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ABSTRACT

The first part of this thesis concerns intersection cohomology sheaves on a smooth projective variety with a torus action that has finitely many fixed points. Under some additional assumptions, we consider tensor products of intersection cohomology sheaves on a Białynicki-Birula stratification of the variety. We give a formula for the hypercohomology of the tensor product in terms of the tensor products of the individual sheaves, as well as the cohomology of the variety. We prove a similar result in the setting of equivariant cohomology.

In the second part of this thesis, we study the Bernstein–Sato polynomial, or the b-function, which is an invariant of singularities of hypersurfaces. We are interested in the b-function of hyperplane arrangements of Weyl arrangements, which are the arrangements of root systems of semi-simple Lie algebras. It has been conjectured that the poles of the local topological zeta function, which is another invariant of hypersurface singularities, are all roots of the b-function. Using the work of Opdam and Budur–Mustaţă–Teitler, we prove this conjecture for all Weyl arrangements. We also give an upper bound for the b-function of the Vandermonde determinant, which cuts out the Weyl arrangement in type A.

1

INTRODUCTION

Complex algebraic varieties are equipped with the Zariski topology, which is defined algebraically. They also inherit a much finer topology from the complex numbers, called the complex or the analytic topology. Even though spaces that are non-isomorphic as algebraic varieties can often be homeomorphic in the complex topology, there are algebraic constructions that sometimes capture complex topological phenomena and vice-versa. In this setting, it is natural to use the language of sheaves and their associated derived categories. Our main focus will be on two fundamental constructions in this subject: perverse sheaves and $\mathcal{D}$-modules.

A connecting thread between these two is provided by the Riemann–Hilbert correspondence, proved in various versions by Deligne [16], Kashiwara [31, 32], and Mebkhout [38, 40, 39]. This theorem gives an equivalence between two categories associated to a smooth complex algebraic variety X. Namely, the derived category of regular holonomic $\mathcal{D}_X$-modules, and the derived category of constructible sheaves on X. Under this equivalence, the abelian category of regular holonomic $\mathcal{D}_X$-modules is sent to the category of perverse sheaves on X. We now give a broad outline of both constructions and the contents of this thesis.

The first part, described in Chapter 2, concerns perverse sheaves on a smooth complex projective variety X. It has been published in self-contained form as [3]. Perverse sheaves arose from the study of intersection homology theory of Goresky–MacPherson [21], which built a new homology theory for singular manifolds. In the language of sheaves, the cohomology of a manifold is simply the sheaf cohomology of the locally constant sheaf on the manifold, which also describes the homology of the manifold by Poincaré duality. Singular manifolds do not satisfy Poincaré duality, and in this case the "correct" replacement for the locally constant sheaf is the intersection cohomology sheaf or IC sheaf on the manifold. The global cohomology (or hypercohomology) of the IC sheaf gives the intersection homology of the manifold, which satisfies a version of Poincaré duality. The IC sheaf on X is a special example of a perverse sheaf. A sheaf-theoretic characterization and foundational theorems about perverse sheaves are explained in a subsequent paper of Goresky–MacPherson [22], as well as a monograph of Beilinson–Bernstein–Deligne [5].

A sheaf on X is called constructible if there is a partition of X into locally closed subsets, such that the sheaf is locally constant on each piece. Perverse sheaves are objects of the derived category of sheaves with constructible cohomology sheaves. In many cases, X is already equipped with a natural choice of partition. In these cases, it is natural to consider perverse sheaves whose cohomology sheaves are constructible along the chosen partition. We consider the case when the variety is equipped with a torus action that has finitely many fixed points. For example, partial flag varieties and Schubert varieties are important examples that satisfy this restriction. By fixing a one-parameter subgroup of T that has the same fixed set as T, we can construct a decomposition of our algebraic variety into locally-closed affine spaces, called the Białynicki-Birula decomposition [8]. Under some reasonable assumptions, we consider perverse sheaves that are constructible along this decomposition. In this context, a theorem of Ginzburg [19] gives a description of the derived morphisms between simple perverse sheaves, which are exactly the IC sheaves on the closures of each locally-closed piece (or stratum). The (hyper)cohomology of each perverse sheaf on X carries an action of the cohomology of X. Moreover, the torus action allows us to construct a Morse function that gives a topological description of this action in terms of intersections of strata with other subspaces. The theorem then states that the derived morphisms between simple perverse sheaves can be computed precisely as those homomorphisms between the corresponding hypercohomology spaces that respect the action of the cohomology of the base. We are interested in a similar question: to give a description of the derived tensor product of a finite collection of simple perverse sheaves. Again, we can reinterpret the action of the cohomology of X on the cohomology of each perverse sheaf in terms of intersection of strata. We describe a Künneth-type formula that describes the cohomology of the tensor product of simple perverse sheaves to the tensor product of their cohomology, modulo the action of the cohomology of X. We also prove a similar result for the torus-equivariant cohomology of the tensor product of a collection of simple perverse sheaves.

The second part of this thesis, described in Chapter 3, is about the Bernstein–Sato polynomial or the b-function, which is an invariant of singularities that arises from the theory of $\mathcal{D}$-modules. This chapter is a joint work with Robin Walters, and one part of it has been written up in [4]. Let $\mathcal{D}$ be the ring of differential operators in n variables, with polynomial coefficients. Modules over this ring are called $\mathcal{D}$-modules, and were studied as an algebraic approach to the theory of linear partial differential equations. We will be most interested in the Bernstein–Sato polynomial or the b-function. Given any polynomial function f in n variables, the b-function of f is a certain polynomial associated to f, defined by Bernstein [6] and Sato–Shintani [52]. Let s be an auxiliary variable commuting with the original n variables, and let $\mathcal{D}[s]$ be the ring of differential operators in the original n variables whose coefficients may also depend polynomially on s. Then the b-function of f is the minimal, monic polynomial $b_f(s)$ that satisfies the following property: there is some $L(s) \in \mathcal{D}[s]$, such that for every natural number m, we have $L(m)f^{m+1} = b_f(m)f^m$. It can be interpreted as the minimal polynomial of an operator on a certain $\mathcal{D}$-module constructed from f. The very existence of b-functions is not evident from the definition, but this is settled by a theorem of Bernstein [6]. Many nice results are known about b-functions. For example, it is known by a theorem of Kashiwara [30] that for any f, all roots of $b_f(s)$ are negative rational numbers. Although defined algebraically in terms of the equation f, the b-function turns out to be an invariant of the hypersurface $V(f)$; it remains unchanged when f is transformed under an algebraic coordinate change. It is also related to other singularity invariants. For example, Malgrange [36, 37] showed that the roots of the b-function of f correspond, under the Riemann–Hilbert equivalence, to eigenvalues of the monodromy operator on the cohomology of the Milnor fiber of f.

Computer algorithms have been proved and implemented to compute b-functions (see, e.g.

[45, 44]). However, a general formula for b-functions is not known, and explicit answers are only known in isolated examples or very special families. Sato [51, 52] originally studied b-functions of semi-invariants of group actions on prehomogeneous vector spaces. We focus on b-functions of hyperplane arrangements, which is an active area of research. Several additional results and methods have been developed for this case (see, e.g., [57, 49, 50, 13, 12, 11]). In particular, we consider "Weyl arrangements", which are the hyperplane arrangements cut out by the roots of a semi-simple Lie algebra. The b-functions of these Weyl arrangements are not known in general. However, we can think about relationships with other singularity invariants. For example, the local topological zeta function is an invariant of hypersurface singularities defined by Denef and Loeser [17], which has a combinatorial description in terms of a resolution of singularities. In this context, the (weak) monodromy conjecture states that the poles of the local topological zeta function, when exponentiated, give the eigenvalues of the monodromy of the Milnor fiber. The strong topological monodromy conjecture states that poles of the local topological zeta function are already the roots of the b-function. This implies the weak monodromy conjecture by the theorem of Malgrange mentioned earlier. Budur–Mustaţă–Teitler [12] proved the weak monodromy conjecture for all hyperplane arrangements. We use additional results of results of Budur–Mustaţă–Teitler [12], together with a formula of Opdam [46] to prove the strong monodromy conjecture for all Weyl arrangements. In the case of the Lie algebra sl_n, the corresponding Weyl arrangement is cut out by the well-known Vandermonde determinant. We also prove an upper bound on the b-function of the Vandermonde determinant.

2

INTERSECTION COHOMOLOGY SHEAVES ON A T-VARIETY

2.1 Background on perverse sheaves and intersection cohomology

In this section we recall some basic definitions and background on perverse sheaves and intersection cohomology [22, 5]. Consider a complex algebraic variety X. We consider the bounded derived category of complexes of sheaves in the analytic topology on X with coefficients in $\mathbb{C}$. Consider the subcategory $D^b_c(X)$ of objects whose cohomology sheaves are constructible. That is, for any object $F^\bullet$ of this category, there is a stratification $X = \coprod_\alpha X_\alpha$, such that the restriction of any cohomology sheaf of $F^\bullet$ to any X_α is locally constant. Let $\mathbf{D}$ denote the Verdier duality functor from $D^b_c(X)$ to itself.

Definition 2.1.1. A *perverse sheaf* is an object of $F^\bullet$ of $D^b_c(X)$ that satisfies both of the following properties.

1. For any j, the complex dimension of the support of $\mathcal{H}^j(F^\bullet)$ is less than or equal to $-j$.
2. For any j, the complex dimension of the support of $\mathcal{H}^j(\mathbf{D}(F^\bullet))$ is less than or equal to $-j$.

For simplicity, we fix a complex topological stratification $\mathbb{S}$ on X (as defined in [22]). If $\dim_{\mathbb{C}} X = n$, let the stratification be described as $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$, where $\dim_{\mathbb{C}} X_i = i$. We work with the bounded derived category of cohomologically constructible sheaves with respect to $\mathbb{S}$. For a stratum S_α, denote by i_{S_α} the inclusion map of S_α into X. In this case, an object $F^\bullet$ is perverse if both of the following properties are true [25, Chapter 8].

1. For any stratum S_α and any integer j, the sheaf $\mathcal{H}^j(i_{S_\alpha}^{-1} F^\bullet)$ is zero whenever $j > -\dim_{\mathbb{C}}(S_\alpha)$.
2. For any stratum S_α and any integer j, the sheaf $\mathcal{H}^j(i_{S_\alpha}^{!} F^\bullet)$ is zero whenever $j < -\dim_{\mathbb{C}}(S_\alpha)$.

For each k, let $U_k = X \backslash X_{k-1}$, and let $j_k \colon U_k \to U_{k-1}$. Finally, let $\tau^{\leq k}$ be the truncation functor that truncates cohomology to degrees less than or equal to k. If $\mathcal{L}$ is any local system on $X_n \backslash X_{n-1}$, we can define the following object, which is perverse:

$$\mathrm{IC}(\mathcal{L}) = (\tau^{\leq -1} Rj_{1*})(\tau^{\leq -2} Rj_{2*}) \cdots (\tau^{\leq -n} Rj_{n*})(\mathcal{L}[n]).$$

This is called the *intersection cohomology* or IC sheaf corresponding to $\mathcal{L}$. Similarly, if $\mathcal{L}$ is any local system on some $X_k \backslash X_{k-1}$, then we can use the construction above to obtain an object on X_k. If $i_{X_k} \colon X_k \to X$ is the inclusion, we write $\mathrm{IC}(\mathcal{L})$ to mean the result of applying $Ri_{X_k!}$ to this object on X_k, and also call it an IC sheaf on X. We will be most interested in IC sheaves corresponding to simple local systems on various strata. If the strata are isomorphic to affine spaces, there are no non-trivial local systems. In that case there is a canonical IC sheaf supported on each X_k, namely the one obtained from the trivial local system on $X_k \backslash X_{k-1}$, and we call it the IC sheaf corresponding to X_k.

