility implies quasi-multiplicativity.

Proposition 2.27. [26] Let $\mathcal{A}\colon \Sigma_q \to \mathrm{GL}_d(\mathbb{R})$ be a locally constant cocycle over a full shift. If $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is quasi-multiplicative, and hence, its norm potential $\Phi_{\mathcal{A}}$ has a unique equilibrium state $\mu_{\mathcal{A}} \in \mathcal{M}(\sigma)$ with the subadditive Gibbs property.

We conclude this preliminary chapter with some comments and discussion on this proposition with regards to the main results of this thesis.

Theorem A can be viewed as a generalization of the above proposition that the norm potentials of "most" fiber-bunched cocycles have unique equilibrium states. A few merits of Theorem A is that this is one of the first few results on the subadditive thermodynamic formalism of cocycles that are not locally constant. Another merit of Theorem A is that it connects a seemingly unrelated topological condition (typicality) on the cocycles to the uniqueness of the equilibrium states for the norm potentials.

Theorem C is another generalization of the above proposition. Instead of having to restrict to fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles, we have a more direct generalization under irreducibility assumption only, which is strictly weaker than typicality. For fiber-bunched $\mathrm{GL}_d(\mathbb{R})$-cocycles, the same result holds true under an extra assumption called the stronger bunching; see spannability introduced in Bochi and Garibaldi [6] and its connection to the uniqueness of the equilibrium states outlined in Section 4.4. We expect that the stronger bunching assumption is unnecessary and that norm potentials of irreducible fiber-bunched $\mathrm{GL}_d(\mathbb{R})$-cocycles have unique equilibrium states, but there is currently no established proof on such a problem.

Once the uniqueness of the equilibrium states $\mu_{\mathcal{A}}$ for irreducible locally constant cocycles was established by Feng, there had been attempts to further investigate various properties on them. For instance, the result of Morris [37, 38, 39] mentioned already in the introduction shows that if $\mu_{\mathcal{A}}$ is totally ergodic, then it is Bernoulli. His result can be compared to the unique equilibrium states for Hölder potentials (from Proposition 2.18) which are Bernoulli without needing to impose any further assumptions. With that said, Morris' result is in some sense similar to how a non-trivial assumption such as quasi-multiplicativity is necessary to extend Bowen's result on the uniqueness of the equilibrium state to subadditive potentials.

Theorem E and F are also results along this direction. While we do not quite obtain Bernoulli property in those theorems, the K-property is a property right below Bernoullicity in the hierarchy of ergodic properties. Hence, we may consider them as partial generalizations of Morris' result to fiber-bunched cocycles.

The complementary results to the above proposition concerning the existence of multiple equilibrium states for the norm potentials of locally constant cocycles have also been studied by Feng and Käenmäki [28]. It is clear from the above proposition that such cocycles are necessarily reducible. In particular, such cocycles can simultaneously be conjugated to upper block triangular matrices where the tuples of diagonal blocks are irreducible. Feng and Käenmäki showed that the number of ergodic equilibrium states for the norm potential is necessarily finite. What is more, they established that each diagonal block can contribute at most one equilibrium state, which then implies that the total number of equilibrium states is bounded above by the number of diagonal blocks.

There is a similar result for the singular value potentials for locally constant cocycles. Bochi and Morris [7] showed that for any locally constant cocycle $\mathcal{A}\colon \Sigma_q \to \mathrm{GL}_d(\mathbb{R})$ and any $s \geq 0$, the number of ergodic equilibrium states for the singular value potential $\Phi^s_{\mathcal{A}}$ is finite. Compared to the norm potentials, their proof for the singular value potentials is more involved and requires in dept analysis on the structure of the cocycles using tools from algebraic geometry.

Lastly, we note that there is no analogue of Bochi-Morris' result for fiber-bunched cocycles. Their method does not readily generalize to fiber-bunched cocycles, and new tools would have to be introduced to establish such a result.

CHAPTER 3

QUASI-MULTIPLICATIVITY OF TYPICAL COCYCLES

This chapter originally appears in [41] [1].

3.1 Preliminary linear algebra

We collect preliminary lemmas and relevant constants needed in the proof of Theorem A.

Definition 3.1. For $A \in \mathrm{GL}_d(\mathbb{R})$, we choose a singular value decomposition (SVD)

$$A = U\Lambda V^*,$$

where the singular values in Λ are listed in a non-increasing order. We define $u(A)$ and $v(A)$ as the first column of U and V, respectively.

If the singular values of A are distinct, then the singular value decomposition of A is unique (up to signs), and hence so are $u(A)$ and $v(A)$. If there are repeated singular values, then the singular value decomposition of A is not necessarily unique. In this case, we simply choose a singular value decomposition of A, and set $u(A)$ and $v(A)$ accordingly.

Roughly speaking, $u(A)$ and $v(A)$ can be thought of as the most expanding direction of A^* and A, respectively. From the definition, we have

$$\|A\|u(A) = Av(A). \tag{3.1}$$

Throughout the section, when we measure the angle between nonzero vectors, we mean the angle between the lines spanned by the vectors. Similarly, when we measure the angle between a nonzero vector v and a hyperplane $\mathbb{W}$, we mean the minimum angle $\measuredangle(v, w)$

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over all $w \in \mathbb{W} \setminus \{0\}$. Also, we will not distinguish between a vector in $\mathbb{V} \setminus \{0\}$ and its corresponding point in the projective space $\mathbb{P}^{d-1}$ when there is no confusion. We have an easy lemma from linear algebra.

Lemma 3.2. Given any $A \in \mathrm{GL}_d(\mathbb{R})$ and any $w \in \mathbb{R}^d$, we have

$$\|Aw\| \geq \cos \measuredangle\big(w, v(A)\big)\|A\| \cdot \|w\|.$$

Proof. Let $v = v(A)$, and write $w = av + v'$ where $|a| = \|w\| \cos \measuredangle(w, v)$ and $v' \in v^{\perp}$. Letting $u = u(A)$, we have from (3.1) that

$$Aw = a\|A\|u + Av'.$$

Since the singular vectors are pairwise orthogonal (i.e., columns of U are pairwise orthogonal), we have $Av' \in u^{\perp}$ and the lemma follows. □

Recall that the *co-norm* $m(A)$ of $A \in \mathrm{GL}_d(\mathbb{R})$ is defined by

$$m(A) := \|A^{-1}\|^{-1}.$$

The following lemma will be useful in proving Theorem A.

Lemma 3.3. Let $\theta > 0$ be given and $A, B, C, D \in \mathrm{GL}_d(\mathbb{R})$ such that

$$\measuredangle\big(B^* v(A), (Cu(D))^{\perp}\big) > \theta.$$

Then,

$$\|ABCD\| \geq \|A\| \cdot \|D\| \cdot \sin(\theta) \cdot \frac{m(B)^2 m(C)}{\|B\|}.$$

Proof. We have

$$\begin{aligned}\|BCu(D)\| \cos \measuredangle\big(BCu(D), v(A)\big) &= |\langle v(A), BCu(D)\rangle| \\ &= |\langle B^* v(A), Cu(D)\rangle| \\ &\geq \|B^* v(A)\| \|Cu(D)\| \sin(\theta).\end{aligned}$$

Hence,

$$\cos \measuredangle\big(BCu(D), v(A)\big) \geq \sin(\theta) \cdot \frac{m(B)}{\|B\|}.$$

It then follows from (3.1) and Lemma 3.2 that

$$\begin{aligned}\|ABCD\| \geq \|ABCDv(D)\| &= \|D\| \cdot \|ABCu(D)\| \\ &\geq \|D\| \cdot \cos \measuredangle\big(BCu(D), v(A)\big) \|A\| \cdot \|BCu(D)\| \\ &\geq \|A\| \cdot \|D\| \cdot \sin(\theta) \cdot \frac{m(B)}{\|B\|} \cdot m(BC).\end{aligned}$$

This completes the proof. □

For $v \in \mathbb{P}^{d-1}$, let the *cone around* v *of size* ε be

$$\mathcal{C}(v, \varepsilon) := \{w \in \mathbb{P}^{d-1} \colon \measuredangle(v, w) < \varepsilon\}.$$

If $P \in \mathrm{GL}_d(\mathbb{R})$ has simple eigenvalues of distinct norms with corresponding eigendirections $\{v_1, \ldots, v_d\}$, then for any $v \in \mathbb{P}^{d-1}$, there exist indices $i, j \in \{1, \ldots, d\}$ such that $P^n v$ converges to v_i as $n \to \infty$ and to v_j as $n \to -\infty$. Given an $\varepsilon > 0$, it then follows that $P^n v$ eventually enters and remains in the ε-cone of $\{v_1, \ldots, v_d\}$ as $|n|$ gets sufficiently large. The following lemma shows that there exists a uniform constant $N \in \mathbb{N}$ such that the number of $n \in \mathbb{Z}$ in which $P^n v$ lies outside of the ε-cone of $\{v_1, \ldots, v_d\}$ is bounded above by N, independent of $v \in \mathbb{P}^{d-1}$.

Lemma 3.4. Suppose $P \in \mathrm{GL}_d(\mathbb{R})$ has simple eigenvalues of distinct norms. Given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for any $v \in \mathbb{P}^{d-1}$,

$$\#\Big\{n \in \mathbb{Z} \colon P^n v \notin \bigcup_{i=1}^{d} \mathcal{C}(v_i, \varepsilon)\Big\} \leq N.$$

Proof. The projectivization of P is a Morse-Smale diffeomorphism on $\mathbb{P}^{d-1}$ with d fixed points $v_1, \ldots, v_d$ and no other periodic points. Hence, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the conclusion of the lemma holds true. □

Remark 3.5. Since the eigenvalues of P from Lemma 3.4 vary continuously in P, we can choose N to work uniformly near P.

In the proof of Theorem A, we will also make use of the adjoint cocycle. For any cocycle $\mathcal{A}$, we define the *adjoint cocycle* $\mathcal{A}_*$ over σ^{-1} generated by $\mathcal{A}_*$ where $\mathcal{A}_*$ is defined by

$$\mathcal{A}_*(x) := [\mathcal{A}(\sigma^{-1}x)]^* \tag{3.2}$$

Suppose $\mathcal{A}$ admits holonomies $H^{s/u}$. Then the adjoint cocycle $\mathcal{A}_*$ also admits holonomies given by

$$H^{s,*}_{x,y} = (H^u_{y,x})^* \quad \text{and} \quad H^{u,*}_{x,y} = (H^s_{y,x})^*.$$

This can be easily seen by plugging $u = (H^s_{x,y})^* \widetilde{u}$ and $v = H^s_{\sigma^{-1}y,\sigma^{-1}x} \widetilde{v}$ into (3.2) for some y in the stable set of x with respect to σ. The following lemma shows that many properties of $\mathcal{A}$ carry over to $\mathcal{A}_*$.

Lemma 3.6. Let $\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$. Then,

1. $\mathcal{A}_*$ is fiber-bunched if and only if $\mathcal{A}$ is fiber-bunched.

2. $\mathcal{A}_*$ is 1-typical if and only if $\mathcal{A}$ is 1-typical.

3. $\mathcal{A}_*$ is typical if and only if $\mathcal{A}$ is typical.

Proof. (1) follows from the fact that a matrix and its transpose have the same singular values. For (2), we note that the eigenvalues of the transpose P^* are equal to the eigenvalues of P; in particular, they are simple and distinct in modulus. Indeed, if we define $\mathbb{W}_j$ to be the hyperplane spanned by all but the j-th eigenvector v_j of P, then the j-th eigendirection of P^* is given by $w_j := (\mathbb{W}_j)^\perp$. The twisting condition (B0) from Definition 2.8 is then equivalent to

$$\langle \psi_p^z(v_i), w_j \rangle \neq 0 \text{ for all } 1 \leq i, j \leq d.$$

Hence, $\langle v_i, (\psi_p^z)^* w_j \rangle \neq 0$ for all $1 \leq i, j \leq d$; this is equivalent to $\mathcal{A}_*$ being 1-typical because $\psi_p^{z,*} = H_{z,p}^{s,*} \circ H_{p,z}^{u,*}$ is equal to $(\psi_p^z)^*$. (3) then trivially follows from (2). □

We end the preliminary set-up with a lemma which allows us to assume that the periodic point p from Theorem A is a fixed point by passing to a power.

Lemma 3.7. Suppose $\mathcal{A}$ is fiber-bunched. For any $n \in \mathbb{N}$, consider $\hat{\sigma} := \sigma^n$ and $\hat{\mathcal{A}} := \mathcal{A}^n$. If $\hat{\mathcal{A}}$ is quasi-multiplicative as a $\mathrm{GL}_d(\mathbb{R})$-cocycle over $\hat{\sigma}$, then $\mathcal{A}$ is quasi-multiplicative as a $\mathrm{GL}_d(\mathbb{R})$-cocycle over σ.

Proof. Let $\mathrm{I}, \mathrm{J} \in \mathcal{L}$ be any two admissible words with respect to σ. Denoting $|\mathrm{I}|$ by $a \in \mathbb{N}$ and writing $\mathrm{I} = i_0 \dots i_{a-1}$, we extend I to $\hat{\mathrm{I}} := i_0 \dots i_{\hat{a}-1} \in \mathcal{L}$ such that the length $|\hat{\mathrm{I}}|$ of $\hat{\mathrm{I}}$ is equal to $\hat{a} := n \cdot \lceil a/n \rceil$. Since $\hat{a}$ is a multiple of n, we may consider $\hat{\mathrm{I}}$ as an admissible word with respect to $\hat{\sigma}$. Similarly, we denote the length $|\mathrm{J}|$ by $b \in \mathbb{N}$, and extend $\mathrm{J} = j_0 \dots j_{b-1}$ to a $\hat{\sigma}$-admissible word $\hat{\mathrm{J}} := j_{b-\hat{b}} \dots j_0 \dots j_{b-1}$ of length $\hat{b} := n \cdot \lceil b/n \rceil$.

Then from quasi-multiplicativity of $\hat{\mathcal{A}}$, there exists a $\hat{\sigma}$-admissible word $\hat{\mathrm{K}}$ of uniformly bounded length so that

$$\|\hat{\mathcal{A}}(\hat{\mathrm{I}}\hat{\mathrm{K}}\hat{\mathrm{J}})\| \geq c \|\hat{\mathcal{A}}(\hat{\mathrm{I}})\| \cdot \|\hat{\mathcal{A}}(\hat{\mathrm{J}})\|.$$

Since $\hat{a} - a$ is bounded above by $n-1$, $\|\hat{\mathcal{A}}(\hat{\mathrm{I}})\|$ is uniformly comparable to $\|\mathcal{A}(\mathrm{I})\|$ upto a constant depending only on $\mathcal{A}$ and n. Similarly, $\|\hat{\mathcal{A}}(\hat{\mathrm{J}})\|$ is uniformly comparable to $\|\mathcal{A}(\mathrm{J})\|$. Noting that $\hat{\mathcal{A}}(\hat{\mathrm{I}}\hat{\mathrm{K}}\hat{\mathrm{J}})$ is equal to $\mathcal{A}(\mathrm{IKJ})$ with $\mathrm{K} := i_a \dots i_{\hat{a}-1} \hat{\mathrm{K}} j_{b-\hat{b}} \dots j_{-1}$, quasi-multiplicativity of $\mathcal{A}$ then follows. □

3.2 Quasi-multiplicativity of typical cocycles

Throughout the section, let $\mathcal{A} \in \mathcal{U}$ be a typical cocycle over (Σ_T, σ). From Lemma 3.7, we may assume that p is a fixed point of σ. Let $\{v_i\}_{1\le i\le d}$ be eigendirections of $P := \mathcal{A}(p)$ listed in the order of decreasing modulus of eigenvalues. Similarly, we denote the eigendirections of $P^* := \mathcal{A}_*(p)$ by $\{w_i\}_{1\le i\le d}$. We define $\mathbb{W}_j$ be the hyperplane in $\mathbb{R}^d$ spanned by all v_i's except v_j.

Remark 3.8. As in the proof of Lemma 3.6, we have $w_j = (\mathbb{W}_j)^\perp$ for each $1 \le j \le d$. It then follows that the angle formed by the top eigendirections v_1 and w_1 of P and P^* is necessarily bounded away from $\pi/2$.

