THE UNIVERSITY OF CHICAGO

MATRIX FACTORIZATIONS FOR QUASI-COHERENT SHEAVES OF CATEGORIES

A DISSERTATION SUBMITTED TO

THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES

IN CANDIDACY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

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CHICAGO, ILLINOIS

AUGUST 2019

A mi Mamá y a mi Papá.

A mi hermana y a mi hermano.

A los amigos.

A las universidades publicas y gratuitas. Siempre.

”Les deseo una vida solidaria y sensible. Solidaria para comprender que nadie es el hacedor absoluto de su propio destino. La vida, nuestra vida, es una gran alquimia, una transmutación maravillosa e increíble del esfuerzo colectivo. Los eternos adoradores del individualismo tratarán de convencerlos de lo contrario. No se dejen engañar: como dijo alguna vez Carpentier, el hombre sólo puede alcanzar su grandeza, su máxima medida en el reino de este mundo; y eso es junto a otros hombres. Sensibles entonces deberán ser para poder percibir que lo que somos en realidad lo debemos y es justicia compartirlo, ponerlo al servicio de los demás.” ... leido por ahi.

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ACKNOWLEDGMENTS

First I would like to thank my advisor, Sasha Beilinson, for his friendship and all this years full of conversations about math, nature, literature, films and life.

Also, I want to thank Nick Rozenblyum for reading the thesis and German Stefanich for so many talks and insights about singular support, sheaves of categories and all that.

Finally, I want to thank the woman at Conicet that, in the name of Justice, refused to accept my papers a day later and, without knowing, started this adventure.

ABSTRACT

We explore some similarities between the theory of D-modules and that of quasi-coherent sheaves of categories. The original motivation is to better understand several results on the literature that relates vanishing cycles with some invariants of the category of matrix factorizations like its periodic cyclic homology or its etale cohomology.

We propose the following explanation: given a function $\phi : X \to \mathbb{A}^1$, the category of matrix factorizations $\mathrm{MF}(X, \phi)$ can be thought as a higher categorical version of vanishing cycles for the sheaf of categories corresponding to $\mathrm{Perf}(X)$. Here, the $2-$periodic structure on matrix factorizations corresponds to the monodromy of vanishing cycles.

A second goal is to find the general phenomena behind the work of A. Preygel that uses derived algebraic geometry to construct $\mathrm{MF}(X, \phi)$ from a cohomological operation of degree 2 on $\mathrm{Coh}(X_0)$. We interpret this as the microlocalization of $\mathrm{Perf}(X)$ over $T_0^*\mathbb{A}^1$ and extend it to arbitrary quasi-coherent sheaves of categories over smooth schemes.

To realize the above we need the theory of quasi-coherent sheaves of categories developed by D.Gaitsgory, J.Lurie, B.Toen and G.Vessozzi. The basic formalism is quite recent and contains pushforward and pullback. We slightly modify it to get an extraordinary pullback for complete intersection morphisms. We also introduce the matrix factorizations functor.

The main result of this thesis is a comparison between the usual and the extraordinary pullback. In rough terms, via Koszul duality, the extraordinary pullback localize over a conormal bundle and the usual pullback correspond to the part supported on the zero section.

We study the case of the closed immersion of a point in a smooth scheme. This defines for every quasi-coherent sheaf of categories its punctual singular support, analogous to the singular support of a D-module. In the end we give a deformation theory interpretation.

This is part of joint work with G. Stefanich. In [dFS] we will define and study the global singular support of a quasi-coherent sheaf of categories. This is a closed conical subset of the cotangent bundle that measures the directions on which the sheaf is proper and smooth.

CHAPTER 1

INTRODUCTION

In the following we explore some similarities between the theory of D-modules and that of quasi-coherent sheaves of categories like in [Lur5], [Gai] and [TV]. The motivation comes from trying to understand the relation between vanishing cycles and matrix factorizations suggested by [BRTV] and [Efi] as well as the general phenomena behind [AG] and [Pre].

Given a smooth scheme X, one can check if a D-module $\mathscr{F}$ is locally constant at a given point x in a given codirection v^*. The collection of all pairs where this does not happen defines its singular support, a closed conical subset of the cotangent bundle T^*X, and ideally one would like to microlocalize the D-module along it.