2.2 Introduction to the problem

Let X be a smooth complex projective variety together with an action of an algebraic torus T with isolated fixed points. We fix a regular algebraic one-parameter subgroup $\lambda \colon \mathbb{C}^* \to T$, which means that the set of λ-fixed points on X equals the set of T-fixed points on X (denoted X^T). Consider the Białynicki-Birula decomposition (see, e.g., [8]) of X: for each $w \in X^T$ define the *plus* and *minus* cells to be respectively

$$U_w = U_w^+ = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x = w\}, \quad t \in \mathbb{C}^*, \text{ and}$$

$$U_w^- = \{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x = w\}, \quad t \in \mathbb{C}^*.$$

Each plus or minus cell is a λ-stable affine space, and hence the decompositions $X = \coprod_{w\in X^T} U_w$ and $X = \coprod_{w\in X^T} U_w^-$ are cell decompositions. For the purposes of this chapter, we make the following additional assumptions on the T-action on X.

Assumption 2.2.1. The cell decompositions $X = \coprod_{w\in X^T} U_w$ and $X = \coprod_{w\in X^T} U_w^-$ are algebraic stratifications of X. In particular, the closure of every plus (resp. minus) cell is a union of plus (resp. minus) cells.

Assumption 2.2.2. For each $w \in X^T$, there is a one-parameter subgroup $\lambda_w\colon \mathbb{C}^* \to T$ and a neighbourhood V_w of w such that $\lim_{t\to 0} \lambda_w(t)\cdot v = w$ for every $v \in V_w$ and $t \in \mathbb{C}^*$.

Through most of this section, we use the word *sheaf* to mean an object in $D^b_{c,\mathrm{BB}}(X,\mathbb{C})$, the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on X that are constructible with respect to the Białynicki-Birula stratification. (Here we make use of Assumption 2.2.1.) Moreover all functors are derived, so for ease of notation we omit the decorations R and L.

For each $w \in X^T$, let IC_w denote the intersection cohomology sheaf on the closure of the cell U_w, extended by zero to all of X. The main theorem of this chapter describes the cohomology of the tensor products of a collection of IC_w, in terms of the tensor products of the cohomologies of the individual IC_w.

2.2.1 *Main result*

Let $\Delta\colon X \to X^m$ be the diagonal embedding. Consider any sheaves $\mathcal{F}_1,\dots,\mathcal{F}_m$ in $D^b_{c,\mathrm{BB}}(X,\mathbb{C})$. Then their (derived) tensor product is also a sheaf in $D^b_{c,\mathrm{BB}}(X,\mathbb{C})$, and will be denoted by $\mathcal{F}_1\otimes\cdots\otimes\mathcal{F}_m$. Recall that

$$\mathcal{F}_1\otimes\cdots\otimes\mathcal{F}_m = \Delta^{-1}(\mathcal{F}_1\boxtimes\cdots\boxtimes\mathcal{F}_m).$$

For any sheaf $\mathcal{F}$, its cohomology $H^\bullet(\mathcal{F}) = H^\bullet(X,\mathcal{F})$ is a graded vector space. There is a natural cup-product $\cup\colon H^\bullet(\mathcal{F}_1)\otimes\cdots\otimes H^\bullet(\mathcal{F}_m) \to H^\bullet(\mathcal{F}_1\otimes\cdots\otimes\mathcal{F}_m)$, defined in Subsection 2.3.2.

Let $\underline{\mathbb{C}}$ denote the constant sheaf on X. For any sheaf $\mathcal{F}$, its cohomology $H^\bullet(\mathcal{F})$ is naturally a (graded) left and right module over the (graded) ring $H(X) = H^\bullet(X, \underline{\mathbb{C}})$, as follows:

$$\cup\colon H(X) \otimes H^\bullet(\mathcal{F}) \to H^\bullet(\underline{\mathbb{C}} \otimes \mathcal{F}) \xrightarrow{\cong} H^\bullet(\mathcal{F}),$$
$$\cup\colon H^\bullet(\mathcal{F}) \otimes H(X) \to H^\bullet(\mathcal{F} \otimes \underline{\mathbb{C}}) \xrightarrow{\cong} H^\bullet(\mathcal{F}).$$

Moreover, the cup-product descends to a morphism

$$H^\bullet(\mathcal{F}_1) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet(\mathcal{F}_m) \to H^\bullet(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_m).$$

Theorem 2.2.3. *Let $(p_1, \ldots, p_m)$ be an m-tuple of T-fixed points of X. If the assumptions 2.2.1 and 2.2.2 hold, then the cup-product map*

$$H^\bullet\left(\mathrm{IC}_{p_1}\right) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet\left(\mathrm{IC}_{p_m}\right) \to H^\bullet\left(\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}\right) \tag{2.1}$$

is an isomorphism.

Since X is a T-space, each IC sheaf IC_{p_j} carries a canonical T-equivariant structure, and so does the tensor product $\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}$. Let $H_T(X) = H_T^\bullet(X, \mathbb{C})$ be the T-equivariant cohomology of X. For any T-equivariant sheaf $\mathcal{F}$ on X, its T-equivariant cohomology $H_T^\bullet(\mathcal{F}) = H_T^\bullet(X, \mathcal{F})$ is a graded $H_T(X)$-module. As before, there is a cup-product map for T-equivariant cohomology, which factors through $H_T(X)$.

Theorem 2.2.4. *Under the assumptions 2.2.1 and 2.2.2, the cup-product map*

$$H_T^\bullet\left(\mathrm{IC}_{p_1}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(\mathrm{IC}_{p_m}\right) \to H_T^\bullet\left(\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}\right)$$

is an isomorphism.

Remark 2.2.5. Even though our results are stated using IC sheaves, it is possible that they generalize to parity sheaves (defined and discussed by Juteau, Mautner, and Williamson in [28]). Our results and proof methods are similar to the main theorem from Ginzburg's paper [19]. In [1, Theorem 4.1], Achar and Rider prove a version of Ginzburg's theorem for parity sheaves on generalized flag varieties of a Kac-Moody group. Similar generalizations may work in our case as well.

2.3 Setup

2.3.1 *The Białynicki-Birula stratification.*

One can find (see, e.g. [55] or [29]) a T-equivariant projective embedding of X into some $\mathbb{P}^N$, such that the action of T on $\mathbb{P}^N$ is linear. Consider the following standard Morse-Bott function on $\mathbb{P}^N$:

$$[z_0 : \cdots : z_N] \mapsto \frac{\sum_{i=0}^N c_i |z_i|^2}{\sum_{i=0}^N |z_i|^2},$$

where c_i are the weights of the λ-action on $\mathbb{P}^N$. The critical sets of this function are precisely the T-fixed points on $\mathbb{P}^N$. The Morse-Bott cells of this function are locally closed algebraic subvarieties of $\mathbb{P}^N$. Since X has isolated T-fixed points, one can show that the composition $f : X \to \mathbb{P}^N \to \mathbb{R}$ is a Morse function with critical set X^T (see, e.g. [2]). Each cell of the Morse decomposition under f is a preimage of a Morse-Bott cell of $\mathbb{P}^N$. Hence it is a locally closed algebraic subvariety of X. Moreover, each cell of the Morse decomposition is known to be a union of Białynicki-Birula plus-cells. A discussion of this may also be found [15, Section 2.4].

The collection of fixed points of the λ-action carries a partial order, where $v < w$ if $U_v \subset \overline{U_w}$. By the previous discussion, we see that $v < w$ iff $f(v) < f(w)$. Fix a weakly increasing enumeration $\{0, 1, \ldots, N\}$ of the points of X^T (sometimes denoted $\{w_0, \ldots, w_N\}$), and set $X_n = \bigcup_{i \leq n} U_i$. Since the closure of every plus cell is a union of plus cells, it follows from the previous discussion that each X_n is a closed subvariety of X.

Similarly, set $X_n^- = \bigcup_{i \geq n} U_i^-$. By using the Morse function $(-f)$ instead of f, we see that each X_n^- is a closed subvariety of X. Hence we obtain two increasing filtrations of X by closed subvarieties: $X_0 \subset \cdots \subset X_N = X$ and $X_N^- \subset \cdots \subset X_0^- = X$.

We have the following inclusions:

$$X_n \overset{i_n}{\hookrightarrow} X, \quad X_{n-1} \overset{v}{\hookrightarrow} X_n \overset{u}{\hookleftarrow} U_n.$$

For any point $p \in X_n^-$, we have $f(w_n) \leq f(p)$, with equality only if $p \in X^T$. For any point $p \in X_n$, we have $f(p) \leq f(w_n)$, with equality only if $p \in X^T$. Hence if $p \in X_n^- \cap X_n$, then $f(p) = f(w_n)$, and $p \in X^T$. But $X_n^- \cap X_n \cap X^T = \{w_n\}$, and it follows that $p = w_n$. Hence for every n, the subvarieties X_n^- and X_n intersect transversally in the single point w_n.

Let $c_n \in H^\bullet(X)$ be the Poincaré dual to the homology class of X_n^-. As a vector space, $H^\bullet(X)$ is generated by the collection $\{c_n\}$. Finally, fix an m-tuple $(p_1, \ldots, p_m)$ of T-fixed points of X, and set $L_{j,n} = i_n^{-1} \mathrm{IC}_{p_j}$ for each j and n.

2.3.2 *The cup-product in cohomology*

Let $\pi\colon X \to \mathrm{pt}$ be the unique morphism to a point. For any sheaf $\mathcal{F}$ on X, its cohomology $H^\bullet(\mathcal{F})$ is a graded vector space, and may be thought of as $\pi_* \mathcal{F}$. We use this to define the cup-product map.

Recall that the functors (π^{-1}, π_*) form an adjoint pair, which has a counit $\pi^{-1} \circ \pi_* \to \mathrm{id}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be sheaves on X. Tensoring the counit maps together, we have a map

$$\pi^{-1} \circ \pi_*(\mathcal{F}_1) \otimes \cdots \otimes \pi^{-1} \circ \pi_*(\mathcal{F}_m) \to \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_m.$$

The left hand side is canonically isomorphic to $\pi^{-1}(\pi_* \mathcal{F}_1 \otimes \cdots \otimes \pi_* \mathcal{F}_m)$. Using the (π^{-1}, π_*) adjunction once more, we obtain the *cup-product*:

$$\cup\colon \pi_* \mathcal{F}_1 \otimes \cdots \otimes \pi_* \mathcal{F}_m \to \pi_*(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_m).$$

The cup-product gives each $H^\bullet(\mathcal{F}_i)$ the structure of a left and right module over $H(X)$. This module structure induces the following map, also called the cup-product:

$$H^\bullet(\mathcal{F}_1) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet(\mathcal{F}_m) \to H^\bullet(\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_m).$$

Proposition 2.3.1. *For every n, the cup-product map*

$$H^\bullet\left(L_{1,n}\right) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet\left(L_{m,n}\right) \to H^\bullet\left(L_{1,n} \otimes \cdots \otimes L_{m,n}\right) \tag{2.2}$$

is an isomorphism.

When $X_n = X$, we have $L_{j,n} = \mathrm{IC}_{p_j}$ for each j. Hence Theorem 2.2.3 follows from this proposition, and we now focus on proving the proposition.

2.4 Proof of the isomorphism

We prove Proposition 2.3.1 by induction on the nth filtered piece of $X_0 \subset \cdots \subset X_N$. In the base case of $n = 0$, the space X_0 is zero-dimensional. Hence each sheaf $L_{j,0}$ is isomorphic to its cohomology. In this case the cup-product map (2.2) reduces to the identity map, which is an isomorphism.

Now we prove the induction step on the filtered piece X_n. We mainly use the following distinguished triangles:

$$u_! u^{-1} L_{j,n} \to L_{j,n} \to v_* v^{-1} L_{j,n}, \tag{2.3}$$

$$v_! v^! L_{j,n} \to L_{j,n} \to u_* u^{-1} L_{j,n}. \tag{2.4}$$

After taking cohomology, each of the above distinguished triangles produces a long exact sequence. In our case, all connecting homomorphisms of these long exact sequences vanish (see, e.g. [53, Lemma 20] and [19, Proposition 3.2]).