Suppose $a, b, c, d \in \Sigma_T$ are related by

$$[c, a] = b \quad \text{and} \quad [a, c] = d,$$

where the bracket operation is defined in (2.1). We say such configuration of points form a *rectangle* with vertices a, b, c, and d, and denote it by $[a, b, c, d]$.

Note that a rectangle is defined by prescribing two opposite vertices. All rectangles appearing in the proof will have one of its vertices at p.

For $q \in \Sigma_T$ in the local neighborhood of p, but not on $\mathcal{W}^s_{\text{loc}}(p) \cup \mathcal{W}^u_{\text{loc}}(p)$, consider the rectangle $[p, x, q, y]$ having p and q as opposite vertices. We define *the holonomy of the rectangle* $[p, x, q, y]$ by

$$R_q := H^u_{y,p} \circ H^s_{q,y} \circ H^u_{x,q} \circ H^s_{p,x}.$$

Since canonical holonomies are Hölder continuous, the holonomy composition R_q uniformly approaches the identity as the rectangle degenerates (i.e., as a pair of opposite sides degenerates to a point) to a line or a point.

Recall that $\theta \in (0,1)$ is the hyperbolicity constant defining the metric on the base Σ_T. The following lemma is immediate from the uniform continuity of the canonical holonomies $H^{s/u}$.

Lemma 3.9. Given $\varepsilon > 0$, there exists $m = m(\varepsilon) \in \mathbb{N}$ such that

(i) If $[p,x,q,y]$ has an edge whose length is at most θ^m, then

$$\measuredangle(R_q(v), v) < \frac{\varepsilon_0}{2} \text{ for any } v \in \mathbb{P}^{d-1}.$$

(ii) If all edges of $[p,x,q,y]$ are no longer than θ^m, then

$$\measuredangle\big(H^u_{x,q} \circ H^s_{p,x}(v), v\big) < \varepsilon_0/2 \quad \text{and} \quad \measuredangle\big(H^s_{y,q} \circ H^u_{p,y}(v), v\big) < \varepsilon_0/2,$$

for any $v \in \mathbb{P}^{d-1}$.

Lemma 3.10. There exists $\varepsilon > 0$ such that for any $\varepsilon_0 > 0$, there exists $\ell_0 := \ell_0(\varepsilon_0) \in \mathbb{N}$ such that for any $\ell \geq \ell_0$ and $v \in \bigcup_{i=1}^{d} \mathcal{C}(v_i, \varepsilon)$, we have

$$P^\ell \psi_p^z(v) \in \mathcal{C}(v_1, \varepsilon_0/2).$$

Proof. The twisting condition (B0) implies that there exists $\varepsilon > 0$ such that

$$\measuredangle\big(\psi_p^z(v), \mathbb{W}_j\big) > \varepsilon,$$

for all $1 \leq j \leq d$ whenever $v \in \bigcup_{i=1}^{d} \mathcal{C}(v_i, \varepsilon)$. Then from the pinching assumption (A0) and the fact that $v_1 \in \mathbb{P}^{d-1}$ is the top eigendirection of P, there exists $\ell_0(\varepsilon_0) \in \mathbb{N}$ such that $P^\ell \psi_p^z(v) \in \mathcal{C}(v_1, \varepsilon_0/2)$ for any $\ell \geq \ell_0$. $\square$

We now begin the proof of Theorem A.

Proof of Theorem A. Let $\mathcal{A} \in \mathcal{U}$ be a typical cocycle, and p and z be the periodic and homoclinic point given by the hypothesis. We may assume that p is a fixed point and that z is on $\mathcal{W}^u_{\mathrm{loc}}(p)$.

Let

$$\beta := \measuredangle(v_1, w_1^\perp) = \measuredangle(v_1, \mathbb{W}_1).$$

Note that β is necessarily positive from Remark 3.8. Let $\varepsilon > 0$ be from Lemma 3.10 (i.e., from the twisting condition (B0)), and fix

$$\varepsilon_0 \in (0, \min(\varepsilon, \beta/8)). \tag{3.3}$$

Letting $m := m(\varepsilon_0)$ and $\ell_0(\varepsilon_0)$ be from Lemma 3.9 and 3.10, fix $\ell \in \mathbb{N}$ such that

$$\ell \geq \ell_0(\varepsilon_0) \text{ and } d(\sigma^\ell z, p) \leq \theta^m. \tag{3.4}$$

Remark 3.11. By slightly relaxing the constants β, ε_0, m, and ℓ (i.e., decrease β, ε_0 and increase m, ℓ) in the sequential order they are defined if necessary, we may assume that they work for the adjoint cocycle $\mathcal{A}_*$ as well. For the adjoint cocycle $\mathcal{A}_*$, we denote the corresponding points (on the orbit of z) playing the role of $z \in \mathcal{W}^u_{\mathrm{loc}}(p)$ and $\sigma^\ell z \in \mathcal{W}^s_{\mathrm{loc}}(p)$ by $\hat{z} \in \mathcal{W}^s_{\mathrm{loc}}(p)$ and $\sigma^{-\ell}\hat{z} \in \mathcal{W}^u_{\mathrm{loc}}(p)$, respectively. In particular, we assume that the corresponding versions of Lemma 3.4, 3.9, 3.10, as well as (3.3) and (3.4) hold for P^*, $\{w_i\}_{1 \leq i \leq d}$, $H^{s/u,*}$, and $(P^*)^\ell \psi_p^{\hat{z},*}$.

Moreover, since all data such as $H^{s/u}$ and P used in defining β, ε_0, m, and ℓ depend continuously on the cocycle $\mathcal{A} \in \mathcal{U}$, we may assume that such constants also work for all cocycles $\mathcal{B} \in \mathcal{U}$ sufficiently close to $\mathcal{A}$.

Since the adjacency matrix T of the subshift Σ_T is primitive, there exists $\bar{\tau} \in \mathbb{N}$ such that $T^{\bar{\tau}} > 0$. Such $\bar{\tau}$ is the mixing rate of the system (Σ_T, σ). Then for any given $\mathrm{I} \in \mathcal{L}$, there exists $\bar{\omega}_0 \in [\mathrm{I}] \cap \mathcal{W}^s(p)$ such that $\sigma^{|\mathrm{I}|+\bar{\tau}}\bar{\omega}_0 \in \mathcal{W}^s_{\mathrm{loc}}(p)$.

We set

$$\omega_0 := \sigma^{\tau}\bar{\omega}_0 \text{ where } \tau = \tau(\mathrm{I}) := |\mathrm{I}| + \bar{\tau} + m.$$

Since $\sigma^{|\mathrm{I}|+\bar{\tau}}\bar{\omega}_0$ is already on the local stable set $\mathcal{W}^s_{\mathrm{loc}}(p)$ of p, we have $d(\omega_0, p) \leq \theta^m$. Define

$$u_{\bar{\omega}_0} := H^s_{\omega_0,p} u(\mathcal{A}^{\tau}(\bar{\omega}_0)) \in \mathbb{P}^{d-1}.$$

Let $N \in \mathbb{N}$ be given by applying Lemma 3.4 to P and $\varepsilon_0/2$. In the same spirit as Remark 3.5 and 3.11, we assume that N also works for P^* as well as for cocycles $\mathcal{B} \in \mathcal{U}$ sufficiently close to $\mathcal{A} \in \mathcal{U}$. Lemma 3.4 applied to $u_{\bar{\omega}_0}$ gives $n := n(u_{\bar{\omega}_0}) \leq N$ such that $P^n u_{\bar{\omega}_0} \in \mathcal{C}(v_i, \varepsilon_0/2)$ for some $1 \leq i \leq d$. Using $u_{\bar{\omega}_0}$ and n, we construct a new point

$$\bar{\omega}_{\mathrm{I}} = \sigma^{-\tau-n}[\sigma^n \omega_0, z];$$

note that $\bar{\omega}_{\mathrm{I}} \in \mathcal{W}^u_{\mathrm{loc}}(\bar{\omega}_0) \cap [\mathrm{I}]$. We set

$$\omega_{\mathrm{I}} := \sigma^{\tau}\bar{\omega}_{\mathrm{I}}, \text{ and } \widetilde{\omega}_{\mathrm{I}} := \sigma^{n+\ell}\omega_{\mathrm{I}}.$$

The forward orbit of $\bar{\omega}_{\mathrm{I}} \in [\mathrm{I}]$ first comes close to p, arriving at ω_{I}, then dwells near p for n iterates, and then shadows the orbit segment from z to $\sigma^{\ell} z$, and finally lands on $\mathcal{W}^s_{\mathrm{loc}}(p)$ at the point $\widetilde{\omega}_{\mathrm{I}}$. Define

$$u_{\bar{\omega}_{\mathrm{I}}} := H^s_{\widetilde{\omega}_{\mathrm{I}},p} \mathcal{A}^{n+\ell}(\omega_{\mathrm{I}}) H^u_{\omega_0,\omega_{\mathrm{I}}} u(\mathcal{A}^{\tau}(\bar{\omega}_0)) \in \mathbb{P}^{d-1}.$$

Lemma 3.12. $u_{\bar{\omega}_{\mathrm{I}}}$ is equal to $P^{\ell}\psi^z_p R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}} P^n u_{\bar{\omega}_0}$

Proof. Consider the rectangle $[p, \sigma^n\omega_0, \sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}, z]$ with opposite vertices at p and $\sigma^n\omega_{\mathrm{I}} =$

$\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}$ and its holonomy rectangle:

$$R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}} = H^u_{z,p} H^s_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}},z} H^u_{\sigma^n\omega_0,\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}} H^s_{p,\sigma^n\omega_0}.$$

Combining this with the relation $H^s_{\widetilde{\omega}_{\mathrm{I}},\sigma^\ell z}\mathcal{A}^\ell(\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}) = \mathcal{A}^\ell(z)H^s_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}},z}$ and (2.5), we obtain

$$\begin{aligned}
H^s_{\widetilde{\omega}_{\mathrm{I}},p}\mathcal{A}^{n+\ell}(\omega_{\mathrm{I}})H^u_{\omega_0,\omega_{\mathrm{I}}} &= H^s_{\widetilde{\omega}_{\mathrm{I}},p}\mathcal{A}^\ell(\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}})\mathcal{A}^n(\omega_{\mathrm{I}})H^u_{\omega_0,\omega_{\mathrm{I}}} \\
&= H^s_{\widetilde{\omega}_{\mathrm{I}},p}H^s_{\sigma^\ell z,\widetilde{\omega}_{\mathrm{I}}}\mathcal{A}^\ell(z)H^s_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}},z}H^u_{\sigma^n\omega_0,\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}\mathcal{A}^n(\omega_0) \\
&= H^s_{\sigma^\ell z,p}\mathcal{A}^\ell(z)H^u_{p,z}R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}H^s_{\sigma^n\omega_0,p}\mathcal{A}^n(\omega_0) \\
&= P^\ell\psi^z_p R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}H^s_{\sigma^n\omega_0,p}\mathcal{A}^n(\omega_0) \\
&= P^\ell\psi^z_p R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}P^n H^s_{\omega_0,p}.
\end{aligned}$$

Then it follows

$$\begin{aligned}
u_{\bar{\omega}_{\mathrm{I}}} &= H^s_{\widetilde{\omega}_{\mathrm{I}},p}\mathcal{A}^{n+\ell}(\omega_{\mathrm{I}})H^u_{\omega_0,\omega_{\mathrm{I}}}u(\mathcal{A}^\tau(\bar{\omega}_0)) \\
&= P^\ell\psi^z_p R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}P^n H^s_{\omega_0,p}u(\mathcal{A}^\tau(\bar{\omega}_0)) \\
&= P^\ell\psi^z_p R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}P^n u_{\bar{\omega}_0},
\end{aligned}$$

as claimed. □

Lemma 3.13. $u_{\bar{\omega}_{\mathrm{I}}} \in \mathcal{C}(v_1, \varepsilon_0/2)$ and $d(\widetilde{\omega}_{\mathrm{I}}, p) \le \theta^m$.

Proof. From the choice of $n = n(u_{\bar{\omega}_0})$, $P^n u_{\bar{\omega}_0} \in \mathcal{C}(v_i, \varepsilon_0/2)$ for some $1 \le i \le d$. Since the edge between p and $\sigma^n\omega_0$ of the rectangle $[p, \sigma^n\omega_0, \sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}, z]$ has length at most θ^m, $R_{\sigma^{-\ell}\widetilde{\omega}_{\mathrm{I}}}P^n u_{\bar{\omega}_0}$ belongs to $\mathcal{C}(v_i, \varepsilon_0)$ from Lemma 3.9. Since $\varepsilon_0 \le \varepsilon$ and $\ell \ge \ell_0(\varepsilon_0)$, the first claim follows from Lemma 3.10. The second claim of the lemma also follows from the choice of ℓ that $d(\widetilde{\omega}_{\mathrm{I}}, p) = d(\sigma^\ell z, p) \le \theta^m$. □

Remark 3.14. In this remark, we summarize what we have done so far. We identify $(\omega, n) \in$

$\Sigma_T \times \mathbb{N}$ with the forward orbit segment starting at ω of length n.

From a given word $\mathrm{I} \in \mathcal{L}$, we construct an orbit segment $(\bar{\omega}_0, \tau)$ starting at $\bar{\omega}_0 \in [\mathrm{I}]$ and ending at $\omega_0 \in \mathcal{W}^s_{\mathrm{loc}}(p)$ using the mixing property of the base system (Σ_T, σ). We do not however have any control of the singular direction $u_{\bar{\omega}_0}$; it could be anywhere in $\mathbb{P}^{d-1}$. So we construct a new orbit segment $(\bar{\omega}_{\mathrm{I}}, \tau + n + \ell)$ which first shadows the orbit of $\bar{\omega}_0$ for time $\tau + n$ and then shadows the orbit of z for time ℓ. By choosing n in such a way that $P^n u_{\bar{\omega}_0}$ is close to one of the eigendirections of P, we ensure that $u_{\bar{\omega}_{\mathrm{I}}}$ is close enough to the top eigendirection v_1 of P.