More precisley, consider a function $\phi : X \to \mathbb{A}^1$ and let $i : X_0 \to X$ be the closed immersion of the special fiber. The vanishing cycles $R\Phi(\mathscr{F})$ of $\mathscr{F}$ with respect to ϕ is a sheaf on the special fiber X_0. It comes with an automorphism, its monodromy, such that its invariants fits into a short exact sequence

$$i^*(\mathscr{F})[-1] \to i^!(\mathscr{F})[1] \to R\Phi(\mathscr{F})^{\mathbb{Z}}. \tag{1.1}$$

The D-module $\mathscr{F}$ is locally constant at x in the codirection of the special fiber if and only if the fiber at x of $R\Phi(\mathscr{F})$ is trivial if and only if $d_x\phi$ is not in its singular support.

Remark 1.1. The invariants $R\Phi(\mathscr{F})^{\mathbb{Z}}$ are Koszul dual to the unipotent part of the sheaf of vanishing cycles $R\Phi(\mathscr{F})$. In particular, they get a square-zero cohomological operation of degree one, Koszul dual to the monodromy of the vanishing cycles.

Now we pass to a higher categorical version of the constant sheaf case. Lets recall that for a given scheme, periodic cyclic homology relates its category of perfect complexes with its de Rham cohomology and its category of coherent complexes with its Borel-Moore de Rham cohomology. Moreover, for a pair (X, ϕ), its vanishing cycles cohomology can be computed from the periodic cyclic homology of its category of matrix factorizations [Efi].

The above suggest that the short exact sequence

$$\mathrm{Perf}(X_0) \to \mathrm{Coh}(X_0) \to \mathrm{MF}(X, \phi) \tag{1.2}$$

for the sheaf of categories whose global sections equals to $\mathrm{Perf}(X)$, should be understood as a higher categorical version of (1.1) for the constant sheaf.

This is our starting point. If we accept this interpretation of matrix factorizations as categorical vanishing cycles, then it is natural to ask: what is the analogue on the category of matrix factorizations of the monodromy on the term of the right of (1.1)?

For this, consider $k[\beta]$ with $|\beta| = 2$. The category $\mathrm{Coh}(X_0)$ has a natural $k[\beta]-$linear structure induced by a cohomological operation $M \to M[2]$. An easy construction of this is as the edge map in the well known distinguished triangle $M[1] \to i^* i_* M \to M$.

The key fact is that the cohomological operation is invertible on the matrix factorizations category and hence gives an $k[\beta, \beta^{-1}]-$linear structure. This is the same as a $2-$periodic structure and should be interpreted as the monodromy on the term on the right of (1.2).

Remark 1.2. As a category, $\mathrm{MF}(X, \phi)$ depends only on X_0, is its singularity category, but its monodromy remembers it is an hypersurface. This is why we talk about matrix factorizations rather than singularity category: to emphasize that it comes with extra structure.

In order to make sense of the above interpretation of (1.2) we need a general formalism for sheaves of categories. We will review precise definitions later, for now we use the following.

Definition 1.3. Given a scheme X, a quasi-coherent sheaf of categories on it is a small idempotent complete stable category with a monoidal action of $\mathrm{Perf}(X)$.

A quasi-coherent sheaf of categories over a scheme X is said to be perfect if it is proper and smooth over $\mathrm{Perf}(X)$ and coherent if it is proper over $\mathrm{Perf}(X)$ and smooth over $\mathrm{Perf}(k)$.

In [Lur5] the basic definitions and properties of f_* and f^* are developed but unfortunately we will need a little bit more: we want to have $f^!$. For our purposes, it suffices to say that the pushforward f_* is the forgetful functor and for f^* and $f^!$ we use the following definition.

Definition 1.4. For $p : Y \to X$ proper and finite tor-dimension and $\mathcal{C}$ a quasi-coherent sheaf of categories on X we define $p^*(\mathcal{C}) = \mathrm{Perf}(Y) \otimes_X \mathcal{C}$ and $p^!(\mathcal{C}) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathcal{C})$.

In sections 5 and 6 we check that this definitions works as expected. In particular, they coincide with the left and right adjoint of p_* and we get base change and projection formula.