For brevity, we will use the following notation through the remainder of the chapter.

$$\begin{aligned} M_{m,n} &= L_{2,n} \otimes \cdots \otimes L_{m,n}, \\ A_{m,n} &= H^\bullet\left(L_{2,n}\right) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet\left(L_{m,n}\right), \\ B_{m,n} &= H^\bullet\left(u_* u^{-1} L_{2,n}\right) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{m,n}\right). \end{aligned} \tag{2.5}$$

The following two lemmas prove the proposition on the open part U_n in X_n.

Lemma 2.4.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be any complexes of sheaves on U_n with locally constant cohomology sheaves. Then the cup-product map*

$$\cup\colon H^\bullet(u_! \mathcal{F}) \otimes H^\bullet(u_* \mathcal{G}) \to H^\bullet(u_! \mathcal{F} \otimes u_* \mathcal{G})$$

is an isomorphism. Since $\cup$ factors through the surjection

$$H^\bullet(u_! \mathcal{F}) \otimes H^\bullet(u_* \mathcal{G}) \twoheadrightarrow H^\bullet(u_! \mathcal{F}) \underset{H(X)}{\otimes} H^\bullet(u_* \mathcal{G}),$$

the induced cup-product

$$\cup\colon H^\bullet(u_! \mathcal{F}) \underset{H(X)}{\otimes} H^\bullet(u_* \mathcal{G}) \to H^\bullet(u_! \mathcal{F} \otimes u_* \mathcal{G})$$

is also an isomorphism.

Proof. Consider the following commutative diagram, where π is the projection to a point.

$$\begin{array}{ccc} U_n & \xrightarrow{\ u\ } & X_n \\ & \searrow^{p=\pi\circ u} & \downarrow \pi \\ & & \mathrm{pt} \end{array}$$

Recall that if A and B are any two complexes on X, then the cup-product is induced by adjunction from the natural map

$$\pi^{-1}(\pi_* A \otimes \pi_* B) \cong \pi^{-1}\pi_* A \otimes \pi^{-1}\pi_* B \to A \otimes B,$$

which may be broken up as follows:

$$\pi^{-1}\pi_* A \otimes \pi^{-1}\pi_* B \to A \otimes \pi^{-1}\pi_* B \to A \otimes B.$$

Therefore the cup-product map may be broken up as follows:

$$\pi_* A \otimes \pi_* B \to \pi_*(A \otimes \pi^{-1}\pi_* B) \to \pi_*(A \otimes B).$$

In our case, this becomes the following sequence of maps:

$$\pi_* u_! \mathcal{F} \otimes \pi_* u_* \mathcal{G} \xrightarrow{\mu_1} \pi_*(u_! \mathcal{F} \otimes \pi^{-1}\pi_* u_* \mathcal{G}) \xrightarrow{\mu_2} \pi_*(u_! \mathcal{F} \otimes u_* \mathcal{G}).$$

Since π is a proper map, we know that $\pi_* \cong \pi_!$, and hence μ_1 is an isomorphism by the projection formula. It remains to show that μ_2 is an isomorphism.

The pair of adjoint functors (π^{-1}, π_*) gives the counit morphism $p^{-1}p_*\mathcal{G} \to u^{-1}u_*\mathcal{G}$. The key observation is that this map is an isomorphism, because $\mathcal{G}$ is a direct sum of its cohomology sheaves on the affine space U_n. Now consider the following commutative diagram.

$$\begin{array}{ccc} u_! \mathcal{F} \otimes \pi^{-1}\pi_* u_* \mathcal{G} & \xrightarrow[\text{(proj.)}]{\cong} & u_!(\mathcal{F} \otimes p^{-1} p_* \mathcal{G}) \\ {\scriptstyle \mu_2} \downarrow {\scriptstyle \text{(counit)}} & & {\scriptstyle \cong} \downarrow {\scriptstyle \text{(counit)}} \\ u_! \mathcal{F} \otimes u_* \mathcal{G} & \xrightarrow[\text{(proj.)}]{\cong} & u_!(\mathcal{F} \otimes u^{-1} u_* \mathcal{G}) \end{array} \tag{2.6}$$

The map μ_2 is obtained by applying the functor π_* to the left vertical map in (2.6) above. The diagram shows that this map is an isomorphism, and hence μ_2 is also an isomorphism. □

Lemma 2.4.2. *The cup-product map induces an isomorphism*

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n} \xrightarrow{\cong} H_c^\bullet\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right).$$

Proof of lemma. Using Lemma 2.4.1 for $\mathcal{F} = u^{-1}L_{1,n}$ and $\mathcal{G} = u^{-1}L_{2,n}$, we obtain an isomorphism

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{2,n}\right) \xrightarrow{\cong} H^\bullet\left(u_! u^{-1} L_{1,n} \otimes u_* u^{-1} L_{2,n}\right).$$

Moreover, we know that $u^{-1}u_*u^{-1}L_{2,n} \cong u^{-1}L_{2,n}$. Using this fact and the projection formula, we have

$$\begin{aligned} H^\bullet\left(u_! u^{-1} L_{1,n} \otimes u_* u^{-1} L_{2,n}\right) &\cong H^\bullet\left(u_!\left(u^{-1} L_{1,n} \otimes u^{-1} u_* u^{-1} L_{2,n}\right)\right) \\ &\cong H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes L_{2,n})\right). \end{aligned}$$

All together, we get an isomorphism

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{2,n}\right) \xrightarrow{\cong} H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes L_{2,n})\right),$$

which can be written in our previously-introduced notation as

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{2,n} \xrightarrow{\cong} H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes M_{2,n})\right).$$

Now we can successively tensor the above map over $H(X)$ with the spaces $H^\bullet\left(u_* u^{-1} L_{i,n}\right)$, with i ranging from 3 to m. Each time, we apply Lemma 2.4.1 for $\mathcal{F} = u^{-1}(L_{1,n} \otimes M_{i-1,n})$ and

$\mathcal{G} = u^{-1}L_{i,n}$ and use the argument above. Ultimately this construction yields

$$\begin{aligned}
H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n} &\xrightarrow{\cong} H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes M_{m-1,n})\right) \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{m,n}\right) \\
&\xrightarrow{\cong} H^\bullet\left(u_!(u^{-1}(L_{1,n} \otimes M_{m,n}))\right) \\
&\cong H_c^\bullet\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right).
\end{aligned}$$

□

The next lemma is a refinement of a standard cohomology exact sequence to our particular case.

Lemma 2.4.3. *There is an exact sequence*

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n} \to H^\bullet\left(L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to 0.$$

Proof. Consider the distinguished triangle (2.3) for the sheaf $L_{1,n}$. Taking cohomology and applying the functor $(-) \underset{H(X)}{\otimes} A_{m,n}$, we obtain the right-exact sequence

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \xrightarrow{f} H^\bullet\left(L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \xrightarrow{g} H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to 0.$$

Using the distinguished triangles (2.4) for each of the sheaves $L_{j,n}$ for $j \geq 2$, we have surjective morphisms

$$H^\bullet\left(L_{j,n}\right) \twoheadrightarrow H^\bullet\left(u_* u^{-1} L_{j,n}\right).$$

Taking the tensor product of all of these along with $H^\bullet\left(u_! u^{-1} L_{1,n}\right)$, we obtain a surjective morphism

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \overset{h}{\twoheadrightarrow} H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n}.$$

We now show that the map f factors through the map h, by showing that $f(\ker h) = 0$. Since all boundary maps in the cohomology long exact sequence of the triangles (2.4) vanish, the following set generates $\ker h$:

$$\{a_1 \otimes a_2 \otimes \cdots \otimes a_n \mid a_j \in H^\bullet\left(v_* v^! L_{j,n}\right) \text{ for some } 2 \le j \le m\}.$$

Consider any element $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \ker h$. Suppose that $a_j \in H^\bullet\left(v_* v^! L_{j,n}\right)$. Recall the following commutative diagram, which is the content of [19, 3.8a].

$$\begin{array}{ccccc} H^\bullet\left(v_* v^! L_{j,n}\right) & \hookrightarrow & H^\bullet\left(L_{j,n}\right) & \twoheadrightarrow & H^\bullet\left(u^{-1} L_{j,n}\right) \\ & & \downarrow c_n & & \downarrow c_n \, \cong \\ & & H^\bullet\left(L_{j,n}\right) & \hookleftarrow & H^\bullet_c\left(u^{-1} L_{j,n}\right) \end{array}$$

From this diagram it follows that $c_n a_j = 0$, and that $a_1 \in c_n H^\bullet\left(L_{1,n}\right)$. Since all tensor products are over $H(X)$, the image of $h(a_1 \otimes \cdots \otimes a_n)$ under f must be zero. Therefore f factors through h, and we obtain the desired short exact sequence. □

Finally, we use the induction hypothesis to tackle the right side of the right-exact sequence from the previous lemma.

Lemma 2.4.4. *The cup-product map induces an isomorphism*

$$H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \xrightarrow{\cong} H^\bullet\left(L_{1,n-1} \otimes M_{m,n-1}\right).$$

Proof of lemma. The cup-product map on the left hand side is the following composition:

$$H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(M_{m,n}\right) \to H^\bullet\left(v_* v^{-1} L_{1,n} \otimes M_{m,n}\right),$$

where the first map is the cup-product on the last $(m-1)$ factors, and the second map is the cup-product of the first factor with the rest. The projection formula also shows that

$$H^\bullet\left(v_* v^{-1} L_{1,n} \otimes M_{m,n}\right) \cong H^\bullet\left(v^{-1} L_{1,n} \otimes v^{-1} M_{m,n}\right) \cong H^\bullet\left(L_{1,n-1} \otimes M_{m,n-1}\right).$$

By induction on m, we may assume that the cup-product $A_{m,n} \to H^\bullet\left(M_{m,n}\right)$ is an isomorphism, and hence the first map above is an isomorphism. It remains to show that the following map is an isomorphism:

$$H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(M_{m,n}\right) \to H^\bullet\left(v_* v^{-1} L_{1,n} \otimes M_{m,n}\right)$$

The element $c_n \in H$ acts on $H^\bullet\left(v_* L_{1,n-1}\right)$ by zero, since $L_{1,n-1}$ is supported on X_{n-1}. Recall from [19] that the cokernel of c_n on $H^\bullet\left(M_{m,n}\right)$ is just $H^\bullet\left(M_{m,n-1}\right)$. Hence

$$H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(M_{m,n}\right) \cong H^\bullet\left(L_{1,n-1}\right) \underset{H(X)}{\otimes} H^\bullet\left(M_{m,n-1}\right).$$

Hence the map above can be rewritten as the cup-product map

$$H^\bullet\left(L_{1,n-1}\right) \underset{H(X)}{\otimes} H^\bullet\left(M_{m,n-1}\right) \to H^\bullet\left(L_{1,n-1} \otimes M_{m,n-1}\right),$$

which is an isomorphism by the induction hypothesis. □

We now apply Saito's theory of mixed Hodge modules ([48, 47]) to obtain another short exact sequence, as follows. Every IC-sheaf has the additional structure of a pure mixed Hodge module, which induces a mixed Hodge structure on tensor products of the $L_{i,n}$.

Lemma 2.4.5.

(i) The cohomology $H^\bullet\left(L_{1,n} \otimes M_{m,n}\right)$ is pure.

(ii) There is a short exact sequence

$$0 \to H_c^\bullet\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right) \to H^\bullet\left(L_{1,n} \otimes M_{m,n}\right) \to H^\bullet\left(L_{1,n-1} \otimes M_{m,n-1}\right) \to 0.$$

Proof. The proof is by induction on n. When $n = 0$, we have $X_{-1} = \emptyset$ and $U = X_0$. The open inclusion u is the identity map, and the closed inclusion v is the zero map, hence (ii) is clear in the base case.