We apply the same argument to the adjoint cocycle $\mathcal{A}_*$ with $\hat{z} \in \mathcal{W}^s_{\mathrm{loc}}(p)$ and $\sigma^{-\ell}\hat{z} \in \mathcal{W}^u_{\mathrm{loc}}(p)$ from Remark 3.11 playing the role of z and $\sigma^\ell z$. Similar to $\bar{\omega}_0$, we obtain $\hat{\iota}_0 \in \sigma^{|\mathrm{J}|}\mathrm{J}$ from the mixing property of (Σ_T, σ) such that

$$\iota_0 := \sigma^{-\tau(\mathrm{J})}\hat{\iota}_0 \in \mathcal{W}^u_{\mathrm{loc}}(p) \text{ where } \tau(\mathrm{J}) = |\mathrm{J}| + \bar{\tau} + m.$$

Applying Lemma 3.4 to P^* and the direction $H^{s,*}_{\iota_0,p} u(\mathcal{A}_*^{\tau(\mathrm{J})}(\hat{\iota}_0))$ gives $\hat{n} \leq N$ such that $(P^*)^{\hat{n}} H^{s,*}_{\iota_0,p} u(\mathcal{A}_*^{\tau(\mathrm{J})}(\hat{\iota}_0))$ belongs to the cone $\mathcal{C}(w_i, \varepsilon_0/2)$ for some $1 \leq i \leq d$. Define

$$\hat{\iota}_{\mathrm{J}} := \sigma^{\tau(\mathrm{J}) + \hat{n}}[\hat{z}, \sigma^{-\hat{n}}\iota_0],$$

and set

$$\iota_{\mathrm{J}} = \sigma^{-\tau(\mathrm{J})}\hat{\iota}_{\mathrm{J}} \text{ and } \widetilde{\iota}_{\mathrm{J}} := \sigma^{-\hat{n}-\ell}\iota_{\mathrm{J}}.$$

Then the analogue of Lemma 3.13 holds for $\mathcal{A}_*$ and ι_{J}:

Lemma 3.15. $d(\widetilde{\iota}_{\mathrm{J}}, p) \leq \theta^m$ and the σ^{-1} orbit of $\hat{\iota}_{\mathrm{J}}$ satisfies

$$H^{s,*}_{\widetilde{\iota}_{\mathrm{J}},p} \mathcal{A}_*^{\hat{n}+\ell}(\iota_{\mathrm{J}}) H^{u,*}_{\iota_0,\iota_{\mathrm{J}}} u(\mathcal{A}_*^{\tau(\mathrm{J})}(\hat{\iota}_0)) \in \mathcal{C}(w_1, \varepsilon_0/2).$$

Having two points $\widetilde{\omega}_{\mathrm{I}} \in \mathcal{W}^s_{\mathrm{loc}}(p)$ and $\widetilde{\iota}_{\mathrm{J}} \in \mathcal{W}^u_{\mathrm{loc}}(p)$ with the desired control on the singular directions (Lemma 3.13 and 3.15), we connect their orbits by taking their bracket

$$\chi := [\widetilde{\omega}_{\mathrm{I}}, \widetilde{\iota}_{\mathrm{J}}],$$

and set $\bar{\chi} := \sigma^{-\tau(\mathrm{I})-n-\ell}\chi \in [\mathrm{I}]$ and $\hat{\chi} := \sigma^{\tau(\mathrm{J})+\hat{n}+\ell}\chi \in \sigma^{|\mathrm{J}|}[\mathrm{J}]$. From the construction, every edge of the rectangle $[p, \widetilde{\omega}_{\mathrm{I}}, \chi, \widetilde{\iota}_{\mathrm{J}}]$ is no longer than θ^m. Then from Lemma 3.9, $H^u_{\widetilde{\omega}_{\mathrm{I}},\chi} \circ H^s_{p,\widetilde{\omega}_{\mathrm{I}}}$ is sufficiently close to the identity in that it does not move any line off itself more than $\varepsilon_0/2$ in angle. It then follows from Lemma 3.13 that

$$\begin{aligned} u_{\bar{\chi}} &:= \mathcal{A}^{n+\ell}(\sigma^{\tau(\mathrm{I})}\bar{\chi}) H^u_{\omega_0,\sigma^{\tau(\mathrm{I})}\bar{\chi}} u(\mathcal{A}^{\tau}(\bar{\omega}_0)) \\ &= H^u_{\widetilde{\omega}_{\mathrm{I}},\chi}\mathcal{A}^{n+\ell}(\omega_{\mathrm{I}}) H^u_{\omega_0,\omega_{\mathrm{I}}} u(\mathcal{A}^{\tau}(\bar{\omega}_0)) \\ &= H^u_{\widetilde{\omega}_{\mathrm{I}},\chi} H^s_{p,\widetilde{\omega}_{\mathrm{I}}} u_{\bar{\omega}_{\mathrm{I}}} \end{aligned}$$

belongs to $\mathcal{C}(v_1, \varepsilon_0)$. Similarly, $H^{u,*}_{\widetilde{\iota}_{\mathrm{J}},\chi} \circ H^{s,*}_{p,\widetilde{\iota}_{\mathrm{J}}}$ does not move any line off itself more than $\varepsilon_0/2$ in angle. From the definition (3.2) of the adjoint cocycle $\mathcal{A}_*$, we have

$$u(\mathcal{A}_*^{\tau(\mathrm{J})}(\hat{\iota}_0)) = v(\mathcal{A}^{\tau(\mathrm{J})}(\iota_0)).$$

Then it similarly follows from Lemma 3.15 that

$$v_{\hat{\chi}} := \mathcal{A}_*^{\hat{n}+\ell}(\sigma^{\hat{n}+\ell}\chi) H^{u,*}_{\iota_0,\sigma^{\hat{n}+\ell}\chi} v(\mathcal{A}^{\tau(\mathrm{J})}(\iota_0))$$

belongs to $\mathcal{C}(w_1, \varepsilon_0)$. By our choice of $\varepsilon_0 \in (0, \beta/8)$, $u_{\bar{\chi}} \in \mathcal{C}(v_1, \varepsilon_0)$ and $v_{\hat{\chi}} \in \mathcal{C}(w_1, \varepsilon_0)$ together give the following uniform gap between the singular directions:

$$\measuredangle\left(v_{\hat{\chi}}, u_{\bar{\chi}}^{\perp}\right) > \frac{3\beta}{4}. \tag{3.5}$$

Using this uniform gap, we now establish quasi-multiplicativity on $\mathcal{A}$. First, we set

$$k := 2m + 2\bar{\tau} + 2N + 2\ell.$$

Note that k is independent of I and J. Then we define the connecting word K using the orbit of χ:

$$\mathrm{K} := [\sigma^{|\mathrm{I}|}\bar{\chi}]^{w}_{\bar{k}},$$

where $\bar{k} = 2m + 2\bar{\tau} + n + \hat{n} + 2\ell$. The length of K is at most k as $n, \hat{n} \leq N$.

Lemma 3.16. There exists $c > 0$ independent of K such that

$$\|\mathcal{A}(\mathrm{IKJ})\| \geq c\|\mathcal{A}(\mathrm{I})\|\|\mathcal{A}(\mathrm{J})\|.$$

Proof. Using the uniform continuity of the canonical holonomies, we fix $C_0 > 1$ that serves as the constant for the bounded distortion property (2.10) for the norm potential $\Phi_{\mathcal{A}}$ as well as the upper bound on $\|H^{s/u}_{x,y}\|$ whenever $y \in \mathcal{W}^{s/u}_{\mathrm{loc}}(x)$. Then we apply Lemma 3.3 with

$$A = \mathcal{A}^{\tau(\mathrm{J})}(\iota_0),\ B = H^s_{\sigma^{\hat{n}+\ell}\chi,\iota_0}\mathcal{A}^{\hat{n}+\ell}(\chi),\ C = \mathcal{A}^{n+\ell}(\sigma^{\tau(\mathrm{I})}\bar{\chi})H^u_{\omega_0,\sigma^{\tau(\mathrm{I})}\bar{\chi}},\text{ and } D = \mathcal{A}^{\tau(\mathrm{I})}(\bar{\omega}_0).$$

Recalling that $H^{s/u,*}_{x,y} = (H^{u/s}_{y,x})^*$, the uniform gap (3.5) implies that such choice of A, B, C

and D satisfies the assumption of Lemma 3.3 with $\theta = 3\beta/4$:

$$
\begin{aligned}
\|\mathcal{A}(\mathrm{IKJ})\| &\geq \|\mathcal{A}^{\bar{k}+|\mathrm{I}|+|\mathrm{J}|}(\bar{\chi})\| \\
&= \|\mathcal{A}^{\tau(\mathrm{J})}(\sigma^{\hat{n}+\ell}\chi)\mathcal{A}^{\hat{n}+\ell}(\chi)\mathcal{A}^{n+\ell}(\sigma^{\tau(\mathrm{I})}\bar{\chi})\mathcal{A}^{\tau(\mathrm{I})}(\bar{\chi})\| \\
&\geq C_0^{-2}\|H^s_{\hat{\chi},\hat{\iota}_0}\mathcal{A}^{\tau(\mathrm{J})}(\sigma^{\hat{n}+\ell}\chi)\mathcal{A}^{\hat{n}+\ell}(\chi)\mathcal{A}^{n+\ell}(\sigma^{\tau(\mathrm{I})}\bar{\chi})\mathcal{A}^{\tau(\mathrm{I})}(\bar{\chi})H^u_{\bar{\omega}_0,\bar{\chi}}\| \\
&= C_0^{-2}\|\mathcal{A}^{\tau(\mathrm{J})}(\iota_0)H^s_{\sigma^{\hat{n}+\ell}\chi,\iota_0}\mathcal{A}^{\hat{n}+\ell}(\chi)\mathcal{A}^{n+\ell}(\sigma^{\tau(\mathrm{I})}\bar{\chi})H^u_{\omega_0,\sigma^{\tau(\mathrm{I})}\bar{\chi}}\mathcal{A}^{\tau(\mathrm{I})}(\bar{\omega}_0)\| \\
&= C_0^{-2}\|ABCD\| \\
&\geq C_0^{-2}\sin(3\beta/4)\|A\|\|D\|\frac{m(B)^2m(C)}{\|B\|}.
\end{aligned}
$$

Setting $\Upsilon := \max\left(\max_{x\in\Sigma_T}\|\mathcal{A}(x)\|, 1\right)$ and $\varrho := \min\left(\min_{x\in\Sigma_T} m(\mathcal{A}(x)), 1\right)$, we have

$$
C_0\Upsilon^{N+\ell} \geq \|B\|, \text{ and } m(B), m(C) \geq C_0^{-1}\varrho^{N+\ell}.
$$

It then follows that

$$
\begin{aligned}
\|\mathcal{A}(\mathrm{IKJ})\| &\geq C_0^{-6}\sin(3\beta/4)\frac{\varrho^{3(N+\ell)}}{\Upsilon^{(N+\ell)}}\|\mathcal{A}^{\tau(\mathrm{J})}(\iota_0)\|\cdot\|\mathcal{A}^{\tau(\mathrm{I})}(\bar{\omega}_0)\| \\
&\geq C_0^{-6}\sin(3\beta/4)\frac{\varrho^{3(N+\ell)+2(\bar{\tau}+m)}}{\Upsilon^{(N+\ell)}}\|\mathcal{A}^{|\mathrm{J}|}(\sigma^{-|\mathrm{J}|}\hat{\iota}_0)\|\cdot\|\mathcal{A}^{|\mathrm{I}|}(\bar{\omega}_0)\| \\
&\geq c\|\mathcal{A}(\mathrm{I})\|\|\mathcal{A}(\mathrm{J})\|,
\end{aligned}
$$

where $c := C_0^{-8}\sin(3\beta/4)\dfrac{\varrho^{3(N+\ell)+2(\bar{\tau}+m)}}{\Upsilon^{(N+\ell)}}$. □

By slightly relaxing them if necessary, the constants c and k uniformly work for cocycles in a small neighborhood of $\mathcal{A}$ due to Remark 3.11. This completes the proof of Theorem A. □

Remark 3.17. Unlike constants c and k, it is clear from the above proof of Theorem A that the connecting word $\mathrm{K} = \mathrm{K}(\mathrm{I}, \mathrm{J}) \in \mathcal{L}$ cannot be chosen uniformly in a small neighborhood of

$\mathcal{A}$. This is because although $\mathcal{B}$ may be arbitrarily close to $\mathcal{A}$, the singular direction $u(\mathcal{B}^\tau(\bar{\omega}_0))$ could be drastically different from $u(\mathcal{A}^\tau(\bar{\omega}_0))$ if the length of I (and, hence, $\tau = m + \bar{\tau} + |\mathrm{I}|$) is arbitrarily large. Then the number of iterates n of P needed to turn $H^s_{\omega_0,p}u(\mathcal{A}^\tau(\bar{\omega}_0))$ close to one of the eigendirections of P would be different from that of $H^s_{\omega_0,p}u(\mathcal{B}^\tau(\bar{\omega}_0))$. Hence we cannot expect K to be chosen uniformly near $\mathcal{A}$.

3.3 Continuity of subadditive pressure

In this section, we prove Theorem B based on the proof of Fekete's lemma. For any $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$, we obtain a subadditive sequence $\{\log \alpha_n(\mathcal{A})\}_{n\in\mathbb{N}}$ where

$$\alpha_n(\mathcal{A}) := \sum_{|\mathrm{I}|=n} \varphi_{\mathcal{A}}(\mathrm{I}) = \sum_{|\mathrm{I}|=n} \max_{x\in[\mathrm{I}]} \varphi(\mathcal{A}^n(x)).$$

Recall from Remark 2.21 that we have

$$\mathsf{P}(\Phi_{\mathcal{A}}) = \lim_{n\to\infty} \frac{1}{n} \log \alpha_n(\mathcal{A}).$$

We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ is *almost superadditive with constant* $C > 0$ if for all $n, m \in \mathbb{N}$, we have

$$a_{n+m} \geq a_n + a_m - C.$$

In the following lemma, we use quasi-multiplicativity from Theorem A to show that given any $\mathcal{A} \in \mathcal{U}$, the sequence $\{\log \alpha_n(\mathcal{B})\}$ is almost superadditive with the uniform constant $C > 0$ for all $\mathcal{B}$ sufficiently close to $\mathcal{A}$.

Lemma 3.18. Let $\mathcal{A} \in \mathcal{U}$ be typical. Then there exists $C > 0$ such that the following holds: there exists a small neighborhood of $\mathcal{A}$ in $\mathcal{U}$ such that for all $\mathcal{B}$ in the neighborhood, the sequence $\{\log \alpha_n(\mathcal{B})\}_{n\in\mathbb{N}}$ is almost superadditive with constant C.

Proof. There exists $C_1 > 0$ such that $\alpha_{n+1}(\mathcal{A}) \leq C_1 \alpha_n(\mathcal{A})$ for any $n \in \mathbb{N}$. In fact, we can set $C_1 = \Upsilon \cdot q$ where $\Upsilon := \max_{x \in \Sigma_T} \|\mathcal{A}(x)\|$ and q is the number of alphabets in Σ_T. We may increase C_1 slightly to ensure that this property also holds for all $\mathcal{B}$ in a small neighborhood of $\mathcal{A}$. After shrinking the neighborhood if necessary, we have

$$c\alpha_n(\mathcal{B})\alpha_m(\mathcal{B}) \leq \sum_{i=0}^{k} \alpha_{n+m+i}(\mathcal{B}) \leq \Big(\sum_{i=0}^{k} C_1^i\Big)\alpha_{m+n}(\mathcal{B})$$

where c and k are the uniform constants from Theorem A. The lemma follows by setting $C = \log\Big(c^{-1} \cdot \sum_{i=0}^{k} C_1^i\Big)$. □

We are now ready to prove Theorem B.

Proof of Theorem B (1). Let $\mathcal{A} \in \mathcal{U}$, and $\varepsilon > 0$ be given. We will show that there exists a small neighborhood of $\mathcal{A}$ such that any $\mathcal{B}$ in that neighborhood satisfies $\Big|\mathsf{P}(\Phi_{\mathcal{A}}) - \mathsf{P}(\Phi_{\mathcal{B}})\Big| < \varepsilon$.

For any $m, n \in \mathbb{N}$, we write $n = tm + r$ with $0 \leq r < m$. For all $\mathcal{B}$ in a small neighborhood of $\mathcal{A}$, Lemma 3.18 gives

$$-C\frac{(t+1)}{n} + \frac{t}{n}\log\alpha_m(\mathcal{B}) + \frac{1}{n}\log\alpha_r(\mathcal{B}) \leq \frac{1}{n}\log\alpha_n(\mathcal{B}) \leq \frac{t}{n}\log\alpha_m(\mathcal{B}) + \frac{1}{n}\log\alpha_r(\mathcal{B}).$$

Notice that as $n \to \infty$, we have $t/n \to 1/m$ and $\frac{1}{n}\log\alpha_r(\mathcal{B}) \to 0$ because there are only m possible values of $\alpha_r(\mathcal{B})$. Sending $n \to \infty$,

$$\Big|\mathsf{P}(\Phi_{\mathcal{B}}) - \frac{1}{m}\log\alpha_m(\mathcal{B})\Big| \leq \frac{C}{m}.$$

We choose $m \in \mathbb{N}$ large so that $C/m < \varepsilon/2$. Then we shrink the neighborhood of $\mathcal{A}$ if necessary such that for any $\mathcal{B}$ in the neighborhood,

$$\Big|\frac{1}{m}\log\alpha_m(\mathcal{A}) - \frac{1}{m}\log\alpha_m(\mathcal{B})\Big| < \varepsilon/2.$$

Then for all $\mathcal{B}$ in such neighborhood of $\mathcal{A}$, the claim follows. □

Proof of Theorem B (2). Suppose $\mathcal{A}_n \in \mathcal{U}$ converges to $\mathcal{A} \in \mathcal{U}$. By passing to a subsequence if necessary, let ν be any weak-$*$ limit of $\mu_{\mathcal{A}_n}$. We recall that two maps $\mu \mapsto h_\mu(\sigma)$ and $(\Phi, \mu) \mapsto \mathcal{F}(\Phi, \mu)$ are upper semi-continuous; the entropy map is upper semi-continuous from the expansivity of the base dynamical system (Σ_T, σ), and $\mathcal{F}$ is upper semi-continuous from being an infimum of continuous functions. From Theorem B (1), ν must be an equilibrium state of $\Phi_{\mathcal{A}}$:

$$\begin{aligned}\mathsf{P}(\Phi_{\mathcal{A}}) = \lim_{n\to\infty} \mathsf{P}(\Phi_{\mathcal{A}_n}) &= \lim_{n\to\infty} h_{\mu_{\mathcal{A}_n}}(f) + \mathcal{F}(\Phi_{\mathcal{A}_n}, \mu_{\mathcal{A}_n}),\\ &\leq h_\nu(\sigma) + \mathcal{F}(\Phi_{\mathcal{A}}, \nu).\end{aligned}$$

Since $\mathcal{A} \in \mathcal{U}$, the equilibrium state $\mu_{\mathcal{A}}$ of $\Phi_{\mathcal{A}}$ is unique. Hence $\nu = \mu_{\mathcal{A}}$, as desired. □

We note that Cao, Pesin, and Zhao [18] recently showed the continuity of the subadditive pressure in a more general setting. In particular, their results imply that the map $\mathcal{A} \mapsto \mathsf{P}(\Phi_{\mathcal{A}})$ is continuous on $C^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$, and Theorem B (1) is implied by their result. However, the methods of proof are different. Cao, Pesin, and Zhao construct a horseshoe with dominated splitting which carries most of the pressure. Using the structural stability of horseshoes, they establish the lower semi-continuity of the pressure. See [18] for details. On the other hand, we compare $\mathsf{P}(\Phi_{\mathcal{A}})$ to $\mathsf{P}(\Phi_{\mathcal{B}})$ using the uniform constants of quasi-multiplicativity from Theorem A.