Example 1.5. In the notation above, $p^*(\mathrm{Perf}(X)) = \mathrm{Perf}(Y)$ and $p^!(\mathrm{Coh}(X)) = \mathrm{Coh}(Y)$. □

Lets point out here that for such definitions it is essential to work with small idempotent complete categories because with their big cocomplete compactly generated counterparts, both pullbacks would agree. In particular, $p^*(\mathcal{C})$ and $p^!(\mathcal{C})$ fully faithful embed into the big functor category

$$\mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}^L_{\mathrm{QCoh}(X)}(\mathrm{QCoh}(Y), \mathrm{Ind}(\mathcal{C})),$$

the first as its compact objects and the second as the functors that preserve compact objects.

The hyphotesis on p implies that there is a fully faithful $p^*(\mathcal{C}) \to p^!(\mathcal{C})$. In some situations this is an equivalence, for example if p is smooth or if $\mathcal{C}$ is dualizable, but in general it is not.

Definition 1.6. For $p : Y \to X$ proper and finite tor-dimension and $\mathcal{C}$ a quasi-coherent sheaf of categories on X we define $\mathrm{MF}_p(\mathcal{C})$ as the cofiber of $p^*(\mathcal{C}) \to p^!(\mathcal{C})$.

This is arguably the most important definition in the text and is motivated by the previous discussion. We will refer to it as the matrix factorizations of $\mathcal{C}$ with respect to p and it comes with a short exact sequence

$$p^*(\mathcal{C}) \to p^!(\mathcal{C}) \to \mathrm{MF}_p(\mathcal{C}). \tag{1.3}$$

Example 1.7. If in the above X is smooth and $\mathcal{C}$ is equal to $\mathrm{Perf}(X) = \mathrm{Coh}(X)$, then it follows that the pullback $p^*(\mathcal{C})$ is equal to $\mathrm{Perf}(Y)$, the exceptional pullback $p^!(\mathcal{C})$ to $\mathrm{Coh}(Y)$ and, moreover, both fully faithful embeds into the big $\mathrm{QCoh}(Y)$. In this case, the matrix factorization category $\mathrm{MF}_p(\mathcal{C})$ is equal to the singularity category $\mathrm{Sing}(Y)$. □

The next step is to define the monodromy in the exact sequence (1.3). For this, we need the convolution groupoid $Y \times_X Y \rightrightarrows Y$. The usual pull-push formalism gives a convolution product of sheaves and we get a short exact sequence of monoidal categories

$$\mathrm{Perf}(Y \times_X Y) \to \mathrm{Coh}(Y \times_X Y) \to \mathrm{Sing}(Y \times_X Y). \tag{1.4}$$

Remark 1.8. We will review the main properties of the convolution groupoid in section 3.

The convolution groupoid naturally acts on the exact sequence (1.3). For example, there is a natural monoidal action, by precomposition, of the monoidal category

$$\mathrm{Coh}(Y \times_X Y) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathrm{Perf}(Y)) \qquad \text{on} \qquad p^!(\mathcal{C}) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathcal{C}),$$

which induces all other actions. We will refer to this as the monodromy of (1.3).

In case $p : Y \to X$ is a complete intersection closed embedding, Koszul duality gives an equivalence between $\mathrm{Coh}(Y \times_X Y)$ with convolution and $\mathrm{Perf}(N_Y^* X[2])$ with the usual tensor product. In particular, the monodromy action can be interpreted as the microlocalization of the short exact sequence (1.3) over the shifted conormal bundle $N_Y^* X[2]$.

Example 1.9. Consider p as the inclusion of the origin in $\mathbb{A}^1$. In this case, $\mathrm{Coh}(0 \times_{\mathbb{A}^1} 0)$ with convolution, is the same as $\mathrm{Perf}(T_0^* \mathbb{A}^1[2]) \simeq \mathrm{Perf}(k[\beta])$ with the usual tensor product.