The set X_0 consists of a single, T-fixed point of X. Call this point w. By Assumption 2.2.2, there exists a neighborhood V_w of w and a one-parameter subgroup $\lambda_w\colon \mathbb{C}^* \to T$ that contracts V_w to w. Let i_w denote the inclusion of $\{w\}$ into the corresponding V_w. Let j_w denote the inclusion of V_w into X. By applying [54, Corollary 1] or [9, Lemma 6] to the sheaves $j_w^{-1}\,\mathrm{IC}_{p_i}$ for each i, we see that

$$H^\bullet\left(V_w, j_w^{-1}\,\mathrm{IC}_{p_i}\right) \cong H^\bullet\left(i_w^{-1} j_w^{-1}\,\mathrm{IC}_{p_i}\right) = H^\bullet(L_{i,0}).$$

The functor $H^\bullet(V_w, j_w^{-1}(-))$ weakly increases weights, while the functor $H^\bullet(i_w^{-1} j_w^{-1}(-))$ weakly decreases weights. Hence $H^\bullet(L_{i,0})$ is pure for each i. Taking the tensor product, we see that $H^\bullet(L_{1,0}) \otimes \cdots \otimes H^\bullet(L_{m,0})$ is pure. Since w is a single point, we can naturally make the following identification:

$$H^\bullet(L_{1,0}) \otimes \cdots \otimes H^\bullet(L_{m,0}) \cong H^\bullet\left(L_{1,0} \otimes \cdots \otimes L_{m,0}\right) = H^\bullet\left(L_{1,0} \otimes M_{m,0}\right).$$

Hence $H^\bullet\left(L_{1,0} \otimes M_{m,0}\right)$ is pure, and (i) is proved in the base case. A similar argument has been used in Lemma 3.5 of [19].

For the induction step, consider the distinguished triangle (2.3) for $L_{1,n}$. Apply the functor $\left(- \otimes L_{2,n} \otimes \cdots \otimes L_{m,n}\right)$, which may be written as $\left(- \otimes M_{m,n}\right)$ in the notation of (2.5). This yields the following distinguished triangle:

$$u_!u^{-1}L_{1,n}\otimes M_{m,n}\to L_{1,n}\otimes M_{m,n}\to v_*v^{-1}L_{1,n}\otimes M_{m,n}.$$

By a repeated application of the projection formula, we may write the first term of this triangle as

$$u_!u^{-1}L_{1,n}\otimes M_{m,n}\cong u_!\left(u^{-1}L_{1,n}\otimes\cdots\otimes u^{-1}L_{m,n}\right)=u_!u^{-1}\left(L_{1,n}\otimes M_{m,n}\right),$$

and the third term of this triangle as

$$v_*v^{-1}L_{1,n}\otimes M_{m,n}\cong v_*\left(v^{-1}L_{1,n}\otimes\cdots\otimes v^{-1}L_{m,n}\right)=v_*\left(L_{1,n-1}\otimes M_{m,n-1}\right).$$

Taking cohomology, we obtain the following long exact sequence:

$$\cdots\to H_c^\bullet\left(u^{-1}(L_{1,n}\otimes M_{m,n})\right)\to H^\bullet\left(L_{1,n}\otimes M_{m,n}\right)\to H^\bullet\left(L_{1,n-1}\otimes M_{m,n-1}\right)\to\cdots.$$

The term $H^\bullet\left(L_{1,n-1}\otimes M_{m,n-1}\right)$ is pure by the induction hypothesis.

From Lemma 2.4.2, we know that

$$H_c^\bullet\left(u^{-1}\left(L_{1,n}\otimes M_{m,n}\right)\right)\cong H_c^\bullet(u^{-1}L_{1,n})\underset{H(X)}{\otimes}H^\bullet(u^{-1}L_{2,n})\underset{H(X)}{\otimes}\cdots\underset{H(X)}{\otimes}H^\bullet(u^{-1}L_{m,n}).$$

Recall that U_n is the Białynicki-Birula plus-cell for the fixed point w_n. Hence the λ-action contracts U_n to w_n. By [54, Corollary 2], we know that $H_c^\bullet\left(u^{-1}L_{1,n}\right)$ is isomorphic to the costalk of $u^{-1}L_{1,n}$ at w_n, which is isomorphic to a shift of the stalk of IC_{p_1} at w_n. For any $i>1$, we know by [54, Corollary 1] that $H^\bullet\left(u^{-1}L_{i,n}\right)$ is isomorphic to the stalk of $u^{-1}L_{i,n}$ at w_n, which is equal to the stalk of IC_{p_i} at w_n. By using Assumption 2.2.2 and the argument used earlier in this proof, we know that the stalk of each IC_{p_i} at any T-fixed point is pure, and hence the spaces $H_c^\bullet\left(u^{-1}L_{1,n}\right)$ as well as $H^\bullet\left(u^{-1}L_{i,n}\right)$ for $i>1$ are all pure. Therefore the tensor product

$H_c^\bullet\left(u^{-1}\left(L_{1,n}\otimes M_{m,n}\right)\right)$ is pure.

Since the terms on either side of the long exact sequence are pure, the connecting homomorphisms are zero, and hence $H^\bullet\left(L_{1,n}\otimes M_{m,n}\right)$ is also pure. This argument completes the induction step, and hence completes the proof. □

Putting together the exact sequences from Lemma 2.4.3 and Lemma 2.4.5, we obtain the following commutative diagram, where the vertical maps are induced by cup-products. In particular, the middle map b is just the map from Proposition 2.3.1.

$$\begin{array}{ccccccc} & H^\bullet\left(u_!u^{-1}L_{1,n}\right)\underset{H(X)}{\otimes}B_{m,n} & \to & H^\bullet\left(L_{1,n}\right)\underset{H(X)}{\otimes}A_{m,n} & \to & H^\bullet\left(v_*v^{-1}L_{1,n}\right)\underset{H(X)}{\otimes}A_{m,n} & \to 0 \\ & \downarrow a & & \downarrow b & & \downarrow c & \\ 0\longrightarrow & H_c^\bullet\left(u^{-1}(L_{1,n}\otimes M_{m,n})\right) & \longrightarrow & H^\bullet\left(L_{1,n}\otimes M_{m,n}\right) & \longrightarrow & H^\bullet\left(L_{1,n-1}\otimes M_{m,n-1}\right) & \longrightarrow 0 \end{array} \tag{2.7}$$

The leftmost map a is an isomorphism by Lemma 2.4.2. The rightmost map c is an isomorphism by Lemma 2.4.4. By the snake lemma, the middle map b is an isomorphism as well, and Proposition 2.3.1 is proved.

2.5 Computation of equivariant cohomology

Consider a smooth complex projective variety X with the same assumptions as in Section 2.2. The goal of this section is to prove Theorem 2.2.4.

First we recall some constructions in equivariant cohomology. The main references are [7] and [20]. Fix a universal principal T-bundle $ET\to BT$, where ET (respectively BT) is the direct limit over m of algebraic approximations ET_m (respectively BT_m). Consider the following diagram, where the map p is the second projection, and the map q is the quotient by the diagonal T-action.

$$\begin{array}{ccc} & ET \times X & \\ & {\scriptstyle p} \swarrow \quad \searrow {\scriptstyle q} & \\ X & & ET \times_T X \end{array}$$

Since each stratum U_n is a locally closed T-invariant affine subvariety of X, the trivial local system on U_n gives rise to a canonically-defined sheaf $\overline{\mathrm{IC}_n}$ on $ET \times_T X$, and a canonical isomorphism $\beta\colon p^{-1}\,\mathrm{IC}_n \xrightarrow{\cong} q^{-1}\overline{\mathrm{IC}_n}$ (see, e.g., [7]). The triple $(\mathrm{IC}_n, \overline{\mathrm{IC}_n}, \beta)$ is called the equivariant IC sheaf corresponding to U_n.

2.5.1 *Equivariant homology and cohomology*

For any variety Y equipped with a T-action, the cohomology of $ET \times_T Y$ is called the *equivariant cohomology* of Y, and is denoted by $H_T^\bullet(Y)$. In particular, since $ET \times_T \mathrm{pt} \cong BT$, we have $H_T^\bullet(\mathrm{pt}) \cong H^\bullet(BT)$. The space $H_T^\bullet(Y)$ is a ring under cup-product, and is also an $H_T(X)$-module via pullback under the projection $Y \to \mathrm{pt}$. For convenience, we will denote $H_T^\bullet(X)$ by $H_T(X)$. In our case, $H_T(X)$ is isomorphic to $H^\bullet(X) \otimes H^\bullet(BT)$ as an $H_T(X)$-module (see, e.g., [20, Theorem 14.1]). Similarly, the equivariant cohomology of any T-equivariant sheaf on X also carries an $H_T(X)$-module structure.

One can define the T-equivariant Borel-Moore homology of X, denoted $H_\bullet^T(X)$. Every T-equivariant closed subvariety Y of X defines a class $[Y]_T$ of degree $2\dim_{\mathbb{C}} Y$ in $H_\bullet^T(X)$. If X is smooth, then every class $[Y]_T$ has an equivariant Poincaré dual cohomology class in $H_T^\bullet(X)$. More details can be found in [23] and [10].

2.5.2 *Proof of the equivariant case*

Consider an m-tuple $(p_1, \ldots, p_m)$ of T-fixed points of X. Then $\mathrm{IC}_{p_1}, \ldots, \mathrm{IC}_{p_m}$ are the IC sheaves corresponding to $U_{p_1}, \ldots, U_{p_m}$ respectively. Let $L_{j,n} = i_n^{-1}\,\mathrm{IC}_{p_j}$ for each j and n.

Proposition 2.5.1. *Under the assumptions 2.2.1 and 2.2.2, the cup-product maps*

$$H_T^\bullet\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(L_{m,n}\right) \to H_T^\bullet\left(L_{1,n} \otimes \cdots \otimes L_{m,n}\right)$$

are isomorphisms for each n.

When $X_n = X$, we have $L_{j,n} = \mathrm{IC}_{p_j}$ for each j. Hence this proposition implies Theorem 2.2.4. To prove the proposition, we first state two general lemmas about T-equivariant cohomology of sheaves.

Lemma 2.5.2. *Consider the fiber bundle $ET \times_T X \to BT$, with fiber X. Let IC_w be the (T-equivariant) IC sheaf on the closure of a stratum X_w, extended by zero to all of X. Then the Leray spectral sequence for the computation of $H_T^\bullet(X; \mathrm{IC}_w) = H^\bullet\left(ET \times_T X; \overline{\mathrm{IC}_w}\right)$ collapses at the E_2-page. Hence $H_T^\bullet(\mathrm{IC}_w)$ is isomorphic to $H^\bullet(\mathrm{IC}_w) \otimes H^\bullet(BT)$ as a graded $H^\bullet(BT)$-module.*

Proof. See [20, Theorem 14.1]. The proof uses the fact that the cohomology of $BT \cong (\mathbb{CP}^\infty)^{\dim T}$ is pure. □

Lemma 2.5.3. *Let Y be any T-space, and let $\mathcal{F}$ be a T-equivariant sheaf on Y such that the space $H^\bullet(Y; \mathcal{F})$ is pure. Then $H_T^\bullet(Y; \mathcal{F})$ is pure as well.*

Proof. Recall that $H_T^\bullet(Y, \mathcal{F}) = H^\bullet\left(ET \times_T X, \overline{\mathcal{F}}\right)$. The result follows from computing the Leray spectral sequence for the fiber bundle $ET \times_T Y \to BT$, and by using that $H^\bullet(BT)$ and $H^\bullet(Y, \mathcal{F})$ are pure. □

We also record some equivariant analogues of results stated in Section 2.4. First note that the boundary maps in the long exact sequences of T-equivariant cohomology for the distinguished triangles (2.3) and (2.4) vanish. The proof is analogous to the non-equivariant case, using Lemma 2.5.3.

The following lemma is an analogue of Lemma 2.4.1.