For similar results in this direction, Feng and Shmerkin [29] showed that locally constant cocycles are continuity points of $\mathsf{P}(\Phi_{\mathcal{A}})$ in $L^\infty(\Sigma_T, M_{d\times d}(\mathbb{R}))$.

3.4 Further applications

In this section, we briefly remark on further applications of Theorem A. For coherence of this thesis, we will only comment on them without providing the proof. For more detailed discussion, we refer the readers to [41].

First, Theorem A holds also for the singular value potentials $\Phi^s_{\mathcal{A}}$. What is more, given any $\mathrm{I}, \mathrm{J} \in \mathcal{L}$, the connecting word $\mathrm{K} = \mathrm{K}(\mathrm{I}, \mathrm{J})$ for the quasi-multiplicativity of $\Phi^s_{\mathcal{A}}$ can be chosen to work simultaneously for all $s \geq 0$. The proof is almost identical to that of Theorem A once we replace Lemma 3.4 to a suitable simultaneous version of the lemma.

The continuity of the pressure also extends to the singular value potentials of typical cocycles. That is, any $(\mathcal{A}, s) \in \mathcal{U} \times \mathbb{R}^+_0$ is a continuity point of the map $(\mathcal{A}, s) \mapsto \mathsf{P}(\Phi^s_{\mathcal{A}})$. This has application to the dimension theory of the fractals. Given a $C^{1+\alpha}$ diffeomorphism f on a closed manifold M, suppose there exists a repeller $\Lambda \subset M$ such that the derivative cocycle Df restricted to Λ is fiber-bunched (in a suitable sense; see [41] for the precise formulation). For each $s \geq 0$ we can associate a subadditive potential Φ^s_{Λ}, and we denote by $s(\Lambda)$ the unique zero of the function $s \mapsto \mathsf{P}(\Phi^s_{\Lambda})$ which serves as an upper bound for the Hausdorff dimension of the repeller Λ. The continuity of $\mathsf{P}(\Phi^s_{\mathcal{A}})$ from Theorem B can be translated to this setting in order to deduce the continuity of the map $g \mapsto s(\Lambda_g)$ on an open and dense subset near f.

Lastly, quasi-multiplicativity reveals information on the shape of the pointwise Lyapunov spectrum. The *pointwise Lyapunov spectrum* $L_{\mathcal{A}}$ of a cocycle $\mathcal{A}$ is the set of all real numbers that can be realized as the Lyapunov exponent for some $x \in \Sigma_T$:

$$L_{\mathcal{A}} := \{\gamma \in \mathbb{R} \mid \gamma = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| \text{ for some } x \in \Sigma_T\}.$$

Feng [25, 26] showed that quasi-multiplicativity of the norm potential implies that the pointwise Lyapunov spectrum is closed and bounded (i.e., a closed interval). After showing that the extremal (i.e., minimum and maximum) Lyapunov exponents α and β are attained by some points x and y respectively, he used quasi-multiplicativity to concatenate the forward orbits of x and y alternatively so that the Lyapunov exponent of the resulting point can be made precisely equal to any given $\gamma \in [\alpha, \beta]$. Using his argument, we can deduce similar results on the pointwise Lyapunov spectrum of typical cocycles. Moreover, there is a natural generalization of such a spectrum by considering all (i.e., not just the top) d-Lyapunov exponents simultaneously. Under typicality assumption, we can show that such an object is closed and bounded as a subset of $\mathbb{R}^d$.

CHAPTER 4

THERMODYNAMIC FORMALISM OF FIBER-BUNCHED $\mathrm{GL}_2(\mathbb{R})$-COCYCLES

This chapter is joint work with Clark Butler and originally appeared in [14][2].

4.1 Reducible fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles

We will first consider the norm potentials of reducible (as in Definition 2.6) fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles. More precisely, we will show that the norm potential $\Phi_{\mathcal{A}}$ of a reducible cocycle $\mathcal{A} \in C_b^{\alpha}(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ has a unique equilibrium state unless the conjugated cocycle $\mathcal{B}$ as in (1.5) satisfies two conditions from Theorem D, in which case there are two ergodic equilibrium states for $\Phi_{\mathcal{A}}$.

For reducible cocycles, we treat them by modifying the results of Feng and Käenmäki [28]. For locally constant cocycles, they showed that after simultaneously conjugating the cocycle into upper block triangle matrices of the same indices such that the tuples of diagonal blocks are irreducible, the number of ergodic equilibrium states for the norm potentials cannot exceed the number of the diagonal blocks. Since the norm potentials of fiber-bunched cocycles have bounded distortion property (2.10), we may modify and apply the result of [28].

Recall that $\lambda_+(\mathcal{A}, \mu)$ denotes the top Lyapunov exponent of $\mathcal{A}$ with respect to μ. When μ is ergodic, then

$$\lambda_+(\mathcal{A}, \mu) = \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\|$$

for μ-a.e. x. The following proposition states that given a $\mathrm{GL}_2(\mathbb{R})$-cocycle taking values in upper triangular matrices, both the top exponent and the subadditive pressure come from the diagonal entries.

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Proposition 4.1. Suppose $\mathcal{B} \in C(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ is of the form (1.5):

$$\mathcal{B}(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & c(x) \end{pmatrix}.$$

Then for any ergodic probability measure μ,

1. the top Lyapunov exponent $\lambda_+(\mathcal{B}, \mu)$ satisfies

$$\lambda_+(\mathcal{B}, \mu) = \max \left\{ \int \log|a| \; d\mu, \int \log|c| \; d\mu \right\}.$$

2. $\mathsf{P}(\Phi_{\mathcal{B}}) = \max \Big\{ \mathsf{P}(\log|a|), \mathsf{P}(\log|c|) \Big\}$.

In order to prove Proposition 4.1, we need a lemma from ergodic theory.

Lemma 4.2. Let $(X, \mathcal{B}, \mu)$ be a probability space, $f \colon X \to X$ an ergodic measure-preserving transformation, and $\varphi \colon X \to \mathbb{R}$ a μ-integrable function with $\sup_{x \in X} |\varphi(x)| < \infty$. Denoting $\alpha := \int \varphi \; d\mu$, for any $\varepsilon > 0$ and μ-almost every $x \in X$, there exists $n_1 = n_1(x) \in \mathbb{N}$ such that

$$|S_n \varphi(f^m x) - n\alpha| \le (n+m)\varepsilon$$

for any $n \ge n_1$ and any $m \in \mathbb{N}$.

Proof. Let $X_0 \subset X$ be a full measure subset from Birkhoff Ergodic Theorem such that the Birkhoff average $\frac{1}{n} S_n \varphi(x)$ converges to α for any $x \in X_0$. For each $x \in X_0$, choose $n_0 = n_0(x) \in \mathbb{N}$ such that

$$\left| \frac{1}{n} S_n \varphi(x) - \alpha \right| < \varepsilon/2$$

for each $n \ge n_0$. Denoting $a_n := S_n \varphi(x) - n\alpha$ for each $n \in \mathbb{N}$, define $n_1 = n_1(x) \ge n_0$ such that

$$n_1 \ge 2/\varepsilon \cdot \left(\max_{1 \le i \le n_0 - 1} |a_i| \right).$$

Consider any $n \geq n_1$ and $m \in \mathbb{N}$. If $m \geq n_0$, then

$$\begin{aligned} |S_n\varphi(f^m x) - n\alpha| &= |\,(S_{n+m}\varphi(x) - (n+m)\alpha) - (S_m\varphi(x) - m\alpha)\,| \\ &\leq (n+2m)\varepsilon/2 \\ &\leq (n+m)\varepsilon. \end{aligned}$$

If $m \leq n_0 - 1$, then

$$\begin{aligned} |S_n\varphi(f^m x) - n\alpha| &= |\,(S_{n+m}\varphi(x) - (n+m)\alpha) - (S_m\varphi(x) - m\alpha)\,| \\ &\leq (n+m)\varepsilon/2 + |a_m| \\ &\leq (n+m)\varepsilon \end{aligned}$$

where the last inequality follows because $n \cdot \varepsilon/2 \geq n_1 \cdot \varepsilon/2 \geq |a_m|$. $\square$

Corollary 4.3. Under the same assumptions of Lemma 4.2, let $C_0 := \sup_{x \in X} |\varphi(x)| < \infty$. Then for any $\varepsilon > 0$ and for μ-almost every $x \in X$, there exists $C(x) > 0$ such that

$$|S_n\varphi(f^m x) - n\alpha| < C(x) + (n+m)\varepsilon$$

for all $n, m \in \mathbb{N}$.

Proof. In view of Lemma 4.2, it suffices to set $C(x) = (C_0 + |\alpha|)(n_1(x) - 1)$ for each $x \in X_0$. $\square$

Proof of Proposition 4.1. By considering $a(x)$ and $c(x)$ as multiplicative cocycles over Σ_T, let $\tau^n(x) := \prod_{i=0}^{n-1} \tau(f^i x)$ for $\tau = \{a, c\}$. Then for any $n \in \mathbb{N}$, we have

$$\mathcal{B}^n(x) = \begin{pmatrix} a^n(x) & \sum_{i=0}^{n-1} a^{n-i-1}(\sigma^{i+1}x)b(\sigma^i x)c^i(x) \\ 0 & c^n(x) \end{pmatrix}.$$

Denoting the (i,j)-entry of a matrix A by $A_{i,j}$, we have

$$\max\Big\{|\mathcal{B}^n(x)_{1,1}|, |\mathcal{B}^n(x)_{2,2}|\Big\} \le \|\mathcal{B}^n(x)\| \le 2^2 \max_{1\le i,j\le 2} |\mathcal{B}^n(x)_{i,j}|. \tag{4.1}$$

Here $\mathcal{B}^n(x)_{1,1} = a^n(x)$ and $\mathcal{B}^n(x)_{2,2} = c^n(x)$.

For any $\varepsilon > 0$, Corollary 4.3 applied to each $\varphi(x) = \log|a(x)|$ and $\varphi(x) = \log|c(x)|$ gives $C(x) > 0$ for μ-almost every $x \in \Sigma_T$ such that

$$|a^{n-i-1}(\sigma^{i+1}x)| \le \exp\Big(C(x) + (n-i-1)\int \log|a|\ d\mu + n\varepsilon\Big)$$

and

$$|c^i(x)| \le \exp\Big(C(x) + i\int \log|c|\ d\mu + i\varepsilon\Big).$$

Denoting $L := \max_{x\in\Sigma_T} |b(x)|$, we have

$$\begin{aligned}
|\mathcal{B}^n(x)_{1,2}| &= \Big|\sum_{i=0}^{n-1} a^{n-i-1}(f^{i+1}x)b(f^i x)c^i(x)\Big|,\\
&\le \sum_{i=0}^{n-1} L\exp\Big(2C(x) + (n-i-1)\int \log|a|\ d\mu + i\int \log|c|\ d\mu + (n+i)\varepsilon\Big),\\
&\le nL\exp\Big(2C(x) + n\max\Big\{\int \log|a|\ d\mu, \int \log|c|\ d\mu\Big\} + 2n\varepsilon\Big).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows from (4.1) that for μ-a.e. $x \in \Sigma_T$, we have

$$\lambda_+(\mathcal{B},\mu) = \lim_{n\to\infty} \frac{1}{n}\log\|\mathcal{B}^n(x)\| = \max\Big\{\int \log|a|\ d\mu, \int \log|c|\ d\mu\Big\},$$

establishing the first statement of the proposition.

From the first statement and the subadditive variational principle (2.9), the second statement also follows. Indeed, let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of ergodic measures in such that

$h_{\mu_n}(\sigma) + \lambda_+(\mathcal{B}, \mu_n)$ limits to $\mathsf{P}(\Phi_{\mathcal{B}})$. By comparing $\int \log|a| \; d\mu_n$ to $\int \log|c| \; d\mu_n$ for each $n \in \mathbb{N}$, without loss of generality, we may assume that there exists $n_k \to \infty$ such that $\int \log|a| \; d\mu_{n_k} \geq \int \log|c| \; d\mu_{n_k}$ for each $k \in \mathbb{N}$. Then from the first statement and the variational priniciple (2.9), we have

$$h_{\mu_{n_k}}(\sigma) + \lambda_+(\mathcal{B}, \mu_{n_k}) = h_{\mu_{n_k}}(\sigma) + \int \log|a| \; d\mu_{n_k} \leq \mathsf{P}(\log|a|).$$

From the choice of μ_n, the left hand side limits to $\mathsf{P}(\Phi_{\mathcal{B}})$ as $k \to \infty$ and this proves $\mathsf{P}(\Phi_{\mathcal{B}}) \leq \max\Big\{\mathsf{P}(\log|a|), \mathsf{P}(\log|c|)\Big\}$. Conversely, applying similar arguments to $\log|a|$ (i.e., by choosing a sequence $\mu_n \in \mathcal{E}(\sigma)$ such that $h_{\mu_n}(\sigma) + \int \log|a| \; d\mu_n$ limits to $\mathsf{P}(\log|a|)$ and making use of the first statement and the subadditive variational principle (2.9)) and $\log|c|$ establishes the reverse inequality. □

If we further suppose that $\mathcal{B}$ from Proposition 4.1 is Hölder continuous, then each $\log|a|$ and $\log|c|$ is a Hölder potential over a mixing hyperbolic system (Σ_T, σ) and has a unique equilibrium state from Proposition 2.18. Moreover, $\mu_{\log|a|}$ is equal to $\mu_{\log|c|}$ if and only if $\log|a|$ and $\log|c|$ are cohomologous. Hence, the following corollary is a consequence of Proposition 4.1. Also, it is clear that this corollary implies Theorem D.

Corollary 4.4. Suppose $\mathcal{B} \in C^\beta(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ is of the form (1.5). Then the following holds:

1. If $\mathsf{P}(\log|a|) \neq \mathsf{P}(\log|c|)$, then $\log|a| \not\sim \log|c|$ and $\Phi_{\mathcal{A}}$ has a unique equilibrium state.

2. If $\log|a| \sim \log|c|$, then $\mathsf{P}(\log|a|) = \mathsf{P}(\log|c|)$ and $\Phi_{\mathcal{A}}$ has a unique equilibrium state $\mu_{\log|a|} = \mu_{\log|c|}$.

3. If $\log|a| \not\sim \log|c|$ and $\mathsf{P}(\log|a|) = \mathsf{P}(\log|c|)$, then $\Phi_{\mathcal{A}}$ has exactly two distinct ergodic equilibrium states $\mu_{\log|a|}$ and $\mu_{\log|c|}$.