If $\mathcal{C}$ is $\mathrm{Perf}(Z)$ given by a proper map $\phi : Z \to \mathbb{A}^1$, then (1.3) recovers (1.2) and moreover, the monodromy on $p^!(\mathcal{C}) \simeq \mathrm{Coh}(Z_0)$ is an $k[\beta]$-linear structure on it. In [Pre] it is proved that $\mathrm{Perf}(Z_0)$ is the full subcategory of $\mathrm{Coh}(Z_0)$ consisting of $\beta-$torsion objects and that the matrix factorization category $\mathrm{MF}(Z, \phi)$ is the localization $\mathrm{Coh}(Z_0) \otimes_{k[\beta]} k[\beta, \beta^{-1}]$. □

The above is the key example which inspires everything. Let emphasize the following point of view: given $\phi : Z \to \mathbb{A}^1$, if we think of the category $\mathrm{Perf}(Z)$ as a quasi-coherent sheaf of categories over $\mathbb{A}^1$, then the $k[\beta]-$linear structure on $\mathrm{Coh}(Z_0)$ is its microlocalization on the cotangent fibre $T_0^* \mathbb{A}^1$ and $\mathrm{MF}(Z, \phi)$, with its $2-$periodic structure, its vanishing cycles.

The next theorem is a direct generalization of 1.9 to arbitrary proper quasi-coherent sheaves of categories over arbitrary smooth schemes. The proof of the claim about properness in the case of a point in a line and an arbitrary sheaf was explained to me by G. Stefanich.

Theorem 1.10. Let $p : Y \to X$ be a complete intersection closed embedding in a smooth scheme, then for every proper quasi-coherent sheaf $\mathcal{C}$ over X, we have that $p^!(\mathcal{C})$ is proper over $\mathrm{Coh}(Y \times_X Y)$ and the short exact sequence

$$p^*(\mathcal{C}) \to p^!(\mathcal{C}) \to \mathrm{MF}_p(\mathcal{C})$$

is obtained from the short exact sequence (1.4) via $- \otimes_{\mathrm{Coh}(Y \times_X Y)} p^!(\mathcal{C})$.

In the rest of this introduction we try to explain the meaning of the theorem and give some examples. First, it implies that

$$p^*(\mathcal{C}) \simeq \mathrm{Perf}(Y \times_X Y) \otimes_{\mathrm{Coh}(Y \times_X Y)} p^!(\mathcal{C}),$$

which can be phrased, via Koszul duality, as "$p^*(\mathcal{C})$ is the part of $p^!(\mathcal{C})$ supported at the zero section of the conormal bundle". In particular, the usual and the exceptional pullback coincide if and only if $p^!(\mathcal{C})$ is supported at the zero section of the conormal bundle.

Example 1.11. Consider p as the inclusion of the origin on $\mathbb{A}^n$ and $\mathcal{C}$ as the quasi-coherent sheaf of categories $\mathrm{Coh}(Z)$ over $\mathbb{A}^n$ given by a smooth scheme Z which is proper over $\mathbb{A}^n$.

Then $p^*(\mathcal{C}) = \mathrm{Perf}(Z_0)$, $p^!(\mathcal{C}) = \mathrm{Coh}(Z_0)$ and $\mathrm{MF}_p(\mathcal{C}) = \mathrm{Sing}(Z_0)$. The monodromy are the cohomological operations on the special fiber Z_0 like in the paper [AG].

The theorem explains two well known facts for complete intersection schemes: perfect complexes are where the cohomological operations are nilpotent and endomorphisms of coherent complexes are finitely generated over the cohomological operations. □

Example 1.12. For a smooth scheme X, the diagonal embedding $X \to X \times X$ may not be a complete intersection but the theorem still holds by [AC]. This will be study in [dFS]. □

Now, we focus on the closed embedding of a point $i_x : x \to X$. For simplicity, lets fix an isomorphism of the formal completion of $\mathcal{O}_{X,x}$ with $k[\![t_1, \dots, t_n]\!]$. Its E_2-Koszul dual is hence naturally isomorphic to $k[\beta_1, \dots, \beta_n]$ with all β_i of degree 2.

Given a quasi-coherent sheaf $\mathcal{C}$ over X, we will refer to $i_x^*(\mathcal{C})$ and $i_x^!(\mathcal{C})$ as its fiber and cofiber at x. The monodromy is an action of $\mathrm{Coh}(\Omega_x X)$ with convolution and Koszul duality translates this into an action of $\mathrm{Perf}(T_x^* X[2])$ i.e. an $k[\beta_1, \dots, \beta_n]-$linear structure.