Lemma 2.5.4. *Let $U = X_n \backslash X_{n-1}$. Let $\mathcal{F}$ and $\mathcal{G}$ be any T-equivariant complexes of sheaves on U. Then the cup-product map*

$$\cup\colon H_T^\bullet(u_!\mathcal{F}) \underset{H^\bullet(BT)}{\otimes} H_T^\bullet(u_*\mathcal{G}) \to H_T^\bullet(u_!\mathcal{F} \otimes u_*\mathcal{G})$$

is an isomorphism. Since $\cup$ factors through the surjection

$$H_T^\bullet(u_!\mathcal{F}) \underset{H^\bullet(BT)}{\otimes} H_T^\bullet(u_*\mathcal{G}) \twoheadrightarrow H_T^\bullet(u_!\mathcal{F}) \underset{H_T(X)}{\otimes} H_T^\bullet(u_*\mathcal{G}),$$

the induced cup-product

$$H_T^\bullet(u_!\mathcal{F}) \underset{H_T(X)}{\otimes} H_T^\bullet(u_*\mathcal{G}) \to H_T^\bullet(u_!\mathcal{F} \otimes u_*\mathcal{G})$$

is also an isomorphism.

Proof. Consider the fiber bundle $ET \times_T X_n \to BT$, with fiber X_n. The E_2 pages of the Leray spectral sequences for $u_!\mathcal{F}$ and $u_*\mathcal{G}$ are as follows:

$$H^p(BT, H^q(u_!\mathcal{F})) \implies H_T^{p+q}(u_!\mathcal{F}),$$
$$H^r(BT, H^s(u_*\mathcal{G})) \implies H_T^{r+s}(u_*\mathcal{G}).$$

On the E_2 page, the cup-product map can be written as the composition of the following two maps. The first map is the cup-product with local coefficients, and the second is the fiber-wise cup-product on the local systems.

$$H^p(BT, H^q(u_!\mathcal{F})) \underset{H^\bullet(BT)}{\otimes} H^r(BT, H^s(u_*\mathcal{G})) \to H^{p+r}(BT, H^q(u_!\mathcal{F}) \otimes H^s(u_*\mathcal{G})),$$
$$H^{p+r}(BT, H^q(u_!\mathcal{F}) \otimes H^s(u_*\mathcal{G})) \to H^{p+r}(BT, H^{q+s}(u_!\mathcal{F} \otimes u_*\mathcal{G})).$$

Since the local systems $H^q(u_! \mathcal{F})$ and $H^s(u_* \mathcal{G})$ are constant on BT, the first map yields isomorphisms

$$H^\bullet(BT, H^q(u_! \mathcal{F})) \underset{H^\bullet(BT)}{\otimes} H^\bullet(BT, H^s(u_* \mathcal{G})) \xrightarrow{\cong} H^\bullet(BT, H^q(u_! \mathcal{F}) \otimes H^s(u_* \mathcal{G})).$$

Finally, we know from Lemma 2.4.1 that $H^\bullet(u_! \mathcal{F}) \otimes H^\bullet(u_* \mathcal{G}) \xrightarrow{\cong} H^\bullet(u_! \mathcal{F} \otimes u_* \mathcal{G})$ via the cup-product map. Altogether, the cup-product maps on the E_2 page yield an isomorphism

$$H^\bullet(BT, H^\bullet(u_! \mathcal{F})) \underset{H^\bullet(BT)}{\otimes} H^\bullet(BT, H^\bullet(u_* \mathcal{G})) \xrightarrow{\cong} H^\bullet(BT, H^\bullet(u_! \mathcal{F} \otimes u_* \mathcal{G})).$$

The left hand side is a tensor product of two free $H^\bullet(BT)$-modules over $H^\bullet(BT)$. Hence it converges to $H_T^\bullet(u_! \mathcal{F}) \underset{H^\bullet(BT)}{\otimes} H_T^\bullet(u_* \mathcal{G})$. The right hand side converges to $H_T^\bullet(u_! \mathcal{F} \otimes u_* \mathcal{G})$. Since the E_2 pages of the left hand side and the right hand side are isomorphic via the cup-product map, the following cup-product map

$$H_T^\bullet(u_! \mathcal{F}) \underset{H^\bullet(BT)}{\otimes} H_T^\bullet(u_* \mathcal{G}) \to H_T^\bullet(u_! \mathcal{F} \otimes u_* \mathcal{G})$$

is an isomorphism. □

Let $\widetilde{c_n} \in H_T(X)$ be the equivariant Poincaré dual of $[X_n^-]_T$. Each $\widetilde{c_n}$ restricts to the class c_n under the map $H_T(X) \to H^\bullet(X)$, hence the collection $\{\widetilde{c_n}\}$ generates $H_T(X)$ over $H^\bullet(BT)$.

The following lemma (analogous to [19, 3.8a]) describes the action of $\widetilde{c_n}$ on the equivariant cohomology of the sheaves $L_{j,n}$ on X.

Lemma 2.5.5. *For every j, the action of $\widetilde{c_n}$ on $H_T^\bullet\left(L_{j,n}\right)$ fits into the following commutative diagram:*

$$\begin{array}{ccc} H_T^\bullet\left(L_{j,n}\right) & \twoheadrightarrow & H_T^\bullet\left(u^{-1} L_{j,n}\right) \\ \downarrow \widetilde{c_n} & & \downarrow \widetilde{c_n}\ \cong \\ H_T^\bullet\left(L_{j,n}\right) & \hookleftarrow & H_{T,c}^\bullet(u^{-1} L_{j,n}) \end{array}$$

Proof. Recall that the intersection of X_n and X_n^- lies away from X_{n-1}. Hence $\widetilde{c}_n$ restricts to zero on X_{n-1}, and cup-product by $\widetilde{c}_n$ annihilates the cohomology of any sheaf supported on X_{n-1}. The kernel of $H_T^\bullet\left(L_{j,n}\right) \twoheadrightarrow H_T^\bullet\left(u^{-1}L_{j,n}\right)$ and the cokernel of $H_{T,c}^\bullet(u^{-1}L_{j,n}) \to H_T^\bullet\left(L_{j,n}\right)$ are both supported on X_{n-1}. So the map of multiplication by $\widetilde{c}_n$ from $H_T^\bullet(X_n)$ to $H_T^\bullet(X_n)$ factors as follows.

$$\begin{array}{ccc} H_T^\bullet\left(L_{j,n}\right) & \twoheadrightarrow & H_T^\bullet\left(u^{-1}L_{j,n}\right) \\ \widetilde{c}_n \downarrow & & \widetilde{c}_n \downarrow \\ H_T^\bullet\left(L_{j,n}\right) & \hookleftarrow & H_{T,c}^\bullet(u^{-1}L_{j,n}) \end{array}$$

It remains to show that the vertical map on the right is an isomorphism. Since X_n and X_n^- intersect transversally in the single point w_n, the restriction of $\widetilde{c}_n$ to X_n is the image in $H_T^\bullet(X_n)$ of a generator of the local cohomology group $H_T^\bullet(X_n, X_n \backslash \{w_n\})$.

Since $w_n \in U_n$, we have $H_T^\bullet(X_n, X_n\backslash\{w_n\}) \cong H_T^\bullet(U_n, U_n\backslash\{w_n\})$ by excision. But U_n is an affine space that is T-equivariantly contractible to w_n, and hence $H_T^\bullet(U_n, U_n\backslash\{w_n\}) \cong H_{T,c}^\bullet(U_n)$. This shows that multiplication by $\widetilde{c}_n$ maps $H_T^\bullet(U_n)$ isomorphically to $H_{T,c}^\bullet(U_n)$.

Since $u^{-1}L_{j,n}$ is T-equivariant, the above argument applies to the cohomology of $u^{-1}L_{j,n}$ as well. This means that $\widetilde{c}_n$ maps $H_T^\bullet\left(u^{-1}L_{j,n}\right)$ isomorphically to $H_{T,c}^\bullet(u^{-1}L_{j,n})$, and the proof is complete. $\square$

Once again, let $M_{m,n}$ denote the sheaf $L_{2,n} \otimes \cdots \otimes L_{m,n}$. For brevity, we set up the following additional notation.

$$\overline{A}_{m,n} = H_T^\bullet\left(L_{2,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(L_{m,n}\right),$$

$$\overline{B}_{m,n} = H_T^\bullet\left(u_* u^{-1}L_{2,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(u_* u^{-1}L_{m,n}\right).$$

The following two lemmas are analogues of Lemma 2.4.3 and Lemma 2.4.5 respectively.

Lemma 2.5.6. *There is an exact sequence*

$$H_T^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{B}_{m,n} \to H_T^\bullet\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} \to H_T^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} \to 0.$$

Proof. The proof is analogous to the proof of Lemma 2.4.3. We use the fact that $H_T^\bullet(X) \cong H^\bullet(X) \otimes H^\bullet(BT)$, and use Lemma 2.5.5 as a substitute for [19, 3.8a]. □

Lemma 2.5.7.

(i) The cohomology $H_T^\bullet\left(L_{1,n} \otimes M_{m,n}\right)$ is pure.

(ii) There is a short exact sequence

$$0 \to H_{T,c}^\bullet\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right) \to H_T^\bullet\left(L_{1,n} \otimes M_{m,n}\right) \to H_T^\bullet\left(L_{1,n-1} \otimes M_{m,n-1}\right) \to 0.$$

Proof. The proofs are analogous to the proofs of their counterparts from Section 2.4, using the observation of Lemma 2.5.3 and the fact that $H^\bullet(BT)$ is pure. □

We now complete the proof of Theorem 2.2.4.

Proof of Theorem 2.2.4. We obtain the following commutative diagram from the exact sequences of Lemma 2.5.6 and Lemma 2.5.7.

$$\begin{array}{ccccccccc}
 & & H_T^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{B}_{m,n} & \to & H_T^\bullet\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} & \to & H_T^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} & \to & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & H_T^\bullet\left(u_! u^{-1} L_{1,n} \otimes M_{m,n}\right) & \longrightarrow & H_T^\bullet\left(L_{1,n} \otimes M_{m,n}\right) & \longrightarrow & H_T^\bullet\left(v_* v^{-1} L_{1,n} \otimes M_{m,n}\right) & \longrightarrow & 0
\end{array} \tag{2.8}$$

First observe that the action of $H_T(X)$ on $H_T^\bullet\left(u_! u^{-1} L_{1,n}\right)$ and on $\overline{B}_{m,n}$ factors through the map $H_T(X) \to H_T^\bullet(U) \cong H^\bullet(BT)$, so

$$H_T^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{B}_{m,n} \cong H_T^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H^\bullet(BT)}{\otimes} \overline{B}_{m,n}.$$

We prove by induction on m that the map a is an isomorphism. As in the proof of Lemma 2.4.2, the case of $m = 2$ is proved by Lemma 2.5.4, and the general case is proved by iterating the argument. An argument similar to the proof of Lemma 2.4.4 proves that the map c is an isomorphism.

Hence by the snake lemma, the middle map b is an isomorphism as well. Consequently, we obtain the following isomorphisms for every n:

$$H_T^\bullet\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(L_{m,n}\right) \to H_T^\bullet\left(L_{1,n} \otimes \cdots \otimes L_{m,n}\right).$$

In particular when $X_n = X$, we see that the cup-product map

$$H_T^\bullet\left(\mathrm{IC}_{p_1}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(\mathrm{IC}_{p_m}\right) \to H_T^\bullet\left(\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}\right)$$

is an isomorphism. □

3

Bernstein–Sato polynomials of hyperplane arrangements

3.1 Background on Bernstein–Sato polynomials

The *Bernstein–Sato polynomial*, also called the *b-function*, is a relatively fine invariant of singularities of hypersurfaces. It is a one-variable polynomial defined in terms of a local equation for the hypersurface. The b-function, specifically the set of its roots, is related to various other invariants of singularities. However, it is very difficult to explicitly compute the b-function of a general hypersurface.

Let f be a polynomial function on an affine space X, and let $\mathcal{D}_X$ be the ring of differential operators on X.

Definition 3.1.1. The b-function of f can be defined as the minimal monic polynomial $b_f(s)$ for the operator of multiplication by s on the holonomic $\mathcal{D}_X[s]$-module $\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}$ [30]. In other words, it is the minimal monic polynomial $b_f(s)$ that satisfies the following equation for some $L(s) \in \mathcal{D}_X[s]$:

$$L(s)f^{s+1} = b_f(s)f^s.$$

Now let p be a point in X, and let $\mathcal{D}_{X,p}$ be the ring of differential operators on X with coefficients in $\mathcal{O}_{X,p}$, the local ring of functions around p.