Remark 4.5. The third alternative from Corollary 4.4 is not a vacuous option in that there are cocycles $\mathcal{B}$ satisfying such conditions. For instance, take any two positive Hölder continuous functions $\log|a|,\ \log|c| \in C^{\beta}(\Sigma_T, \mathbb{R}^+)$ such that there exist two periodic points $p, q \in \Sigma_T$ of some periods $n, m \in \mathbb{N}$ such that the Birkhoff sum $(S_n \log|a|)(p)$ equals $(S_n \log|c|)(p)$ while $(S_m \log|a|)(q)$ differs from $(S_m \log|c|)(q)$. The assumption on the Birkhoff sums along the orbit of q ensures that $\log|a|$ is not cohomologous to $\log|c|$.

If $\mathsf{P}(\log|a|) = \mathsf{P}(\log|c|)$, then by setting $b \equiv 0$, the cocycle $\mathcal{B}$ satisfies the conditions from the third alternative of Corollary 4.4. If not, then suppose $\mathsf{P}(\log|a|) > \mathsf{P}(\log|c|)$ without loss of generality. Since $\log|c|$ is a positive function, from the variational principle (2.9), $\mathsf{P}(s\log|c|)$ limits to ∞ as $s \to \infty$. So there exists $s_0 > 1$ such that $\mathsf{P}(\log|a|) = \mathsf{P}(s_0 \log|c|)$, and the assumption on the Birkoff sums along the orbit of p ensures that $\log|a|$ is not cohomologous to $s_0 \log|c|$. Then setting $b \equiv 0$ again and replacing the function $\log|c|$ by $s_0 \log|c|$, the cocycle $\mathcal{B}$ satisfies the conditions from the third alternative from Corollary 4.4.

We may also choose such functions so that $\mathcal{B}$ is fiber-bunched as well. Indeed, start with any constant function $\log|c| \equiv k$ with $k \in \mathbb{R}^+$ sufficiently large compared to the entropy $h_{\text{top}}(\sigma)$ of (Σ_T, σ), and let $\log|a|$ be a small perturbation of $\log|c|$ obtained by slightly increasing the function in a neighborhood of some periodic orbit. If the perturbation is small enough, then s_0 is sufficiently close to 1, and the resulting cocycle $\mathcal{B}$ will be fiber-bunched.

4.2 Trichotomy for fiber-bunched irreducible $\mathrm{GL}_2(\mathbb{R})$-cocycles

In the rest of the chapter, we will prove Theorem C. To do so, we introduce a class of weakly typical cocycles in $C^{\alpha}_b(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$.

Definition 4.6. We say $\mathcal{A} \in C^{\alpha}_b(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ is *weakly typical* if

1. (pinching) There exists a periodic point $p \in \Sigma_T$ such that $\mathcal{A}^{\mathrm{per}(p)}(p)$ has simple eigenvalues of distinct norms with corresponding eigendirections $v_+, v_- \in \mathbb{P}$;

2. (twisting) There exist homoclinic points $z_+, z_- \in \mathcal{W}^s(p) \cap \mathcal{W}^u(p) \setminus \{p\}$ such that for each $\tau \in \{+, -\}$, the holonomy loop $\psi_p^{z_\tau} := H^s_{z_\tau, p} \circ H^u_{p, z_\tau}$ twists v_τ:

$$\psi_p^{z_\tau}(v_\tau) \neq v_\tau.$$

We denote the set of weakly typical cocycles by

$$\mathcal{U}_w := \{\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_2(\mathbb{R})) \colon \mathcal{A} \text{ is weakly typical}\}.$$

As in the case for typical cocycles, the set of weakly typical cocycles $\mathcal{U}_w$ is open and dense in $C^\alpha_b(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$, and has infinite codimension. In the following remark, we point out the differences between the typicality assumption from Definition 2.11 and the weak typicality assumption defined above.

Remark 4.7. The main difference between typicality and weak typicality lies in the formulation of the twisting assumption. The twisting assumption for typical cocycles (Definition 2.11) requires that there exists a *single* homoclinic point z whose holonomy loop ψ_p^z twists all eigendirections.

On the other hand, the weak typicality assumption (Definition 4.6) allows each $v_+, v_- \in \mathbb{P}$ to be twisted under holonomy loops of different homoclinic points z_+, z_-, respectively. In particular, since we only require that $\psi_p^{z_\tau}(v_\tau) \neq v_\tau$ for each $\tau \in \{+, -\}$, it could happen that $\psi_p^{z_+}(v_+) = v_-$ and $\psi_p^{z_-}(v_-) = v_+$. Moreover, as we will see, the weak typicality is flexible enough to establish the trichotomy in Theorem 4.8 below, and yet has enough structures to guarantee the uniqueness of the equilibrium state.

Next theorem establishes a trichotomy among irreducible cocycles in $C^\alpha_b(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ with weak typicality being one of the alternatives.

Theorem 4.8. Suppose $\mathcal{A} \in C^\alpha_b(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ is irreducible. Then either

1. $\mathcal{A}$ is weakly typical (i.e., $\mathcal{A} \in \mathcal{U}_w$), or

2. there exist two bi-holonomy invariant line bundles interchanged by $\mathcal{A}$, or

3. there is a Hölder conjugacy of $\mathcal{A}$ into the group of linear conformal transformations of $\mathbb{R}^2$.

The content of Theorem 4.8 is similar to the fact that a subset of $\mathrm{GL}_2(\mathbb{R})$ which does not preserve a common line either generates a Zariski dense subgroup, preserves a union of two lines, or belongs to a subgroup of the form $O(2) \times \mathbb{R}^*$ in some inner product.

As (Σ_T, σ) is a mixing hyperbolic system, for any periodic point $p \in \Sigma_T$ the set of homoclinic points of p is dense in Σ_T.

Lemma 4.9. Let $\mathcal{A} \in C_b^\alpha(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ be an irreducible cocycle. For any fixed point $p \in \Sigma_T$ and any line $L \in \mathbb{P}$, either

1. $\mathcal{A}(p)(L) \neq L$, or

2. there exists a homoclinic point z of p such that $\psi_p^z(L) \neq L$.

Proof. Suppose the conclusion of the lemma does not hold. Then there exists an $\mathcal{A}(p)$-invariant line $L \in \mathbb{R}^2$ that is preserved under ψ_p^z for all homoclinic points z of p. For each homoclinic point z, we define

$$L_z := H^s_{p,z}(L) = H^u_{p,z}(L).$$

The second equality holds because L is invariant under ψ_p^z.

We will show that such extension of L to the set of homoclinic points of p is Hölder continuous. Suppose x, y are homoclinic points of p with $d(x, y)$ small. Setting $z := [y, x]$, z is also a homoclinic point of p. Then $H^s_{x,z}$ maps L_x to L_z:

$$L_z = H^s_{p,z}(L) = H^s_{x,z} H^s_{p,x}(L) = H^s_{x,z}(L_x).$$

Similarly, $L_z = H^u_{y,z}(L_y)$. Hence,

$$L_y = H^u_{z,y} H^s_{x,z}(L_x).$$

Since $H^{s/u}_{x,y}$ varies α-Hölder continuously in x and y from (2.3), there exists $C > 0$ depending only on $\mathcal{A}$ such that

$$\measuredangle(L_x, L_y) \leq C d(x, y)^\alpha.$$

Since the set of homoclinic points of p is dense in Σ_T, it follows that L can be uniquely extended to an $\mathcal{A}$-invariant and $H^{s/u}$-invariant line bundle over Σ_T, contradicting the irreducibility assumption on $\mathcal{A}$. □

The following corollary is an immediate consequence of Lemma 4.9.

Corollary 4.10. Let $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ be a fiber-bunched irreducible cocycle, $p \in \Sigma_T$ a fixed point, and $L \in \mathbb{P}$ an eigendirection of $\mathcal{A}(p)$. Then there exists a homoclininc point z of p such that $\psi^z_p(L) \neq L$.

The following proposition from Kalinin and Sadovskaya [33] produces an $\mathcal{A}$-invariant conformal (not necessarily non-trivial) sub-bundle when the extremal Lyapunov exponents of $\mathcal{A}$ coincide for all periodic points.

Proposition 4.11. [33, Proposition 2.1, 2.7] Let f be a transitive C^2 Anosov diffeomorphism on a compact manifold M, $\mathcal{E}$ a finite-dimensional vector bundle over M, and $\mathcal{A}\colon \mathcal{E} \to \mathcal{E}$ an α-Hölder linear cocycle. Suppose for every periodic point $p \in M$, the invariant measure μ_p supported on the orbit of p satisfies

$$\lambda_+(\mathcal{A}, p) = \lambda_-(\mathcal{A}, p). \tag{4.2}$$

Then either $\mathcal{A}$ preserves an α-Hölder continuous conformal structure on $\mathcal{E}$ or $\mathcal{A}$ preserves an α-Hölder continuous proper non-trivial sub-bundle $\mathcal{E}' \subset \mathcal{E}$ and an α-Hölder continuous conformal structure on $\mathcal{E}'$.

Although it is not formulated in the statement of Proposition 4.11, the assumption (4.2) has other consequences as well. First of all, it implies that the canonical holonomies $H^{s/u}$ for $\mathcal{A}$ converge and are as regular as the cocycle $\mathcal{A}$ (see the proof of Corollary 3.6 in [32]). Moreover, the sub-bundle $\mathcal{E}'$ from Proposition 4.11 is $H^{s/u}$-invariant.

For fiber-bunched cocycles, the following proposition from Bochi and Garibaldi [6] shows that the converse also holds:

Proposition 4.12. [6, Corollary 3.5] Let $\mathcal{A}$ be an α-Hölder fiber-bunched cocycle of a vector bundle $\mathcal{E}$ over a hyperbolic homeomorphism. An $\mathcal{A}$-invariant sub-bundle $\mathcal{F} \subset \mathcal{E}$ is α-Hölder if and only if it is $H^{s/u}$-invariant.

Remark 4.13. While [6, Corollary 3.5] is stated for fiber-bunched cocycles over general hyperbolic homeomorphisms, the same result readily extend to our setting where the base dynamical system is a mixing subshift of finite type (Σ_T, σ).

Hence, the conclusion of Proposition 4.11 for $\mathcal{A} \in C^{\alpha}(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ satisfying (4.2) may be stated as follows: either $\mathcal{A}$ preserves an α-Hölder continuous conformal structure on $\Sigma_T \times \mathbb{R}^2$ or $\mathcal{A}$ is reducible. The former alternative is equivalent to the existence of an α-Hölder continuous conjugacy of $\mathcal{A}$ into the group of linear conformal transformations of $\mathbb{R}^2$. With this observation at hand, the proof for Theorem 4.8 now easily follows.

Proof of Theorem 4.8. Let $\mathcal{A} \in C_b^{\alpha}(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ be an irreducible cocycle. We divide the proof into a few cases.

1. There exists a periodic point $p \in \Sigma_T$ of period n such that $\mathcal{A}^n(p)$ has two eigenvalues of distinct absolute values. Let $\mathcal{B}$ be a cocycle over (Σ_T, σ^n) defined by $\mathcal{B}(x) := \mathcal{A}^n(x)$.

(a) In the case where $\mathcal{B}$ is irreducible, then Corollary 4.10 applies to p which is now a fixed point with respect to σ^n. Hence, $\mathcal{B}$ is weakly typical, which then implies that $\mathcal{A}$ is weakly typical. This gives the first alternative of Theorem 4.8.

(b) In the case where $\mathcal{B}$ is reducible, we get the second alternative of Theorem 4.8; see Lemma 4.14 below for the proof.

2. The absolute value of two eigenvalues of $\mathcal{A}^{\mathrm{per}(p)}(p)$ are equal for every periodic point $p \in \Sigma_T$. In this case, the assumption (4.2) is satisfied. Proposition 4.11 and 4.12 then imply that either there exists an α-Hölder continuous conjugacy of $\mathcal{A}$ into the group of conformal linear transformations of $\mathbb{R}^2$ or $\mathcal{A}$ is reducible. Since $\mathcal{A}$ is irreducible, it must be that $\mathcal{A}$ falls into the third alternative of Theorem 4.8.

This completes the proof. $\square$

Lemma 4.14. Let $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_2(\mathbb{R})$ be an irreducible fiber-bunched cocycle. Suppose there exists a periodic point $p \in \Sigma_T$ of some period $n \in \mathbb{N}$ such that $\mathcal{A}^n(p)$ has two eigenvalues of distinct absolute values. If $\mathcal{B} := \mathcal{A}^n$ is reducible, then $\mathcal{A}$ interchanges two bi-holonomy invariant line bundles.

Proof. Let L be the bi-holonomy invariant and $\mathcal{B}$-invariant line bundle. Then there are n bi-holonomy invariant (but not $\mathcal{A}$-invariant nor necessarily distinct) line bundles $\{L_1, \dots, L_n\}$ defined by $L_i := \mathcal{A}^i L$; that is, $L_i(x) := \mathcal{A}^i(\sigma^{-i}x)L(\sigma^{-i}x)$. Some of these line bundles might coincide with one another, so we denote the distinct lines bundles among them by $\{\mathcal{L}_1, \dots, \mathcal{L}_k\}$.

Note $k \geq 2$ because otherwise the irreducibility assumption on $\mathcal{A}$ would be violated. By distinct line bundles, we mean that for $i \neq j$, there exists $x \in \Sigma_T$ such that $\mathcal{L}_i(x) \neq \mathcal{L}_j(x)$. In this case, we will show that if $i \neq j$, then in fact $\mathcal{L}_i(x)$ differs from $\mathcal{L}_j(x)$ at every $x \in \Sigma_T$.

Claim: For $i \neq j$, we have $\mathcal{L}_i(x) \neq \mathcal{L}_j(x)$ for every $x \in \Sigma_T$.

Proof of Claim. Suppose, for the sake of contradiction, that $\mathcal{L}_{i_0}(x) = \mathcal{L}_{j_0}(x)$ for some $i_0 \neq j_0$ and $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma_T$. From bi-holonomy invariance of $\mathcal{L}_i$'s, it follows that $\mathcal{L}_{i_0}$ and $\mathcal{L}_{j_0}$ agree on 1-cylinder $[x_0]$.

Letting $i_1, j_1 \in \{1,\dots,k\}$ be the indices such that $\mathcal{L}_{\chi_1} = \mathcal{A}\mathcal{L}_{\chi_0}$ for $\chi \in \{i,j\}$. From the previous paragraph, it follows that $\mathcal{L}_{i_1}$ and $\mathcal{L}_{j_1}$ agree on all y with $\sigma^{-1}y \in [x_0]$. Notice also that $i_1 \neq j_1$ because if they were the same, then this would imply that $\mathcal{L}_{i_0}$ and $\mathcal{L}_{j_0}$ agree everywhere, contradicting the fact that $\mathcal{L}_i$'s are distinct line bundles.

Repeating this argument iteratively for each $m \in \mathbb{N}$ gives distinct indices $i_m, j_m \in \{1,\dots,k\}$ such that $\mathcal{L}_{\chi_m} = \mathcal{A}\mathcal{L}_{\chi_{m-1}}$ for $\chi \in \{i,j\}$ and that $\mathcal{L}_{i_m}$ and $\mathcal{L}_{j_m}$ agree on all y with $\sigma^{-m}y \in [x_0]$.

Recall that (Σ_T, σ) is a mixing subshift of finite type with q letters defined by a primitive matrix T. Letting $m_0 \in \mathbb{N}$ be the mixing rate of (Σ_T, σ), we can find $y^{(r)} \in [r]$ for each $r \in \{1,\dots,q\}$ such that $\sigma^{-m_0}y^{(r)} \in [x_0]$. This implies that $\mathcal{L}_{i_{m_0}}$ and $\mathcal{L}_{j_{m_0}}$ agree at $y^{(r)}$ for each $r \in \{1,\dots,q\}$. From bi-holonomy invariance of $\mathcal{L}_i's$, two line bundles $\mathcal{L}_{i_{m_0}}$ and $\mathcal{L}_{j_{m_0}}$ agree everywhere on Σ_T. However, this is a contradiction to the fact that $\mathcal{L}_i$'s are distinct line bundles. □

We now conclude that $k = 2$. This is because if $k \geq 3$, then $\mathcal{A}^n(p)$ preserves the union of k-distinct lines $\{\mathcal{L}_1(p), \dots, \mathcal{L}_k(p)\}$, and hence, it is conjugated to a conformal linear transformation. However, this is contradictory to the assumption that $\mathcal{A}^n(p)$ has two eigenvalues of distinct absolute values. Therefore, $k = 2$ and $\mathcal{A}$ interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$ because otherwise the irreducibility assumption of $\mathcal{A}$ would be violated. □

4.3 Irreducible fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles

We prove Theorem C in this section. We begin by establishing the uniqueness of the equilibrium state for $\Phi_{\mathcal{A}}$ in the three case of Theorem 4.8.