In section 9 we review the notion of support of a $k[\beta_1, \dots, \beta_n]-$linear category following the expositions in [AG] and [BIK]. For the rest of the introduction it is enough to say the following: for proper $k[\beta_1, \dots, \beta_n]-$linear categories its support can be computed as the support of the graded Homs of the associated triangulated category.

Definition 1.13. Given a point x in a smooth scheme X and a proper quasi-coherent sheaf of categories $\mathcal{C}$ over it, we define $SS_x(\mathcal{C})$ as the $k[\beta_1, \dots, \beta_n]-$linear support of $i_x^!(\mathcal{C})$.

The above defines the punctual singular support $SS_x(\mathcal{C})$ of a proper quasi-coherent sheaf of categories $\mathcal{C}$, as a closed conical subset of the cotangent fibre $T_x^* X$.

For the following recall that for proper $k-$linear categories there are different notions of smoothness: smooth, regular and right saturated. For precise definitions check section 10.

Proposition 1.14. Given a coherent sheaf of categories $\mathcal{C}$ over a smooth scheme X, the following conditions are all equivalent:

- The punctual singular support $SS_x(\mathcal{C})$ is contained in the zero section.
- The functor $i_x^*(\mathcal{C}) \to i_x^!(\mathcal{C})$ is an equivalence.
- The $k-$linear category $i_x^*(\mathcal{C})$ is right saturated.

Example 1.15. Consider a smooth scheme X. For a proper and smooth quasi-coherent sheaf of categories $\mathcal{C}$, all its fibers are also proper and smooth. In particular, such sheaves have right saturated fibers and its punctual singular support is contained in the zero section. $\square$

Now, we give criteria to decide when a covector (x, v^*) is in the punctual singular support of a coherent sheaf of categories. It is analogous to the vanishing cycles test. For a given function $\phi : X \to \mathbb{A}^1$ we denote by $i_\phi : X_0 \to X$ the closed immersion of the special fiber.

Proposition 1.16. Consider a smooth scheme X with a function $\phi : X \to \mathbb{A}^1$ and a coherent sheaf of categories $\mathcal{C}$. Given a point x, if the fiber $i^*_{x,X_0}(\mathrm{MF}_{i_\phi}(\mathcal{C}))$ is non-trivial then the covector $(x, d_x\phi)$ is in $SS_x(\mathcal{C})$. If $i^!_{x,X}(\mathcal{C})$ has a compact generator the converse holds.

To finish we give a deformation theory interpretation. Recall that for a $k-$linear category, its curved deformations over an Artinian E_2-algebra A are classified by $\mathrm{KD}_2(A)-$linear structures on it, with $\mathrm{KD}_2(A)$ the E_2-Koszul dual of A. The unobstructed objects are those on which the augmentation ideal of $\mathrm{KD}_2(A)$ acts trivially. This is in [Lur2].

The point is that the $k[\beta_1, \dots, \beta_n]-$linear structure on the cofiber $i^!_x(\mathcal{C})$ is encoding a deformation over Spf $\widehat{\mathcal{O}}_{X,x}$. The unobstructed objects are those on which $\beta_1, \dots, \beta_n$ acts trivially and the proof of 1.10 will show that the full subcategory generated by them is $i^*_x(\mathcal{C})$.

In particular, 1.14 says that the punctual singular support is contained in the zero section if and only if, the whole cofiber is generated by unobstructed objects. Moreover, in any curved deformation of a right saturated category, every object is a retract of a finite colimit of unobstructed object. The criteria in 1.16 says that when (x, v^*) is not in the punctual singular support $SS_x(\mathcal{C})$, if an object in $i^!_x(\mathcal{C})$ is unobstructed along X_0, then it is unobstructed.

This is part of joint work with G. Stefanich. In [dFS] we will define a better invariant of a coherent sheaf of categories: its global singular support which is a closed conical subset of the cotangent bundle analogous to the singular support of a D-module.

Conventions and Notation

1. We fix a field k of characteristic 0.
2. We work as derived as possible. In particular, category means $(\infty, 1)-$category, groupoid means $\infty-$groupoid and scheme means derived scheme.
3. We denote by $\widehat{\mathrm{Cat}}_\infty$ the category of all categories and arbitrary functors.