Definition 3.1.2. The local b-function of f at a point p is the minimal monic polynomial $b_{f,p}(s)$ that satisfies the following equation for some $L(s) \in \mathcal{D}_{X,p}[s]$:

$$L(s)f^{s+1} = b_{f,p}(s)f^s.$$

For example, if p is a point not on the zero locus of f, then $b_{f,p}(s) = 1$; this is seen by simply setting $L(s) = 1/f$. If p is a point on the zero locus of f, then any equation of the form $P(s)f^{s+1} = b(s)f^s$ with $P(s) \in \mathcal{D}_X[s]$ can be thought of as an equation with $P(s) \in \mathcal{D}_{X,p}[s]$. By minimality, we see that $b_{f,p}(s) \mid b_f(s)$. In particular, for a fixed f, there are finitely many choices for $b_{f,p}(s)$ over all points $p \in X$, and it is clear that the least common multiple of $b_{f,p}(s)$ over all $p \in X$ divides $b_f(s)$. In fact, it is known (see, e.g., [41, Section 4.2]) that $b_f(s)$ equals the lcm of $b_{f,p}(s)$ over all points $p \in X$.

Examples 3.1.3.

1. Consider the one-variable polynomial $f(x) = x$. Then we know that

$$\frac{d}{dx}(x^{s+1}) = (s+1)(x^s).$$

It is easy to verify that $(s+1)$ is the minimal polynomial that satisfies an equation as above, and therefore $b_f(s) = (s+1)$.

2. Consider the polynomial in n^2 variables (x_{ij}) defined as $f((x_{ij})) = \det((x_{ij}))$. Then the b-function of f is $(s+n)(s+n-1)\cdots(s+1)$, given by the following equation:

$$\det((\partial_{ij}))f^{s+1} = (s+n)(s+n-1)\cdots(s+1)f^s.$$

3.2 Goals and setup

The goal of this chapter is to prove two results about b-functions of hyperplane arrangements of Weyl type, which are defined later in this section. We give a brief overview. The first result is the Strong Topological Monodromy Conjecture for these arrangements, which links the roots of the b-function with another invariant of hypersurface singularities, namely the local topological zeta function. The second result is an upper bound on the b-function of the Vandermonde determinant, which is a special case of a hyperplane arrangement of Weyl type.

Studying b-functions of hyperplane arrangements has proved particularly tractable for computation, especially to compute and relate singularity invariants such as b-functions, zeta functions, Milnor monodromy, and jumping coefficients [49, 50, 57, 12, 13, 11]. We now define hyperplane arrangements of Weyl type, which is our main focus.

3.2.1 Weyl arrangements

Let G be a complex connected reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let $R \subset \mathfrak{h}^*$ be the associated root system with Weyl group W. Define ξ to be the product of the positive roots:

$$\xi = \prod_{\alpha \in R^+} \alpha.$$

The zero locus $V(\xi)$ is a union of hyperplanes, and is known as the *Weyl arrangement* for the Lie algebra $\mathfrak{g}$. This is the arrangement we wish to study.

The function ξ is anti-symmetric with respect to the W-action on $\mathfrak{h}$, and is the Jacobian determinant of the quotient map $\mathfrak{h} \to \mathfrak{h}/W$. The set $V(\xi)$ consists of points fixed by at least one non-trivial element of W. Thus $V(\xi)$ is the complement of $\mathfrak{h}^{\mathrm{reg}}$. The W-invariant function ξ^2 is called the *discriminant* of the root system R.

When the root system R is of type A_{n-1}, this polynomial is recognized as the *Vandermonde determinant*:

$$\xi_n = \prod_{1 \le i < j \le n} (x_i - x_j).$$

3.2.2 Motivation from previous work

Some of our motivation for studying b-functions of Weyl arrangements stems from a theorem of Eric Opdam, which computes the b-function of a related polynomial.

Since ξ^2 is in $\mathbb{C}[\mathfrak{h}]^W$, we can consider its image in $\mathbb{C}[\mathfrak{h}/W]$. Specifically, by the Chevalley-

Shephard-Todd theorem, $\mathfrak{h}/W$ is an n-dimensional affine space, where $n = \mathrm{rk}(G) = \dim(\mathfrak{h})$. Hence $\mathbb{C}[\mathfrak{h}/W]$ is a polynomial ring in n variables. Fix a homogeneous free set of generators for this polynomial ring, so that $\mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[e_1, \ldots, e_n]$. We write $\mathbb{C}[\mathfrak{h}/W]$ to mean polynomials in the generators $\{e_1, \ldots, e_n\}$, and $\mathbb{C}[\mathfrak{h}]^W$ to mean polynomials in the generators $\{x_1, \ldots, x_n\}$ of $\mathbb{C}[\mathfrak{h}]$. Let g be the polynomial corresponding to ξ^2 in $\mathbb{C}[\mathfrak{h}/W]$, that is, $g(e_1, \ldots, e_n) = \xi^2(x_1, \ldots, x_n)$.

In [46], Eric Opdam found the b-function for g.

Theorem 3.2.1 (Opdam [46]). *Let g be as above, let $n = \mathrm{rk}(\mathfrak{g})$, and let $(d_1, \ldots, d_n)$ be the list of fundamental invariants of $\mathfrak{g}$. Then*

$$b_g(s) = \prod_{i=1}^{n} \prod_{j=1}^{d_i - 1} \left(s + \frac{1}{2} + \frac{j}{d_i} \right).$$

We show in the next section that $b_g(s)$ divides $b_{\xi^2}(s)$, but evidence suggests that it falls far short of equality. Moreover, for a general f, it is always true that $b_{f^2}(s) \mid b_f(2s+1) b_f(2s)$, but equality does not always hold.

3.3 The strong monodromy conjecture

In this section we describe the connection between the roots of the b-function and the poles of the local topological zeta function. The *local topological zeta function* associated to a hypersurface $V(f)$ is a function $Z_{\mathrm{top},f}(s)$ on $\mathbb{C}$. Defined by Denef and Loeser [17], it is computed in terms of the Euler-Poincaré characteristic of the irreducible components of an embedded resolution of singularities of the hypersurface $V(f)$. Thus it forms a topological analog to the more analytic *local Igusa zeta function* [27].

In the case of f a relative invariant on a prehomogeneous vector space, poles of the Igusa zeta function correspond to roots of the b-function [27]. Consequently, by work of Malgrange [36, 37] and Kashiwara [30], the poles also give the eigenvalues of the monodromy operator on the cohomology of the Milnor fiber. The Topological Monodromy Conjecture of Denef and

Loeser [17] is an analog of this work for topological zeta functions. The weak form states that poles of $Z_{\text{top},f}$ give eigenvalues of the monodromy operator. The strong form states that poles of $Z_{\text{top},f}$ give roots of b_f, which, by Malgrange and Kashiwara, implies the weak version.

In particular, Budur, Mustaţă, and Teitler have proved the weak version of the Topological Monodromy Conjecture for hyperplane arrangements [12, Theorem 1.3(a)]. We prove the strong version for the class of Weyl arrangements.

Theorem 3.3.1. *Let $\mathfrak{h}$ be a Cartan subalgebra of a simple complex Lie algebra $\mathfrak{g}$. Let $\xi \in \mathbb{C}[\mathfrak{h}]$ be the product of the positive roots. If c is a pole of $Z_{\text{top},\xi}(s)$, then $b_\xi(c) = 0$.*

3.3.1 Proofs

In [12, Theorem 1.3(b)], Budur, Mustaţă, and Teitler reduce the Strong Monodromy Conjecture to the so-called *n/d conjecture* [12, Conjecture 1.2]. We prove Theorem 3.3.4, which is the *n/d* conjecture in the case of Weyl arrangements. As a corollary, we deduce Theorem 3.3.1, which is the Strong Monodromy Conjecture in this case.

We begin by proving the following relationship between the b-functions of g and ξ.

Theorem 3.3.2. *The function $b_g(s)$ divides the function $b_\xi(2s+1)$.*

Proof. The inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces a restriction map $\rho\colon \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$, which is an isomorphism by the Chevalley restriction theorem. Let $\Delta = \rho^*(\xi^2)$, which is an element of $\mathbb{C}[\mathfrak{g}]^G$.

Let $L_{\xi^2}(s) \in D(\mathfrak{h})[s]$ be an operator that satisfies $L_{\xi^2}(s)(\xi^{2(s+1)}) = b_{\xi^2}(s) \cdot (\xi^2)^s$. Since ξ^2 is W-invariant, we may assume (by averaging) that $L_{\xi^2}(s) \in D(\mathfrak{h})^W[s]$.

The space $D(\mathfrak{h})^W$ of W-invariant operators acts on $\mathbb{C}[\mathfrak{h}/W]$, by pulling back via the isomorphism $\mathbb{C}[\mathfrak{h}/W] \cong \mathbb{C}[\mathfrak{h}]^W$. For any $L \in D(\mathfrak{h})^W$, let $\varphi(L)$ be the corresponding differential operator in $D(\mathfrak{h}/W)$. Clearly, φ extends to a map $\varphi\colon D(\mathfrak{h})^W[s] \to D(\mathfrak{h}/W)[s]$. Applying φ to $L_{\xi^2}(s)$, we see that $\varphi(L_{\xi^2}(s))(g^{s+1}) = b_{\xi^2}(s) \cdot g^s$.

This equation shows that the b-function of g divides $b_{\xi^2}(s)$, that is,

$$b_g(s) \mid b_{\xi^2}(s). \tag{3.1}$$

Similarly, we have a map $D(\mathfrak{g})^G[s] \to D(\mathfrak{g} /\!/ G)[s]$. Let $L_\Delta(s)$ be an operator that satisfies $L_\Delta(s)(\Delta^{s+1}) = b_\Delta(s) \cdot \Delta^s$. Since the action of G on $D(\mathfrak{g})[s]$ is locally finite, we may assume by averaging that $L_\Delta(s) \in D(\mathfrak{g})^G[s]$. By a similar argument as above for the quotient $\mathfrak{g} \to \mathfrak{g} /\!/ G$ instead of $\mathfrak{h} \to \mathfrak{h}/W$, we see that

$$b_g(s) \mid b_\Delta(s). \tag{3.2}$$

Let $L_\xi(s) \in D(\mathfrak{h})$ such that $L_\xi(s)(\xi^{s+1}) = b_\xi(s) \cdot \xi^s$. Observe that

$$L_\xi(2s)L_\xi(2s+1)(\xi^{2(s+1)}) = b_\xi(2s)b_\xi(2s+1) \cdot (\xi^2)^s.$$

Therefore the b-function of ξ^2 divides $b_\xi(2s)b_\xi(2s+1)$, that is,

$$b_{\xi^2}(s) \mid b_\xi(2s)b_\xi(2s+1). \tag{3.3}$$

From (3.1) and (3.3), we see that

$$b_g(s) \mid b_\xi(2s)b_\xi(2s+1). \tag{3.4}$$

We use the following theorem. The existence is due to Harish-Chandra [24], and the surjectivity is due to Wallach [56], and Levasseur–Stafford [34].

Proposition 3.3.3. *Conjugating the radial part map* Rad *by* ξ *yields a surjective homomorphism of algebras* $\mathrm{HC}\colon D(\mathfrak{g})^G \to D(\mathfrak{h})^W$, *called the* Harish-Chandra homomorphism.

Clearly, HC extends to a map $\mathrm{HC}\colon D(\mathfrak{g})^G[s] \to D(\mathfrak{h})^W[s]$. Recall that $L_\Delta(s)$ is in $D(\mathfrak{g})^G[s]$, and was chosen such that $L_\Delta(s)(\Delta^{s+1}) = b_\Delta(s) \cdot \Delta^s$. Since Δ corresponds to the function ξ^2 under the Chevalley restriction map, we have

$$\begin{aligned}\mathrm{HC}(L_\Delta(s-1/2))\cdot(\xi^2)^{s+1} &= \xi\circ\mathrm{Rad}(L_\Delta(s-1/2))\circ\xi^{-1}(\xi^2)^{s+1}\\ &= \xi\circ\mathrm{Rad}(L_\Delta(s-1/2))(\xi^2)^{(2s+1)/2}\\ &= \xi\cdot b_\Delta(s-1/2)\cdot(\xi^2)^{(2s-1)/2}\\ &= b_\Delta(s-1/2)\cdot\xi^{2s},\end{aligned}$$

which shows that $b_{\xi^2}(s)\mid b_\Delta(s-1/2)$.