The first case of Theoorem 4.8 concerns with weakly typical cocycles. Using the identical argument as in the proof of Theorem A, such cocycles can be shown to be quasi-multiplicative. Indeed, in the proof of Theorem A we used the same homoclinic point z to build the paths from I to p and J to p. For weakly typical cocycles, the only difference is that we need to use z_- to build a path from I to p and use z_+ to build a path from J to p separately. We omit the details and state it as a proposition:

Proposition 4.15. Let $\mathcal{A} \in \mathcal{U}_w$ be a weakly typical cocycle. Then $\mathcal{A}$ is quasi-multiplicative, and its norm potential $\Phi_{\mathcal{A}}$ has a unique equilibrium state.

For the second case of Theorem 4.8 where there exist two bi-holonomy invariant line bundles interchanged by the action of $\mathcal{A}$, by conjugating $\mathcal{A}$ if necessary, we may assume that $\mathcal{A}$ takes the following form:

$$\mathcal{A}(x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}.$$

Then consider a cocycle $\mathcal{B}$ over (Σ_T, σ^2) defined by $\mathcal{B}(x) := \mathcal{A}^2(x)$; then $\mathcal{B}(x)$ is a diagonal matrix with entries given by $a(\sigma x)b(x)$ and $a(x)b(\sigma x)$. From Theorem C, the norm potential $\Phi_{\mathcal{B}}$ has a unique equilibrium state unless two additive potentials

$$f(x) := \log |a(\sigma x)b(x)| \text{ and } g(x) := \log |a(x)b(\sigma x)|$$

have equal pressures but are not cohomologous with respect to σ^2.

If $\Phi_{\mathcal{B}}$ has a unique equilibrium state, such equilibrium state must also be the unique equilibrium state for $\Phi_{\mathcal{A}}$. This is because $\mathsf{P}(\Phi_{\mathcal{B}}, \sigma^2) = 2\mathsf{P}(\Phi_{\mathcal{A}}, \sigma)$ (see Lemma 5.11 for instance), and hence, any equilibrium state for $\Phi_{\mathcal{A}}$ is an equilibrium state for $\Phi_{\mathcal{B}}$; see for instance [16, Lemma 4.10].

If instead $\Phi_{\mathcal{B}}$ has two distinct equilibrium states $\mu_1, \mu_2 \in \mathcal{M}(\sigma^2)$, each corresponding to f and g, we will show that $\Phi_{\mathcal{A}}$ has a unique equilibrium state given by the average of μ_1

and μ_2.

Lemma 4.16. If $\Phi_{\mathcal{B}} = \Phi_{\mathcal{A}^2}$ has two distinct equilibrium states $\mu_1, \mu_2 \in \mathcal{M}(\sigma^2)$, then $\Phi_{\mathcal{A}}$ has a unique equilibrium state given by $\dfrac{\mu_1 + \mu_2}{2}$.

Proof. Suppose that $\Phi_{\mathcal{A}}$ has two distinct equilibrium states $\mu, \nu \in \mathcal{M}(\sigma)$. Considered as equilibrium states of $\Phi_{\mathcal{B}}$, each has to be a linear combination of μ_1 and μ_2. From Lemma 4.17 below, this implies that μ_1 and μ_2 are σ-invariant. In particular, μ_1 is an equilibrium state for $f/2$ over σ. This follows because μ_1 is an equilibrium state for $\Phi_{\mathcal{B}}$ which is also σ-invariant. Then

$$\begin{aligned}\mathsf{P}(\Phi_{\mathcal{A}^2}, \sigma^2) = \mathsf{P}(f, \sigma^2) = h_{\mu_1}(\sigma^2) + \int f\, d\mu_1 &= 2\Big(h_{\mu_1}(\sigma) + \int f/2\, d\mu_1\Big) \\ &\leq 2\mathsf{P}(f/2, \sigma) \leq \mathsf{P}(f, \sigma^2)\end{aligned}$$

where the last inequality is due to the fact that any σ-invariant measure, including any equilibrium state for $f/2$, can be thought of as a σ^2-invariant measure. Hence, all inequalities are indeed equalities, and μ_1 is an equilibrium state for $f/2$ over σ. Likewise, analogous argument shows that μ_2 is an equilibrium state for $g/2$ over σ.

However, this is a contradiction to μ_1 and μ_2 being distinct measures because considered as potentials over (Σ_T, σ), $f/2$ and $g/2$ are cohomologous:

$$f/2 - g/2 = h \circ \sigma - h$$

where $h := \dfrac{1}{2}(\log|a| - \log|b|)$. Hence $\Phi_{\mathcal{A}}$ has a unique equilibrium state $\mu_{\mathcal{A}} \in \mathcal{M}(\sigma)$.

In order to show that $\mu_{\mathcal{A}}$ is the average of μ_1 and μ_2, first notice that $\sigma_*\mu_1$ coincides with μ_2. This follows because $\sigma_*\mu_1$ is an ergodic equilibrium state for $\Phi_{\mathcal{B}}$ distinct from μ_1 (if it were equal to μ_1 itself, then μ_1 would be σ-invariant, and by applying the same argument to μ_2, we would end up in the contradictory setting of the previous paragraph), so it must be μ_2. Then notice that $\dfrac{\mu_1 + \sigma_* \mu_1}{2} = \dfrac{\mu_1 + \mu_2}{2}$ is σ-invariant from the σ^2-invariance of μ_1 and an equilibrium state for $\Phi_{\mathcal{A}}$. From the uniqueness of the equilibrium state for $\Phi_{\mathcal{A}}$ established in the previous paragraph, it follows that $\mu_{\mathcal{A}}$ is equal to $\dfrac{\mu_1 + \mu_2}{2}$. □

Lemma 4.17. Let μ_1, μ_2 be σ^2-invariant. If there exist more than one $\gamma \in [0, 1]$ such that $\gamma\mu_1 + (1 - \gamma)\mu_2$ is σ-invariant, then μ_1, μ_2 are σ-invariant.

Proof. Suppose there exist distinct $\gamma_1, \gamma_2 \in [0, 1]$ such that both $\mu = \gamma_1\mu_1 + (1 - \gamma_1)\mu_2$ and $\nu = \gamma_2\mu_1 + (1 - \gamma_2)\mu_2$ are σ-invariant.

If one of γ_1 and γ_2 is 0 or 1, suppose without loss of generality that $\gamma_1 = 1$, then $\mu = \mu_1$ is σ-invariant. Since both ν and μ_1 is σ-invariant, so is μ_2.

If neither γ_1 and γ_2 are 0 nor 1, then we have $\dfrac{1}{\gamma_i}\mu = \mu_1 + \dfrac{1 - \gamma_i}{\gamma_i}\mu_2$ for $i = 1, 2$ and by subtracting the equation for $i = 1$ from the equation for $i = 2$ we get that

$$\frac{1}{\gamma_2}\nu - \frac{1}{\gamma_1}\mu = \Big(\frac{1 - \gamma_2}{\gamma_2} - \frac{1 - \gamma_1}{\gamma_1}\Big)\mu_2.$$

The left hand side is σ-invariant and the coefficient of μ_2 is nonzero (as $\gamma_1 \neq \gamma_2$), we have that μ_2 is σ-invariant. Similarly, we get that μ_1 is also σ-invariant. □

For the third case of Theorem 4.8 where there exists a α-Hölder conjugacy $\mathcal{C}\colon \Sigma_T \to \mathrm{GL}_2(\mathbb{R})$ such that $\mathcal{B}(x) = \mathcal{C}(\sigma x)\mathcal{A}(x)\mathcal{C}(x)^{-1}$ is conformal, the conformality of $\mathcal{B}$ implies that the norm of $\mathcal{B}^n$ is *multiplicative*:

$$\|\mathcal{B}^n(x)\| = \prod_{i=0}^{n-1} \|\mathcal{B}(\sigma^i x)\|$$

for any $x \in \Sigma_T$ and $n \in \mathbb{N}$. Then $\Phi_{\mathcal{B}} = \{\log \|\mathcal{B}^n(\cdot)\|\}_{n \geq 0}$ becomes a Hölder continuous additive cocycle generated by $\varphi_{\mathcal{B}}(x) := \log \|\mathcal{B}(x)\|$ in the sense that $S_n\varphi(x) = \log \|\mathcal{B}^n(x)\|$. Hence, $\Phi_{\mathcal{B}}$ has a unique equilibrium state $\mu \in \mathcal{M}(\sigma)$ from Proposition 2.18. Since $\mathcal{A}$ and $\mathcal{B}$ are conjugated by a continuous conjugacy $\mathcal{C}$, the set of equilibrium states for their norms potentials are the same (see Remark 2.22), and hence μ is the unique equilibrium state for $\Phi_{\mathcal{A}}$. This completes the proof of Theorem C.

4.4 Alternate approach via spannability

In this section, we explain how Theorem C may alternatively be established based on the result of Bochi and Garibaldi [6]. This method employs techniques used to prove Theorem A, but the approach in obtaining quasi-multiplicativity circumvents invoking Theorem 4.8.

Bochi and Garibaldi [6, Proposition 3.11] showed that irreducible and strongly fiber-bunched automorphisms of Hölder vector bundles over hyperbolic homeomorphisms are uniformly spannable. While they established this results for other usages, we explain below how it applies to thermodynamic formalism of the norm potentials of $\mathrm{GL}_d(\mathbb{R})$-cocycles.

In our context of $\mathrm{GL}_2(\mathbb{R})$-cocycles, the strongly fiber-bunching condition coincides with the usual fiber-bunching condition. For $\mathrm{GL}_d(\mathbb{R})$-cocycles with $d \geq 3$, the *strong fiber-bunching* requires that for all $x \in \Sigma_T$,

$$\|\mathcal{A}(x)\| \cdot \|\mathcal{A}(x)^{-1}\| \cdot \theta^{\alpha/3} < 1.$$

Note that the exponent $\alpha/3$ is specific to our case of subshifts of finite type. For more general hyperbolic homeomorphisms, the exponent $\alpha/3$ needs to be replaced to another constant which depends only on the base dynamic system and the Hölder exponent of the cocycle; see [6] for precise definition.

Definition 4.18. [6, Section 3.4] A fiber-bunched $\mathrm{GL}_d(\mathbb{R})$-cocycle $\mathcal{A}$ over Σ_T is *spannable* if for any $x, y \in \Sigma_T$ and $u \in \mathbb{R}^d$, there exists

1. $x_1, \ldots, x_d \in \mathcal{W}^u(x)$, and

2. $n_1, \ldots, n_d \in \mathbb{N}_0$ such that the points $y_i := \sigma^{n_i} x_i$ all belong to $\mathcal{W}^s(y)$,

with the property that $\{v_i\}_{i=1}^d$ defined by

$$v_i := H^s_{y_i,y} \circ \mathcal{A}^{n_i}(x_i) \circ H^u_{x,x_i}(u) \tag{4.3}$$

forms a basis of $\mathbb{R}^d$.

In the context of $\mathrm{GL}_d(\mathbb{R})$-cocycles, the relevant result of [6] can be formulated as follow, where the irreducibility is defined in Definition 2.6.

Proposition 4.19. [6, Theorem 3.7] Let $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_d(\mathbb{R})$ be a strongly fiber-bunched and irreducible cocycle. Then $\mathcal{A}$ is spannable.

In fact, Bochi and Garibaldi proved a stronger statement [6, Proposition 3.9] under the same assumptions of Proposition 4.19: using the compactness of Σ_T, they showed that $\mathcal{A}$ is *uniformly spannable*: there exist $k \in \mathbb{N}$ and $C_0 > 0$ such that

- $H^{s/u}$ from (4.3) are local holonomies,
- Each $n_1, \ldots, n_d$ can be chosen to be at most k,
- A linear map $L\colon \mathbb{R}^d \to \mathbb{R}^d$ sending $\{v_i\}_{i=1}^d$ to an orthonormal basis of $\mathbb{R}^d$ satisfies $\|L\| < C_0$.

Proposition 4.20. Suppose that a fiber-bunched cocycle $\mathcal{A} \in C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ is uniformly spannable. Then $\mathcal{A}$ is quasi-multiplicative.

Proof. Let k and C_0 be the constants from uniform spannability of $\mathcal{A}$. Given any $\mathrm{I}, \mathrm{J} \in \mathcal{L}$, let $\bar{x} \in [\mathrm{I}]$ be the point such that $\|\mathcal{A}(\mathrm{I})\| = \|\mathcal{A}^{|\mathrm{I}|}(\bar{x})\|$, and set $x := \sigma^{|\mathrm{I}|}(\bar{x})$. We similarly let $y \in [\mathrm{J}]$ such that $\|\mathcal{A}(\mathrm{J})\| = \|\mathcal{A}^{|\mathrm{J}|}(y)\|$. Applying uniform spannability to $x, y \in \Sigma_T$ and $u = u(\mathcal{A}^n(\bar{x}))$ gives vectors $\{v_i\}_{i=1}^d$ defined by (4.3) that span $\mathbb{R}^d$.

From the condition $\|L\| < C_0$ on a linear map L straightening out $\{v_i\}_{i=1}^d$ into an orthonormal basis, the angle between each pair v_i and v_j, $i \neq j$, is uniformly bounded below by some constant $\varepsilon > 0$ depending only on C_0. In particular, at least one of them, say v_t, satisfies $\measuredangle(v(\mathcal{A}^{|\mathrm{J}|}(y))^{\perp}, v_t) > \varepsilon/2$; this is similar to the angle gap obtained from (3.5).

Then Lemma 3.3 applied to $A = \mathcal{A}^{|\mathrm{J}|}(y)$, $B = \mathrm{id}$, $C = H^s_{y_t,y}\mathcal{A}^{n_t}(x_t)H^u_{x,x_t}$, and $D = \mathcal{A}^{|\mathrm{I}|}(\overline{x})$ gives

$$\begin{aligned}\|\mathcal{A}^{|\mathrm{J}|}(y)H^s_{y_t,y}\mathcal{A}^{n_t}(x_t)H^u_{x,x_t}\mathcal{A}^{|\mathrm{I}|}(\overline{x})\| &\geq c\|\mathcal{A}^{|\mathrm{J}|}(y)\|\|\mathcal{A}^{|\mathrm{I}|}(\overline{x})\| \\ &= c\|\mathcal{A}(\mathrm{I})\|\|\mathcal{A}(\mathrm{J})\|\end{aligned}$$

for some $c > 0$ that only depends on $\mathcal{A}$, ε, and k. Denoting the cylinder of length n_t containing x_t by [K], we have $\overline{x}_t := \sigma^{-|\mathrm{I}|}x_t \in [\mathrm{IKJ}]$ with $|\mathrm{K}| \leq k$. Since the left hand side of the above inequality is equal to $\|H^s_{\sigma^{n_t+|\mathrm{J}|}x_t,\sigma^{|\mathrm{J}|}y}\mathcal{A}^{|\mathrm{I}|+n_t+|\mathrm{J}|}(\overline{x}_t)H^u_{\overline{x},\overline{x}_t}\|$, it is uniformly comparable to $\|\mathcal{A}(\mathrm{IKJ})\|$ due to the bounded distortion (2.10) of $\Phi_{\mathcal{A}}$ and Hölder continuity of the canonical holonomies. This establishes the quasi-multiplicativity of $\mathcal{A}$. $\square$

Since the strong fiber-bunching is merely the usual fiber-bunching for $\mathrm{GL}_2(\mathbb{R})$-cocycles, in view of Proposition 2.26 it is clear that Proposition 4.19 and 4.20 provide an alternative proof that irreducible fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles are quasi-multiplicative.

CHAPTER 5

KOLMOGOROV PROPERTY

This chapter is joint work with Benjamin Call, and originally appeared in [16][3].