4. We will assume all small stable categories to be $k-$linear and idempotent complete and all big stable categories to be $k-$linear and presentable.

5. We denote by DG-Cat_A the category whose objects are $A-$linear presentable stable categories and whose morphisms are functors that preserve colimits. It is a closed symmetric monodial category with the tensor product constructed by Lurie.

6. We denote by DG-Cat_A^{sm} the category whose objects are $A-$linear small idempotent complete stable categories and whose morphisms are $A-$linear functors between them.

   This category coincides with the subcategory of DG-Cat_A consisting of compactly generated categories and functors whose right adjoint preserves colimits.

   The category DG-Cat_A^{sm} is closed symmetric monoidal with tensor product the compact objects of the tensor product of their ind-completions.

7. We denote by PreStacks_k the category of all prestacks over Spec k.

8. All schemes are assumed of finite type over k. In particular, for every such scheme X, we have $\text{QCoh}(X)$ and $\text{IndCoh}(X)$ with the usual functorialities [GR, 3.1.1].

9. For a closed subscheme S of a scheme X, we denote by $\text{Perf}_S(X)$ the full subcategory of perfect complexes in $\text{Perf}(X)$ that vanish on $X - S$.

10. For a scheme X, we use the notation $(-) \otimes_X (-)$ as a shorthand for the relative tensor product over $\text{QCoh}(X)$ or $\text{Perf}(X)$ depending if we are dealing with big or small categories respectively. Similarly we use $\text{Fun}_X(-,-)$ for the relative inner Hom.

11. A morphism $p : Y \to X$ is complete intersection if $p = i \circ f$ with i a complete intersection closed embedding and f a smooth map. The expression "Y is a complete intersection in X" means complete intersection closed embedding.

CHAPTER 2

PRELIMINARIES I: FOURIER-MUKAI TRANSFORM

We explain the tensor product and the functor category theorems. It will be important to pay special attention to the difference between the versions with big and small categories.

Given a map $p : Y \to X$ and a QCoh(X)−linear category $\mathcal{C}$, the Fourier-Mukai transform is the functor

$$\mathrm{FM} : \mathrm{QCoh}(Y) \otimes_X \mathcal{C} \longrightarrow \mathrm{Fun}^L_X(\mathrm{QCoh}(Y), \mathcal{C})$$

such that for a compact object M of QCoh(Y) and a compact object C of $\mathcal{C}$ we have

$$\mathrm{FM}(M \otimes_X C)(-) = p_*((-) \otimes_Y M) \otimes_X C. \tag{2.1}$$

The Fourier-Mukai transform is an equivalence. This is because QCoh(Y) is dualizable as a QCoh(X)−linear category, with dual canonically identified with itself via naive duality.

Remark 2.1. If p is proper and finite tor-dimension, the Fourier-Mukai transform is well defined with small categories, but it is not an equivalence. For this we need to prove that for compact bimodules, its associated functor preserves compact objects. This is because under the hyphotesis of p proper and finite tor-dimension, p_* preserves compact objects.

Example 2.2. If we consider $\mathcal{C} = \mathrm{QCoh}(Z)$ with $q : Z \to X$, a straightforward computation using base change and projection formula shows that

$$\mathrm{FM}(M \otimes_X N)(-) = p_*((-) \otimes_Y M) \otimes_X N = q^*(p_*((-) \otimes_Y M)) \otimes_Z N$$

$$\mathrm{FM}(M \otimes_X N)(-) = p_*(q^*((-) \otimes_Y M)) \otimes_Z N = p_*(q^*(-) \otimes_{Y \times_X Z} (M \boxtimes_X N)),$$

which is the usual Fourier-Mukai transform. This is where the name comes from and it can be checked that composition of functors can be interpreted as convolution of sheaves. □

Example 2.3. If $\mathcal{C} = \mathrm{QCoh}(Y)$ and $M = C = \mathcal{O}_Y$ then

$$\mathrm{FM}(\mathcal{O}_Y \otimes_X \mathcal{O}_Y)(-) = p_* p^*(-) = p^* p_*(-)$$

as can be checked using the last formula of the previous example. □

Now we state the two theorems. It is important to keep in mind that behind the scenes, there is an identification of $\mathrm{QCoh}(Y)$ with its dual category $\mathrm{QCoh}(Y)^{op}$.