Since HC is surjective, $L_{\xi^2}(s)\in D(\mathfrak{h})^W$ can be lifted to an operator in $D(\mathfrak{g})^G$. By running the previous argument in reverse, we can see that $b_\Delta(s-1/2)\mid b_{\xi^2}(s)$. We conclude that $b_{\xi^2}(s)=b_\Delta(s-1/2)$, and by changing variables that

$$b_{\xi^2}(s+1/2)=b_\Delta(s). \tag{3.5}$$

From (3.2), (3.3), and (3.5), we see that

$$b_g(s)\mid b_\xi(2s+1)b_\xi(2s+2). \tag{3.6}$$

Suppose that $b_g(s)\nmid b_\xi(2s+1)$. This means that there is some c that is a root of $b_g(s)$ of some multiplicity m, but is a root of $b_\xi(2s+1)$ of multiplicity $k<m$ (where k may be zero). By (3.4), c must be a root of $b_\xi(2s)$, and by (3.6), c must be a root of $b_\xi(2s+2)$.

By [49, Theorem 1], the difference between any two roots of the b-function of ξ, a hyperplane arrangement, is less than 2. So c cannot be a root of both $b_\xi(2s)$ and $b_\xi(2s+2)$, and we have a contradiction. This argument proves that $b_g(s)\mid b_\xi(2s+1)$. □

The proof of the n/d conjecture for Weyl arrangements now follows quite easily, which also proves Theorem 3.3.1.

Theorem 3.3.4. *Let $\mathfrak{h}$ be a Cartan subalgebra of a simple complex Lie algebra $\mathfrak{g}$. Let $\xi \in \mathbb{C}[\mathfrak{h}]$ be the product of the positive roots as defined earlier. Let $d = \deg(\xi)$ and let $n = \dim(\mathfrak{h})$. Then $-n/d$ is always a root of the b-function of ξ.*

Proof. Let $d_1 \leq \cdots \leq d_n$ be a list of the degrees of the fundamental invariants of the Lie group G. The degree of the highest fundamental invariant is equal to the Coxeter number. Recall that n is the rank of the root system, and the total number of roots equals $2d$. It is known (see, e.g., [26, Section 3.18]) that $d_n \cdot n = 2d$.

From [46], we know that

$$b_g(s) = \prod_{i=1}^{n} \prod_{j=1}^{d_i-1} \left(s + \frac{1}{2} + \frac{j}{d_i}\right).$$

Notice that one of the factors above is

$$\left(s + \frac{1}{2} + \frac{1}{d_n}\right) = \left(s + \frac{1}{2} + \frac{n}{2d}\right).$$

So $-(1/2+n/(2d))$ is a root of $b_g(s)$ and hence of $b_\xi(2s+1)$, which precisely means that $b_\xi(-n/d) = 0$. □

3.4 The b-function of the Vandermonde determinant

In this section we describe progress towards computing the b-function of the Vandermonde determinant, which corresponds to the Weyl arrangement in type A. We denote the Vandermonde determinant in n variables by vm_n:

$$\mathrm{vm}_n(x_1,\ldots,x_n) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1\le j<i\le n} (x_i - x_j).$$

For notational convenience, we write $c(a,b)$ to mean the binomial coefficient $\binom{a}{b}$. We have the following recursive upper bound for b_{vm_n}.

Theorem 3.4.1. *For a partition $\lambda = (\lambda_1,\ldots,\lambda_k)$ of n, denoted $\lambda \vdash n$, let $b_\lambda(s)$ denote the product of the b-functions of vm_{λ_i} for each i. We have the following divisibility relation:*

$$b_{\mathrm{vm}_n}(s) \,\Big|\, \operatorname*{lcm}_{\substack{\lambda\vdash n \\ \lambda\neq(n)}} b_\lambda(s)\cdot \prod_{i=(n-1)}^{(n-1)^2}\left(s+\frac{i}{c(n,2)}\right). \tag{3.7}$$

The remainder of this section will be devoted to proving this theorem. We begin by proving a much cruder upper bound for vm_n, which we will later refine. Kashiwara [30] proved that the roots of any b-function are negative rational numbers. The method of proof is to consider an embedded resolution of singularities of the hypersurface, and compare the b-function of the hypersurface to that of its preimage. We apply a similar argument in our case. Consider the hyperplane arrangement cut out by vm_n, and consider the (singular) locus of points $x_1 = x_2 = \cdots = x_n = 0$. Blow up $\mathbb{C}^n$ along this locus, and let $\widetilde{\mathrm{vm}}_n$ denote the pullback of vm_n to the blowup. In any affine open subset U of the blowup, it makes sense to compute the b-function of $\widetilde{\mathrm{vm}}_n|_U$. We will denote the least common multiple of this b-function (over all choices of U) by $b_{\widetilde{\mathrm{vm}}_n}$, and call it the b-function of $\widetilde{\mathrm{vm}}_n$. The precise relationship between the b-functions of vm_n and of $\widetilde{\mathrm{vm}}_n$ is given by the following theorem.

Theorem 3.4.2 (Kashiwara [30]). *In the setup above, the following is true for some non-negative integer N:*

$$b_{\mathrm{vm}_n}(s) \mid b_{\widetilde{\mathrm{vm}}_n}(s) b_{\widetilde{\mathrm{vm}}_n}(s+1) \cdots b_{\widetilde{\mathrm{vm}}_n}(s+N).$$

The exceptional locus occurs with multiplicity exactly equal to $c(n,2)$, which is the total number of hyperplanes in vm_n. Locally in an affine chart on the blowup, it is possible to write $\widetilde{\mathrm{vm}}_n$ as $e \cdot f$, where e is the $c(n,2)$th power of a local equation for the exceptional divisor, and f is a function in variables disjoint from those of e. The local b-function of $\widetilde{\mathrm{vm}}_n$ then divides $b_e(s)b_f(s)$, and the (global) b-function of $\widetilde{\mathrm{vm}}_n$ is the lcm of all of the terms $b_e(s)b_f(s)$.

Notice that the zero locus $V(e)$ of e is smooth and has multiplicity $c(n,2)$, so

$$b_e(s) = \prod_{i=1}^{c(n,2)} \left(s + \frac{i}{c(n,2)} \right), \tag{3.8}$$

independent of the chart. So the lcm of all the products $b_e(s)b_f(s)$ is simply $b_e(s)$ times the lcm of all the possible $b_f(s)$. Equivalently, it is $b_e(s)$ times the lcm of the local b-functions at all points of $V(\widetilde{\mathrm{vm}}_n)$ away from the exceptional locus.

Recall that the projection from the blowup to the base is an isomorphism away from the exceptional locus. So for any point in $V(\widetilde{\mathrm{vm}}_n)$ away from the exceptional locus, its local b-function can be computed on the base. This amounts to computing the local b-functions at all points $p \neq 0$ in $V(\mathrm{vm}_n)$.

Consider some point $p \neq 0$ in $V(\mathrm{vm}_n)$. At p, some subsets of coordinates (but not all coordinates) are equal. Up to reordering coordinates, we can assume that the first λ_1 coordinates are equal, the next λ_2 coordinates are equal, and so on, such that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a non-trivial partition of k. This means that locally around p, the equation for vm_k can be written as the product of vm_{λ_i} for each i. Recall that the product of the b-functions of all the vm_{λ_i} is denoted by b_λ. Since the variables in the different vm_{λ_i} do not interact, we have the relationship $b_{\mathrm{vm}_k,p}(s) \mid b_\lambda(s)$. Taking the lcm at all such points p and combining this with (3.8) via the previous argument, we see that

$$b_{\widetilde{\mathrm{vm}}_n}(s) \mid \operatorname*{lcm}_{\substack{\lambda\vdash n\\ \lambda\neq(n)}} b_\lambda(s)\cdot \prod_{i=1}^{c(n,2)}\left(s+\frac{i}{c(n,2)}\right). \tag{3.9}$$

This equation, together with Theorem 3.4.2, gives the following lemma.

Lemma 3.4.3 (Crude upper bound on b_{vm_n})**.** *There are some non-negative integers M and N for which the following relationship holds:*

$$b_{\mathrm{vm}_n}(s) \mid \left(\prod_{i=0}^{M} \operatorname*{lcm}_{\substack{\lambda\vdash n\\ \lambda\neq(n)}} b_\lambda(s+i)\right)\cdot\left(\prod_{j=1}^{N}\left(s+\frac{j}{c(n,2)}\right)\right).$$

Given this crude upper bound on the b-function of vm_n, we now prove Theorem 3.4.1 by induction on n. Assume the divisibility relation in (3.7) holds for $b_{\mathrm{vm}_{n-1}}(s)$.

Recall the relationship between b-functions and another set of singularity invariants, namely the *jumping coefficients*. For a function f and a positive rational number p, the *multiplier ideal* is the ideal in $\mathcal{O}_X$ of functions h for which $|h|^2/|f|^{2p}$ is locally integrable. The multiplier ideal can also be defined algebro-geometrically (see, e.g., [18]). The values of p at which the multiplier ideal changes size are called jumping coefficients. By the work of Yano [58], Lichtin [35], and Kollár [33, Theorem 10.6], it is known that the smallest jumping coefficient α satisfies $b_f(-\alpha)=0$, and that in fact $-\alpha$ is the largest root of the b-function.

Let $\mathcal{A}$ be a central hyperplane arrangement of d hyperplanes embedded essentially into $\mathbb{C}^m$ (i.e. the arrangement does not embed into $\mathbb{C}^r$ for any $r<m$). Let f be a function that cuts out $\mathcal{A}$. For $S\subset\mathcal{A}$, denote $W_S=\bigcap_{H\in S}H$. By Mustaţă [42, Corollary 0.3], the minimal jumping coefficient of $\mathcal{A}$ is

$$\alpha=\min_{\emptyset\neq S\subset\mathcal{A}}\frac{\operatorname{codim}(W_S)}{|\{H\mid H\supset W_S\}|}.$$

Similarly to [49, Equation 0.2] and using the relationship between b-functions and the minimal jumping coefficient, we can rewrite α as follows:

$$\alpha = \min\left(\left\{\frac{m}{d}\right\} \cup \left\{\bigcup_{0\neq x\in\mathbb{C}^m} \left\{s \mid b_{f,x}(-s) = 0\right\}\right\}\right). \tag{3.10}$$

Lemma 3.4.4. *For any positive integer k, the minimal jumping coefficient of* vm_k *equals*

$$\frac{(k-1)}{c(k,2)}.$$

Proof. Consider the arrangement cut out by vm_k. Any point on this arrangement other than zero can be described as the set where some subsets of coordinates (but not all coordinates) are equal. Consider some point $p \neq 0$. Up to reordering coordinates, we can assume that the first λ_1 coordinates are equal, the next λ_2 coordinates are equal, and so on, such that $\lambda = (\lambda_1, \dots, \lambda_r)$ is a non-trivial partition of k. This means that locally around p, the equation for vm_k can be written as the product of vm_{λ_i} for each i. Recall that the product of the b-functions of each of the vm_{λ_i} is denoted by b_λ. Since the variables in the different vm_{λ_i} do not interact, we have the relationship $b_{\mathrm{vm}_k,p} \mid b_\lambda$.