5.1 Criteria for K-property

The goal of this chapter is to prove the remaining theorems from the introduction. First, we will prove Theorem 5.6 which establishes sufficient conditions for an invariant measure to have the K-property. This will be based on Ledrappier's criterion suitably generalized for the subadditive setting. Throughout this section, we will consider (X, f) to be an expansive homeomorphism on a compact metric space. Throughout the chapter, we will also use the following notation $\mathsf{P}_\mu(\Phi) := h_\mu(f) + \lim\limits_{n\to\infty} \frac{1}{n}\int \log \varphi_n \, d\mu$.

5.1.1 Uniqueness of Equilibrium States

In this subsection, we establish sufficient conditions for subadditive equilibrium states to be unique, based on [11]. In doing so, we will need to make use of the Kolmogorov-Sinai entropy of a transformation. We assume that every partition appearing here onward is *measurable*, which as we restrict ourselves to finite partitions, means only that each element is measurable. Furthermore, we recall that a measurable partition ξ *generates* the Borel σ-algebra $\mathcal{B}$, if, as $n \to \infty$, the σ-algebras associated with $\bigvee_{i=-n}^{n} f^i\xi$ generate $\mathcal{B}$. In what follows, we use the convention that $0 \log 0 = 0$, and note this defines a continuous function $x \mapsto -x \log x$ on $[0, 1]$.

For any measure ν on X and any finite partition ξ of X, define

$$H_\nu(\xi) = -\sum_{A\in\xi} \nu(A) \log \nu(A)$$

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and

$$h_\nu(f,\xi) := \lim_{n\to\infty} \frac{1}{n} H_\nu(\bigvee_{i=0}^{n-1} f^{-i}\xi) = \inf_{n\to\infty} \frac{1}{n} H_\nu(\bigvee_{i=0}^{n-1} f^{-i}\xi) \tag{5.1}$$

where the infimum is due to subadditivity. Then the Kolmogorov-Sinai entropy of ν is defined by

$$h_\nu(f) = \sup_{\text{finite partitions } \xi} h_\nu(f,\xi).$$

By Sinai's theorem, if a partition ξ generates the Borel σ-algebra, then $h_\nu(f) = h_\nu(f,\xi)$.

We say a subadditive potential $\Phi = \{\log \varphi_n\}$ on X has *bounded distortion* if there exists $C \geq 1$ such that for all $\varepsilon > 0$ sufficiently small, $x \in X$, $n \in \mathbb{N}$, and $y, z \in B_n(x,\varepsilon)$, we have

$$C^{-1} \leq \frac{\varphi_n(y)}{\varphi_n(z)} \leq C. \tag{5.2}$$

Here B_n is denotes the ball with respect to the metric d_n, and this definition is a generalization of the bounded distortion (2.10) defined for the subshift (Σ_T, σ).

For the following lemma, we only need the lower inequality of the Gibbs property stated in Proposition 2.26. We call such a property by the *lower Gibbs property*.

Lemma 5.1. Let $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ be a subadditive potential on X with bounded distortion and suppose $\eta \in \mathcal{M}(f)$ is an ergodic equilibrium state of Φ with the subadditive lower Gibbs property. Then η is the unique equilibrium state of Φ.

Proof. We follow the proof of [11, Lemma 8] closely. Assume for the sake of contradiction that $\nu \in \mathcal{M}(f)$ is an ergodic equilibrium state not equal to η. Then ν and η are mutually singular, and so there exists a $(\nu+\eta)$-measurable set $B \subset X$ such that $f(B) = B$, $\eta(B) = 0$ and $\nu(B) = 1$. For instance, we could take B to be the set of generic points for ν.

Let $4\varepsilon > 0$ be smaller than the expansivity constant of (X, f) and small enough for the bounded distortion (5.2) to hold. For each $n \in \mathbb{N}$ we fix a maximal $(n, 2\varepsilon)$-separated set $E_n \subset X$. Then we fix an adapted partition $\xi_n := \{A_x : x \in E_n\}$ of X such that

$B_n(x,\varepsilon) \subseteq A_x \subseteq \overline{B_n(x,2\varepsilon)}$ for each $x \in E_n$.

In order to make use of the expansivity assumption, define for all n, the partition $\Omega_n := f^{[n/2]}\xi_n$ and denote the element of Ω_n containing $y \in X$ by $\omega_n(y)$. From the construction of Ω_n, for any $y \in X$ there exists some $x \in E_n$ such that $B_n(x,\varepsilon) \subseteq f^{-[n/2]}\omega_n(y) \subseteq \overline{B_n(x,2\varepsilon)}$. It then follows that $f^{-[n/2]}\omega_n(y) \subseteq \overline{B_n(y,4\varepsilon)}$. Therefore expansivity gives $\bigcap_{n\in\mathbb{N}} \omega_n(y) = \{y\}$ for all $y \in X$, and by [20, Lemma 3.14] there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ where C_n is a union of elements of Ω_n such that $\lim_{n\to\infty} (\nu+\eta)(C_n \triangle B) \to 0$. Since B is f-invariant, setting

$$\mathcal{U}_n := f^{-[n/2]}C_n,$$

which consists of a union of elements of ξ_n, we have $(\nu+\eta)(\mathcal{U}_n \triangle B) \to 0$. From the assumptions on B, this is equivalent to $\eta(\mathcal{U}_n) \to 0$ and $\nu(\mathcal{U}_n) \to 1$.

As (X, f) is expansive, ξ_n is a generator under f^n by observing that given $y, z \in \bigcap_{k\in\mathbb{Z}} f^{-kn}\overline{B_n(x_k, 2\varepsilon)}$ for some $\{x_k\}_{k\in\mathbb{Z}} \subset X$, we have that $d(f^k y, f^k z) \le 4\varepsilon$ for all $k \in \mathbb{Z}$. Consequently,

$$nh_\nu(f) = h_\nu(f^n) = h_\nu(f^n, \xi_n) \le H_\nu(\xi_n)$$

where the last inequality is from (5.1). Moreover, from the subadditivity of Φ, we have

$$\lim_{k\to\infty} \frac{1}{k} \int \log \varphi_k \, d\nu = \inf_{k\to\infty} \frac{1}{k} \int \log \varphi_n \, d\nu \le \frac{1}{n} \int \log \varphi_n \, d\nu$$

for each $n \in \mathbb{N}$. Hence,

$$\begin{aligned} n\mathsf{P}(\Phi) &= n\Big(h_\nu(f) + \lim_{k\to\infty} \frac{1}{k} \int \log \varphi_k \, d\nu\Big) \\ &\le H_\nu(\xi_n) + \int \log \varphi_n \, d\nu \\ &= \sum_{A_x \in \xi_n} \Big(-\nu(A_x) \log \nu(A_x) + \int \log \varphi_n \cdot \chi_{A_x} \, d\nu \Big). \end{aligned}$$

Let C be the constant given by the bounded distortion (5.2) on Φ. Then

$$\int \log \varphi_n \cdot \chi_{A_x} \, d\nu \leq \nu(A_x)\big(C + \log \varphi_n(x)\big)$$

for all n sufficiently large. In particular, we have

$$n\mathsf{P}(\Phi) \leq C + \sum_{A_x \subseteq \mathcal{U}_n} \nu(A_x)\Big(-\log \nu(A_x) + \log \varphi_n(x)\Big) + \sum_{A_x \cap \mathcal{U}_n = \emptyset} \nu(A_x)\Big(-\log \nu(A_x) + \log \varphi_n(x)\Big).$$

Applying a Jensen-type inequality (see [11, Lemma 7]) to each sum, we have

$$n\mathsf{P}(\Phi) - C \leq 2C^* + \nu(\mathcal{U}_n) \log \left(\sum_{A_x \subseteq \mathcal{U}_n} \varphi_n(x) \right) + \nu(\mathcal{U}_n^c) \log \left(\sum_{A_x \cap \mathcal{U}_n = \emptyset} \varphi_n(x) \right),$$

where $C^* := \max_{t \in [0,1]} -t \log t$.

Let C_0 be the constant from the subadditive lower Gibbs property of η. Then after rearranging the terms, we have

$$\begin{aligned} -2C^* - C &\leq \nu(\mathcal{U}_n) \left(\log \sum_{A_x \subseteq \mathcal{U}_n} \varphi_n(x) e^{-nP(\Phi)} \right) + \nu(\mathcal{U}_n^c) \log \left(\sum_{A_x \cap \mathcal{U}_n = \emptyset} \varphi_n(x) e^{-nP(\Phi)} \right) \\ &\leq \nu(\mathcal{U}_n) \log(C_0 \eta(\mathcal{U}_n)) + \nu(\mathcal{U}_n^c) \log(C_0 \eta(\mathcal{U}_n^c)) \\ &= \log C_0 + \nu(\mathcal{U}_n) \log \eta(\mathcal{U}_n) + \nu(\mathcal{U}_n^c) \log \eta(\mathcal{U}_n^c). \end{aligned}$$

This, however, is a contradiction because as we send $n \to \infty$, the lower bound $-2C^* - C$ is independent of $n \in \mathbb{N}$ while $\nu(\mathcal{U}_n) \log \eta(\mathcal{U}_n) \to -\infty$ and $\nu(\mathcal{U}_n^c) \log \eta(\mathcal{U}_n^c) \to 0$. Hence, ν cannot be an equilibrium state of Φ. □

5.1.2 *Subadditive generalization of Ledrappier's criterion*

In order to state Ledrappier's criterion for the K-property, we need to consider the product space $(X \times X, f \times f)$. On the product space, we take the metric to be the maximum of the distance in each coordinate:

$$d((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), d(y_1, y_2)\}.$$

Ledrappier [36] then showed that these equilibrium states have the K-property by means of the following proposition.

Proposition 5.2. [36, Proposition 1.4] Let (X, f) be asymptotically entropy expansive and let $\varphi : X \to \mathbb{R}$ be continuous. Suppose that $(X \times X, f \times f)$ has a unique equilibrium state for the potential $\Phi(x, y) = \varphi(x) + \varphi(y)$. Then the unique equilibrium state for φ has the K-property.

Lemma 5.3. For any subadditive potential $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ on (X, f), consider a sequence of continuous functions $\Psi = \{\log \psi_n\}_{n\in\mathbb{N}}$ on $(X \times X, f \times f)$ defined by

$$\psi_n(x, y) := \varphi_n(x) \cdot \varphi_n(y). \tag{5.3}$$

Then Ψ is subadditive and $\mathsf{P}(\Psi) = 2\mathsf{P}(\Phi)$.

Proof. Subadditivity of Ψ follows immediately: as for all $n, m \in \mathbb{N}$,

$$\begin{aligned}\log \psi_{n+m}(x, y) &= \log \varphi_{n+m}(x) + \log \varphi_{n+m}(y)\\ &\le \log \varphi_n(x) + \log \varphi_n \circ f^m(x) + \log \varphi_n(y) + \log \varphi_n \circ f^m(y)\\ &= \log \psi_n(x, y) + \log \psi_n(f^m x, f^m y).\end{aligned}$$

For the second statement, let μ be an equilibrium state for Φ. Then $\mu \times \mu \in \mathcal{M}(f \times f)$, and we also have

$$\mathsf{P}_{\mu\times\mu}(\Psi) = h_{\mu\times\mu}(f\times f) + \lim_{n\to\infty}\frac{1}{n}\int \log\psi_n \, d\mu\times\mu = 2h_\mu(f) + 2\lim_{n\to\infty}\frac{1}{n}\int \log\varphi_n \, d\mu = 2\mathsf{P}_\mu(\Phi).$$

Therefore, by the variational principle (2.9), we see that $\mathsf{P}(\Psi) \geq 2\mathsf{P}(\Phi)$.

For the reverse direction, we again proceed by the variational principle. Let $\nu \in \mathcal{M}(f\times f)$ be arbitrary, and write ν_1 and ν_2 to be the projections of ν onto the first and second coordinate, respectively. Each ν_i is a f-invariant measure on X. An elementary calculation shows that $h_\nu(f\times f) \leq h_{\nu_1}(f) + h_{\nu_2}(f)$ (see for instance, [22, Fact 4.4.3]), and

$$\lim_{n\to\infty}\frac{1}{n}\int \log\psi_n \, d\nu = \lim_{n\to\infty}\frac{1}{n}\Big(\int \log\varphi_n \, d\nu_1 + \int \log\varphi_n \, d\nu_2\Big).$$

Therefore,

$$\mathsf{P}_\nu(\Psi) \leq h_{\nu_1}(f) + \lim_{n\to\infty}\frac{1}{n}\int \log\varphi_n \, d\nu_1 + h_{\nu_2}(f) + \lim_{n\to\infty}\frac{1}{n}\int \log\varphi_n \, d\nu_2 \leq 2\mathsf{P}(\Phi).$$

This completes the proof. □

Corollary 5.4. If $\mu \in \mathcal{M}(f)$ is an equilibrium state for Φ, then $\mu\times\mu \in \mathcal{M}(f\times f)$ is an equilibrium state for Ψ.

We can now state the subadditive generalization of Proposition 5.2 for establishing the K-property. Recall that we call a measure μ is K if and only if it has no nontrivial zero entropy factors.

Proposition 5.5. Let $\Phi = \{\log\varphi_n\}_{n\in\mathbb{N}}$ be a subadditive potential on X with unique equilibrum state $\mu \in \mathcal{M}(f)$. If $\mu\times\mu \in \mathcal{M}(f\times f)$ is the unique equilibrium state for Ψ, then μ has the K-property.

Proof. We follow the original proof of Ledrappier, and prove the contrapositive. Let $\mu \in \mathcal{M}(f)$ be the unique equilibrium state for Φ, and suppose it is not K. Then the Pinsker factor Π for μ is non-trivial. We therefore can define $m \in \mathcal{M}(f \times f)$ different from $\mu \times \mu$ to be

$$m(A \times A') = \int_A \mathbb{E}[\chi_{A'} \mid \Pi]\, d\mu$$

for all measurable $A, A' \subset X$. To see this is different from $\mu \times \mu$, take A to be Π-measurable with $0 < \mu(A) < 1$, and observe that $m(A \times A) = \mu(A) \neq \mu(A)^2 = (\mu \times \mu)(A \times A)$. For those familiar with joinings, this is the relatively independent self-joining of μ over Π.

The entropy calculation from [36] is purely dependent on the measure, and so is unaffected by the subadditive setting. For a reference where this calculation is carried out in full, see [15]. Hence, $h_m(f \times f) = 2h_\mu(f)$. Now because $m(A \times X) = m(X \times A) = \mu(A)$, and ψ is is defined independently in each coordinate, we observe that for all $n \in \mathbb{N}$, $\int \log \psi_n \, dm = 2 \int \log \varphi_n \, d\mu$. Therefore,

$$\mathsf{P}_m(\Psi) = h_m(f \times f) + \lim_{n\to\infty} \frac{1}{n} \int \log \psi_n \, dm = 2h_\mu(f) + 2 \lim_{n\to\infty} \frac{1}{n} \int \log \varphi_n \, d\mu = 2\mathsf{P}_\mu(\Phi) = 2\mathsf{P}(\Phi).$$

Hence, m is an equilibrium state for Ψ in $\mathcal{M}(f \times f)$, as is $\mu \times \mu$. So there exist multiple equilibrium states for the product system. □

Now, recall from Proposition 2.14 that a measure $\mu \in \mathcal{M}(f)$ is weak mixing if and only if $\mu \times \mu \in \mathcal{M}(f \times f)$ is ergodic. Using this fact, we obtain the following theorem:

Theorem 5.6. Let (X, f) be an expansive homeomorphism on a compact metric space and $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ be a subadditive potential on X with bounded distortion. Suppose $\eta \in \mathcal{M}(f)$ is a weak mixing equilibrium state of Φ with the lower subadditive Gibbs property. Then η has the K-property.

Proof. First, as η is a weak mixing equilibrium state, $\eta \times \eta$ is an ergodic equilibrium state. Therefore, if we can show Lemma 5.1 holds for the system $(X \times X, f \times f)$ with potential Ψ defined as (5.3), then it follows that $\eta \times \eta$ is the unique equilibrium state. Therefore, by the subadditive version of Ledrappier's criterion (Proposition 5.5), it immediately follows that η is K.

We now verify the assumptions in Lemma 5.1. First, $(X \times X, f \times f)$ is still an expansive homeomorphism on a compact metric space. Thus, we only need to check that Ψ has the bounded distortion and the subadditive Gibbs property.