Further note that the arrangement cut out by vm_k has degree $d = c(k,2)$. It embeds essentially in $(k-1)$ dimensions via the map from $\mathbb{C}^k$ to $\mathbb{C}^{k-1}$, equivalently from $\mathbb{C}[y_1,\dots,y_k]$ to $\mathbb{C}[x_1,\dots,x_k]$, defined as $y_i \mapsto x_i - x_{i+1}$. So the number "$m/d$" from (3.10) equals $(k-1)/c(k,2)$. For p and λ as above, we showed that the local b-function of vm_k at p divides b_λ. Since λ is a non-trivial partition, we have $\lambda_i < k$ for each i. By induction, the minimal s such that $b_{\mathrm{vm}_{\lambda_i}}(-s) = 0$ is $(\lambda_i - 1)/c(\lambda_i, 2)$. Therefore

$$\min\left\{\bigcup_{0\neq p\in\mathbb{C}^{k-1}} \left\{s \mid b_{\mathrm{vm}_k,p}(-s) = 0\right\}\right\} = \min_i \left\{\frac{(\lambda_i - 1)}{c(\lambda_i, 2)}\right\} > \frac{(k-1)}{c(k,2)}.$$

Thus $m/d = (k-1)/c(k,2)$ is the minimal element in (3.10), and the lemma is proved. □

Corollary 3.4.5. *For each positive integer k, the roots of b_{vm_k} lie in the following interval:*

$$\left[\frac{-(k-1)^2}{c(k,2)}, \frac{-(k-1)}{c(k,2)}\right].$$

Proof. The previous lemma implies that the roots of b_{vm_k} are bounded above by the number $-(k-1)/c(k,2)$. By combining results of Narváez Macarro [43] and Calderón Moreno–Narváez Macarro [14], we can verify that the b-function of every vm_k is "symmetric around the point -1". More precisely, this means that

$$b_{\mathrm{vm}_k}(s) = \pm b_{\mathrm{vm}_k}(-s-2). \tag{3.11}$$

Using the previous lemma, we compute that the roots of b_{vm_k} are bounded below by the following number:

$$\frac{(k-1)}{c(k,2)} - 2 = -\frac{(k-1)^2}{c(k,2)}.$$

□

Now we can finish the proof of Theorem 3.4.1. Recall Lemma 3.4.3, our crude upper bound for b_{vm_n}. Using Corollary 3.4.5, we can create a tighter upper bound for b_{vm_n} by pruning factors from the second expression that correspond to roots outside the appropriate interval. This yields the following relationship:

$$b_{\mathrm{vm}_n}(s) \mid \left(\operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s)\right) \cdot \left(\prod_{j=1}^{M} \operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s+j)\right) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{c(n,2)}\right). \tag{3.12}$$

Recall the symmetry property of (3.11). We will use this to eliminate the middle term of the dividend in the above equation. First note that we can also eliminate all factors beyond $j = 1$: Corollary 3.4.5 implies that all roots of $b_{\mathrm{vm}_n}(s)$ lie in the interval $(-2, 0)$, but for any $j > 1$ and any non-trivial partition $\lambda \vdash n$, all roots of $b_\lambda(s+j)$ are less than -2. So we can simplify the above equation:

$$b_{\mathrm{vm}_n}(s) \mid \left(\operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \right) \cdot \left(\operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s+1) \right) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{c(n,2)} \right). \tag{3.13}$$

For convenience, set up the following notation:

$$A_n = \left(\operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \right), \quad B_n = \left(\operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s+1) \right), \quad C_n = \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{c(n,2)} \right).$$

Suppose for contradiction that $b_{\mathrm{vm}_n}(s) \nmid A_n C_n$. This means that there is some root ρ of $b_{\mathrm{vm}_n}(s)$ of multiplicity m, which occurs in $A_n C_n$ with multiplicity strictly less than m. Consequently, ρ is a root of B_n. Note that $\rho \neq -1$, because -1 is not a root of B_n. Since $b_{\mathrm{vm}_n}(s)$ is symmetric around the point -1, the multiplicity of $-\rho-2$ in $b_{\mathrm{vm}_n}(s)$ is also equal to m. On the right hand side, note that A_n and C_n are both symmetric around the point -1. So the multiplicity of $-\rho-2$ in $A_n C_n$ is strictly less than m. This implies that $-\rho-2$ is a root of B_n. By the bounds already established, we know that the roots of $A_n C_n$ lie in $(-2,0)$, and those of B_n lie in $(-3,-1)$. Therefore $-2 < \rho < -1$, and $-1 < -\rho - 2 < 0$, and therefore $-\rho-2$ cannot be a root of B_n. This is a contradiction.

The argument outlined above proves that $b_{\mathrm{vm}_n}(s) \mid A_n C_n$, which completes the proof of Theorem 3.4.1.

References

[1] Pramod N Achar and Laura Rider. Parity sheaves on the affine Grassmannian and the Mirković-Vilonen conjecture. arXiv:1305.1684 [math.RT], to appear in Acta Math., 2013.

[2] Michèle Audin. *Torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, revised edition, 2004.

[3] Asilata Bapat. Torus actions and tensor products of intersection cohomology. *Pacific J. Math.*, 276(1):19–34, 2015.

[4] Asilata Bapat and Robin Walters. The strong topological monodromy conjecture for Coxeter hyperplane arrangements. http://math.uchicago.edu/~asilata/papers/bapat-walters-2015.pdf, to appear in Mathematical Research Letters, 2015.

[5] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[6] Joseph Bernstein. Modules over a ring of differential operators. An investigation of the fundamental solutions of equations with constant coefficients. *Funkcional. Anal. i Priložen.*, 5(2):1–16, 1971.

[7] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[8] Andrzej Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[9] Tom Braden. Hyperbolic localization of intersection cohomology. *Transform. Groups*, 8(3):209–216, 2003.

[10] Michel Brion. Poincaré duality and equivariant (co)homology. *Michigan Math. J.*, 48:77–92, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[11] Nero Budur. Singularity invariants related to Milnor fibers: survey. In *Zeta functions in algebra and geometry*, volume 566 of *Contemp. Math.*, pages 161–187. Amer. Math. Soc., Providence, RI, 2012.

[12] Nero Budur, Mircea Mustaţă, and Zach Teitler. The monodromy conjecture for hyperplane arrangements. *Geom. Dedicata*, 153:131–137, 2011.

[13] Nero Budur and Morihiko Saito. Jumping coefficients and spectrum of a hyperplane arrangement. *Math. Ann.*, 347(3):545–579, 2010.

[14] Francisco Calderón Moreno and Luis Narváez Macarro. The module dfs for locally quasi-homogeneous free divisors. *Compositio Mathematica*, 134:59–74, 10 2002.

[15] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2010. Reprint of the 1997 edition.

[16] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[17] Jan Denef and François Loeser. Caractéristiques d'Euler-Poincaré, fonctions zéta locales et mod- ifications analytiques. *J. Amer. Math. Soc.*, 5(4):545–579, 1992.

[18] Lawrence Ein, Robert Lazarsfeld, Karen E. Smith, and Dror Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.*, 123(3):469–506, 2004.

[19] Victor Ginzburg. Perverse sheaves and $\mathbf{C}^*$-actions. *J. Amer. Math. Soc.*, 4(3):483–490, 1991.

[20] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.

[21] Mark Goresky and Robert MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980.

[22] Mark Goresky and Robert MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[23] William Graham. Positivity in equivariant Schubert calculus. *Duke Math. J.*, 109(3):599–614, 2001.

[24] Harish-Chandra. Invariant differential operators and distributions on a semisimple Lie algebra. *Amer. J. Math.*, 86:534–564, 1964.

[25] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi.

[26] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[27] Jun-ichi Igusa. *An introduction to the theory of local zeta functions*, volume 14. Cambridge Univ. Press, 2000.

[28] Daniel Juteau, Carl Mautner, and Geordie Williamson. Parity sheaves. *J. Amer. Math. Soc.*, 27(4):1169–1212, 2014.

[29] Tatsuji Kambayashi. Projective representation of algebraic linear groups of transformations. *Amer. J. Math.*, 88:199–205, 1966.

[30] Masaki Kashiwara. B-functions and holonomic systems. Rationality of roots of B-functions. *Invent. Math.*, 38(1):33–53, 1976/77.

[31] Masaki Kashiwara. Faisceaux constructibles et systèmes holonômes d'équations aux dérivées partielles linéaires à points singuliers réguliers. In *Séminaire Goulaouic-Schwartz, 1979–1980 (French)*, pages Exp. No. 19, 7. École Polytech., Palaiseau, 1980.

[32] Masaki Kashiwara. The Riemann-Hilbert problem for holonomic systems. *Publ. Res. Inst. Math. Sci.*, 20(2):319–365, 1984.

[33] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[34] Thierry Levasseur and John Tobias Stafford. Invariant differential operators and an homomorphism of Harish-Chandra. *J. Amer. Math. Soc.*, 8(2):365–372, 1995.

[35] Ben Lichtin. Poles of $|f(z,w)|^{2s}$ and roots of the b-function. *Ark. Mat.*, 27(2):283–304, 1989.

[36] Bernard Malgrange. Le polynôme de Bernstein d'une singularité isolée. In *Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974)*, pages 98–119. Lecture Notes in Math., Vol. 459. Springer, Berlin, 1975.

[37] Bernard Malgrange. Polynômes de Bernstein-Sato et cohomologie évanescente. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 243–267. Soc. Math. France, Paris, 1983.

[38] Zoghman Mebkhout. Sur le problème de Hilbert-Riemann. In *Complex analysis, microlocal calculus and relativistic quantum theory (Proc. Internat. Colloq., Centre Phys., Les Houches, 1979)*, volume 126 of *Lecture Notes in Phys.*, pages 90–110. Springer, Berlin-New York, 1980.

[39] Zoghman Mebkhout. Une autre équivalence de catégories. *Compositio Math.*, 51(1):63–88, 1984.

[40] Zoghman Mebkhout. Une équivalence de catégories. *Compositio Math.*, 51(1):51–62, 1984.

[41] Zoghman Mebkhout and Luis Narváez Macarro. La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer. *Ann. Sci. École Norm. Sup. (4)*, 24(2):227–256, 1991.

[42] Mircea Mustaţă. Multiplier ideals of hyperplane arrangements. *Trans. Amer. Math. Soc.*, 358(11):5015–5023 (electronic), 2006.

[43] Luis Narváez Macarro. A duality approach to the symmetry of Bernstein-Sato polynomials of free divisors. arXiv:1201.3594 [math.AG], 2012.

[44] Masayuki Noro. An efficient modular algorithm for computing the global b-function. In *Mathematical software (Beijing, 2002)*, pages 147–157. World Sci. Publ., River Edge, NJ, 2002.

[45] Toshinori Oaku. An algorithm of computing b-functions. *Duke Math. J.*, 87(1):115–132, 1997.

[46] Eric M. Opdam. Some applications of hypergeometric shift operators. *Invent. Math.*, 98(1):1–18, 1989.

[47] Morihiko Saito. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.*, 24(6):849–995 (1989), 1988.

[48] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.

[49] Morihiko Saito. Bernstein-Sato polynomials of hyperplane arrangements. arXiv:math/0602527 [math.AG], 2006.

[50] Morihiko Saito. Multiplier ideals, b-function, and spectrum of a hypersurface singularity. *Compos. Math.*, 143(4):1050–1068, 2007.

[51] Mikio Sato. Theory of prehomogeneous vector spaces (algebraic part)—the English translation of Sato's lecture from Shintani's note. *Nagoya Math. J.*, 120:1–34, 1990. Notes by Takuro Shintani, Translated from the Japanese by Masakazu Muro.

[52] Mikio Sato and Takuro Shintani. On zeta functions associated with prehomogeneous vector spaces. *Ann. of Math. (2)*, 100:131–170, 1974.

[53] Wolfgang Soergel. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3(2):421–445, 1990.

[54] T. A. Springer. A purity result for fixed point varieties in flag manifolds. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(2):271–282, 1984.

[55] Hideyasu Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14:1–28, 1974.

[56] Nolan R. Wallach. Invariant differential operators on a reductive Lie algebra and Weyl group representations. *J. Amer. Math. Soc.*, 6(4):779–816, 1993.

[57] Uli Walther. Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements. *Compos. Math.*, 141(1):121–145, 2005.

[58] Tamaki Yano. b-functions and exponents of hypersurface isolated singularities. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 641–652. Amer. Math. Soc., Providence, RI, 1983.