Since the metric on our product space is the maximum of the distance in each coordinate, it follows that

$$B_n((x,y),\varepsilon) = B_n(x,\varepsilon) \times B_n(y,\varepsilon).$$

From this, it follows that the subadditive Gibbs property on η and the bounded distortion of Φ induce the corresponding properties on $\eta \times \eta$ and Ψ. □

We note that weak mixing is a natural assumption to impose in this theorem, as one can easily define a system which is not weak mixing and satisfies all other conditions of this theorem.

5.2 Total ergodicity to K-property

In this section, we prove Theorem E and F by applying Theorem 5.6. Note that these theorems concern with subadditive potentials over a subshift (Σ_T, σ), unlike the previous section where the base dynamical system was arbitrary.

We recall the setting of Theorem E. Let $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ be a quasi-multiplicative subadditive potential on Σ_T with bounded distortion (2.10). Let $\mu \in \mathcal{M}(\sigma)$ be the unique equilibrium state for Φ with the Gibbs property from Proposition 2.26, and suppose that μ is totally ergodic. We wish to show that μ is K. By Theorem 5.6, it suffices to show that μ is weak mixing.

The following proposition is essentially a reformulation of [37, Theorem 5 (ii)]. The setting there is for norm potentials of irreducible locally constant cocycles; however, the proof generalizes easily to any quasi-multiplicative subadditive potentials with bounded distortion.

Lemma 5.7. Let $\Phi = \{\log \varphi_n\}_{n\in\mathbb{N}}$ be a quasi-multiplicative subadditive potential on Σ_T with bounded distortion. Suppose the unique equilibrium state $\mu \in \mathcal{M}(\sigma)$ of Φ from Proposition 2.26 is totally ergodic. Then μ is mixing.

Proof. The proof of [37, Theorem 5 (ii)] extends without much modification; we only point out minor modifications required to extend the proof. From Proposition 2.26, it follows that μ has the Gibbs property with constant C_0.

As in Definition 2.25, we will make use of the subadditive potential $\Phi = \log\varphi$ on $\mathcal{L}$ defined by $\varphi(\mathrm{I}) := \max\limits_{x\in[\mathrm{I}]} \varphi_{|\mathrm{I}|}(x)$. Denoting the constant from the bounded distortion (2.10) of Φ by C_1, for any $n \in \mathbb{N}$, $\mathrm{I} \in \mathcal{L}(n)$, and $x \in [\mathrm{I}]$ we have the following bounds on $\mu([\mathrm{I}])/\big(e^{-n\mathsf{P}(\Phi)}\Phi(\mathrm{I})\big)$:

$$(C_0C_1)^{-1} \le C_1^{-1} \cdot \frac{\mu([\mathrm{I}])}{e^{-n\mathsf{P}(\Phi)}\varphi_n(x)} \le \frac{\mu([\mathrm{I}])}{e^{-n\mathsf{P}(\Phi)}\Phi(\mathrm{I})} \le C_0$$

Then for any cylinders $\mathrm{I}, \mathrm{J} \in \mathcal{L}$ of length n and m, we have for any $k > n$,

$$\begin{aligned}
\mu([\mathrm{I}] \cap \sigma^{-k}[\mathrm{J}]) &= \sum_{\substack{|\mathrm{K}|=k-n\\ \mathrm{IKJ}\in\mathcal{L}}} \mu([\mathrm{IKJ}])\\
&\le C_0 \sum_{\substack{|\mathrm{K}|=k-n\\ \mathrm{IKJ}\in\mathcal{L}}} e^{-(k+m)\mathsf{P}(\Phi)}\Phi(\mathrm{IKJ})\\
&\le C_0 \sum_{\substack{|\mathrm{K}|=k-n\\ \mathrm{IKJ}\in\mathcal{L}}} e^{-(k+m)\mathsf{P}(\Phi)}\Phi(\mathrm{I})\Phi(\mathrm{K})\Phi(\mathrm{J})\\
&\le C_0^4C_1^3\mu([\mathrm{I}])\mu([\mathrm{J}])\Big(\sum_{\substack{|\mathrm{K}|=k-n\\ \mathrm{IKJ}\in\mathcal{L}}} \mu([\mathrm{K}])\Big)\\
&\le C_0^4C_1^3\mu([\mathrm{I}])\mu([\mathrm{J}]).
\end{aligned}$$

This gives $\limsup\limits_{k\to\infty} \mu([\mathrm{I}] \cap \sigma^{-k}[\mathrm{J}]) \le C\mu([\mathrm{I}])\mu([\mathrm{J}])$ where $C = C_0^4C_1^3$. Using this property together with total ergodicity of μ, the rest of the proof from here on (i.e., promoting total ergodicity to weak mixing, and then to mixing) follows that of [37, Theorem 5 (ii)] verbatim, following the method of Ornstein [40]. □

Theorem E now follows as an easy consequence of Proposition 2.26 which gives an equilibrium state with the Gibbs property, Lemma 5.7 which shows that total ergodicity is enough to get weak mixing, and Theorem 5.6 which lifts these together to K.

In view of Theorem A and E, the only missing ingredient in proving Theorem F is the total ergodicity of $\mu_{\mathcal{A}}$ which we establish below. For each $n \in \mathbb{N}$, consider $\mathcal{A}^n$ as a cocycle over (Σ_T, σ^n) and denote the corresponding norm potential by $\Phi_{\mathcal{A}^n}$. It can be easily checked from the definition that if $\mathcal{A}$ is fiber-bunched over (Σ_T, σ), then so is $\mathcal{A}^n$ over (Σ_T, σ^n). The idea of the proof is similar to that of [38, Theorem 5 (i)].

Proposition 5.8. Let $\mathcal{A} \in C_b^\alpha(\Sigma_T, \mathrm{GL}_d(\mathbb{R}))$ be typical. Then the unique equilibrium state $\mu_{\mathcal{A}} \in \mathcal{M}(\sigma)$ of $\Phi_{\mathcal{A}}$ is totally ergodic.

Proof. Note that the holonomies $H^{s/u}$ for $\mathcal{A}$ are also holonomies for $\mathcal{A}^n$ for every $n \in \mathbb{N}$. Moreover, $\mathcal{A}^n$ is typical with respect to (Σ_T, σ^n) via the same periodic and the homoclinic points p and z from the definition of typical cocycles. Applying Theorem A to $\mathcal{A}^n$ and (Σ_T, σ^n), $\Phi_{\mathcal{A}^n}$ has a unique equilibrium state $\mu_{\mathcal{A}^n} \in \mathcal{M}(\sigma^n)$. In particular, $\mu_{\mathcal{A}^n}$ is ergodic with respect to (Σ_T, σ^n). From Lemma 5.11 which also applies to $\Phi_{\mathcal{A}^n}$, it follows that $\mu_{\mathcal{A}^n}$ coincides with $\mu_{\mathcal{A}}$. Hence, $\mu_{\mathcal{A}}$ is totally ergodic. □

Remark 5.9. As noted in Section 3.4, Theorem A and B hold also for the singular value potentials of typical cocycles. In the same spirit, the same argument of Proposition 5.8 also applies the unique equilibrium states for such potentials. Hence, Theorem F holds for such equilibrium states also.

5.3 Application to fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycles

Let $\mathcal{A}\colon \Sigma_T \to \mathrm{GL}_2(\mathbb{R})$ be a Hölder continuous and fiber-bunched cocycle.

Lemma 5.10. If $\mathcal{A} \in C_b^\alpha(\Sigma_T, \mathrm{GL}_2(\mathbb{R}))$ is reducible, then all ergodic equilibrium states of $\Phi_{\mathcal{A}}$ are Bernoulli.

Proof. Since the set of equilibrium states of $\Phi_{\mathcal{A}}$ is a subset of $\{\mu_{\log|a|}, \mu_{\log|c|}\}$ from Theorem D and both $\mu_{\log|a|}$ and $\mu_{\log|c|}$ are Bernoulli from Proposition 2.18, our claim follows. □

Lemma 5.11. For any $n \in \mathbb{N}$, we have $\mathsf{P}(\Phi_{\mathcal{A}^n}) = n\mathsf{P}(\Phi_{\mathcal{A}})$. Moreover, any equilibrium state $\mu \in \mathcal{M}(\sigma)$ of $\Phi_{\mathcal{A}}$ is an equilibrium state of $\Phi_{\mathcal{A}^n}$.

Proof. We proceed via the variational principle (2.9). Observe that $\varphi_{\mathcal{A}^n,m} = \varphi_{\mathcal{A},mn}$. Then, we see that if $\mu \in \mathcal{M}(\sigma)$,

$$h_\mu(\sigma^n) + \lim_{m\to\infty} \frac{1}{m} \int \log \varphi_{\mathcal{A}^n,m}\, d\mu = nh_\mu(\sigma) + n \lim_{m\to\infty} \frac{1}{mn} \int \log \varphi_{\mathcal{A},mn}\, d\mu.$$

Considering μ as a σ^n-invariant measure, we have just shown that $\mathsf{P}_\mu(\Phi_{\mathcal{A}^n}) = n\mathsf{P}_\mu(\Phi_{\mathcal{A}})$ and the variational principle implies that $n\mathsf{P}(\Phi_{\mathcal{A}}) \leq \mathsf{P}(\Phi_{\mathcal{A}^n})$.

For the reverse inequality, take $\mu \in \mathcal{M}(\sigma^n)$ and define $\nu = \sum_{i=0}^{n-1} \frac{(\sigma^i)_*\mu}{n}$. Then ν is σ-invariant, and furthermore, $h_\mu(\sigma^n) = h_\nu(\sigma^n) = nh_\nu(\sigma)$. Since $\mathcal{A}$ is continuous and Σ_T is compact, for any $0 \leq i \leq n-1$, two functions $\log \varphi_{\mathcal{A}^n,m} \circ \sigma^i$ and $\log \varphi_{\mathcal{A}^n,m}$ are uniformly comparable. Hence,

$$\lim_{m\to\infty} \frac{1}{m} \int \log \varphi_{\mathcal{A}^n,m} \circ \sigma^i\, d\mu = \lim_{m\to\infty} \frac{1}{m} \int \log \varphi_{\mathcal{A}^n,m}\, d\mu.$$

Then it follows that

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{mn}\int\log\varphi_{\mathcal{A},mn}d\nu &= \lim_{m\to\infty}\frac{1}{mn}\sum_{i=0}^{n-1}\frac{1}{n}\int\log\varphi_{\mathcal{A}^n,m}\circ\sigma^i\,d\mu\\ &= \frac{1}{n}\lim_{m\to\infty}\frac{1}{m}\int\log\varphi_{\mathcal{A}^n,m}\,d\mu.\end{aligned}$$

Therefore, $\mathsf{P}_\mu(\Phi_{\mathcal{A}^n}) = n\mathsf{P}_\nu(\Phi_{\mathcal{A}})$. This gives the reverse inequality, and the result follows. That any equilibrium state of $\Phi_{\mathcal{A}}$ is an equilibrium state of $\Phi_{\mathcal{A}^n}$ is a direct consequence. □

Proof of Theorem G. In view of Lemma 5.10 it suffices to focus on irreducible $\mathrm{GL}_2(\mathbb{R})$-cocycles. Let $\mathcal{A}\in C_b^\alpha(\Sigma_T,\mathrm{GL}_2(\mathbb{R}))$ be irreducible, and $\mu_{\mathcal{A}}\in\mathcal{M}(\sigma)$ be the unique equilibrium state for $\Phi_{\mathcal{A}}$ from Theorem C.

We then consider the cocycle $\mathcal{A}^2$ over (Σ_T,σ^2). Noting that Theorem D also applies to $\mathcal{A}^2$, we divide into two cases depending on the number of equilibrium states of $\Phi_{\mathcal{A}^2}$.

Case 1: $\Phi_{\mathcal{A}^2}$ has a unique equilibrium state.

Such a unique equilibrium state must be $\mu_{\mathcal{A}}$ by Lemma 5.11. Uniqueness then implies that $(\Sigma_T,\sigma^2,\mu_{\mathcal{A}})$ is ergodic. We claim that in fact, $\mu_{\mathcal{A}}$ is totally ergodic, which by Theorem E, would imply that $\mu_{\mathcal{A}}$ is K.

It suffices to show that $\Phi_{\mathcal{A}^n}$ has a unique equilibrium state for each $n\in\mathbb{N}$ because this will then imply $(\Sigma_T,\sigma^n,\mu_{\mathcal{A}})$ is ergodic for each $n\in\mathbb{N}$. Assume for the sake of contradiction that there exists $n\in\mathbb{N}$ such that $\Phi_{\mathcal{A}^n}$ has at least two ergodic equilibrium states. Since $\mathcal{A}^n$ is still a fiber-bunched $\mathrm{GL}_2(\mathbb{R})$-cocycle over (Σ_T,σ^n), $\Phi_{\mathcal{A}^n}$ can have at most two distinct ergodic equilibrium states by Theorem D. So $\Phi_{\mathcal{A}^n}$ must have exactly two distinct equilibrium states $\mu_1,\mu_2\in\mathcal{M}(\sigma^n)$. Consider the map

$$\sigma_*\colon\{\mu_1,\mu_2\}\to\{\mu_1,\mu_2\}.$$

Since the n-th power of this map is the identity by σ^n-invariance, σ_* must be injective. Consequently, either both μ_1 and μ_2 are σ-invariant or σ_* interchanges μ_1 and μ_2. In the latter case, μ_1 and μ_2 are distinct, σ^2-invariant, and equilibrium states for $\Phi_{\mathcal{A}^2}$, a contradiction. Meanwhile, the former case contradicts uniqueness of the equilibrium state for $\Phi_{\mathcal{A}}$. This shows that $\mu_{\mathcal{A}}$ is the unique equilibrium state for $\Phi_{\mathcal{A}^n}$ for all $n \in \mathbb{N}$, which in turn implies total ergodicity.

Case 2: $\Phi_{\mathcal{A}^2}$ has multiple equilibrium states. This case is similar to the proof of Theorem 4.8. From Theorem D, $\mathcal{A}^2$ over (Σ_T, σ^2) must be reducible and $\Phi_{\mathcal{A}^2}$ must have two distinct ergodic equilibrium states $\mu_1, \mu_2 \in \mathcal{M}(\sigma^2)$. In fact, denoting the $\mathcal{A}^2$-invariant and $H^{s/u}$-invariant line bundle by L_1, consider another $\mathcal{A}^2$-invariant and $H^{s/u}$-invariant line bundle L_2 defined by $L_2(\sigma x) := \mathcal{A}(x)L_1(x)$. Since $\mathcal{A}$ is irreducible, L_1 and L_2 are distinct bundles. Then as in the proof of Lemma 4.14, $L_1(x)$ differs from $L_2(x)$ for all $x \in \Sigma_T$.

For each $x \in \Sigma_T$, let $\mathcal{C}(x) \in \mathrm{GL}_2(\mathbb{R})$ be the unique linear map that takes the standard basis of $\mathbb{R}^2$ into $\{L_1(x), L_2(x)\}$. Then $\mathcal{B}(x) := \mathcal{C}(\sigma x)^{-1}\mathcal{A}(x)\mathcal{C}(x)$ exchanges the coordinate axes of $\mathbb{R}^2$, and hence must be of the form specified in Theorem G:

$$\mathcal{B}(x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}.$$

Then $\mathcal{B}^2(x)$ is the diagonal matrix given by $\mathrm{diag}(a(\sigma x)b(x), a(x)b(\sigma x))$. Moreover, two potentials $\log|a(\sigma x)b(x)|$ and $\log|b(\sigma x)a(x)|$ have the same pressure (with respect to σ^2), and their σ^2-ergodic equilibrium states are μ_1 and μ_2, respectively, each of which is Bernoulli by Lemma 5.10. From the assumption that μ_1 and μ_2 are distinct, and Lemma 4.16 shows that $\frac{1}{2}(\mu_1 + \mu_2)$ is the unique equilibrium state $\mu_{\mathcal{A}}$ for $\Phi_{\mathcal{A}}$. It is then clear that such $\mu_{\mathcal{A}}$ is not totally ergodic, and this is the only case where $\mu_{\mathcal{A}}$ fails to be K. $\square$

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