Proposition 2.4. [BZFN, 1.2] Given schemes X, Y and Z and maps $Y \to X$ and $Z \to X$, we have the following identities:

$$\mathrm{QCoh}(Y \times_X Z) = \mathrm{QCoh}(Y) \otimes_X \mathrm{QCoh}(Z) = \mathrm{Fun}_X^L(\mathrm{QCoh}(Y), \mathrm{QCoh}(Z)).$$

For a bimodule on the left, the composition is the following functor: for a quasi-coherent sheaf in Y, we pullback to $Y \times_X Z$, then tensor with the bimodule and pushforward to Z. The functor from the second to the third term is the Fourier-Mukai transform.

Example 2.5. Given $p : Y \to X$, if $Z = Y$, the diagonal bimodule $\mathcal{O}_Y$ in $Y \times_X Y$ corresponds to the identity functor and the bimodule $\mathcal{O}_{Y \times_X Y}$ to the functor $p^* p_*(-)$.

Finally we state the version with small categories. The first claim holds in general but for the second we need $Y \to X$ to be proper. Several improvements are discussed in [BZNP].

Proposition 2.6. [BZNP 1.1.3] Given schemes X, Y and Z, a proper map $Y \to X$ and an arbitrary map $Z \to X$, we have the following identities:

$$\mathrm{Perf}(Y \times_X Z) = \mathrm{Perf}(Y) \otimes_X \mathrm{Perf}(Z) \quad \text{and} \quad \mathrm{Coh}(Y \times_X Z) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathrm{Coh}(Z)).$$

Remark 2.7. The difference between the two propositions is the main reason we will choose to work with small categories later on. The point is that with small categories we see more.

CHAPTER 3

PRELIMINARIES II: CONVOLUTION

Consider a groupoid scheme $G \rightrightarrows S$ with $m, p_1, p_2 : G \times_S G \to G$ where m denotes the multiplication and p_1 and p_2 the projections. The convolution of two quasi-coherent sheaves is defined by the formula

$$\mathscr{F}_1 * \mathscr{F}_2 = m_*(p_1^*(\mathscr{F}_1) \otimes p_2^*(\mathscr{F}_2)).$$

This gives QCoh(G) a monoidal structure. In case m, p_1 and p_2 are proper the convolution of coherent sheaves is coherent and if moreover they are finite tor-dimension, the convolution of perfect sheaves is perfect. The only caveat is that the unit may not be perfect. [GR, 5.5]

There are two examples of groupoids we care about: vector bundles and convolution groupoids. We review its definitions and basic facts and state a version of Koszul duality.

Definition 3.1. Given a scheme X and a perfect sheaf $\mathscr{F}$ cohomologically supported in non-positive degrees, we define the vector bundle $\mathrm{Vect}_X(\mathscr{F})$ as $\mathrm{Spec}_X(Sym_{\mathcal{O}_X}(\mathscr{F}))$.

We refer to the above as the vector bundle associated with $\mathscr{F}$. The sum and multiplication by scalars on its fibers gives vector bundles the structure of a group scheme over X and an action of $\mathbb{G}_m$ respectively. In particular, they are groupoids schemes over X.

Remark 3.2. Although $\mathrm{Vect}_X(\mathscr{F})$ makes no sense as a scheme if $\mathscr{F}$ is not cohomologically supported in non-positive degrees, we can define $\mathrm{Perf}(\mathrm{Vect}_X(\mathscr{F}))$ in full generality as the compact objects of the category of $Sym_{\mathcal{O}_X}(\mathscr{F})-$modules in QCoh($X$).

Example 3.3. Given a map $Y \to X$ with $L_{Y/X}[-1]$ a locally free sheaf, we define the normal and conormal bundles $N_{Y/X}$ and $N^*_{Y/X}$ as $\mathrm{Vect}_Y(L_{Y/X}[-1])$ and $\mathrm{Vect}_Y(L^\vee_{Y/X}[1])$.

There are also shifted versions of the above. For instance we define the shifted normal bundle $N_{Y/X}[-1]$ as the vector bundle associated with the cotangent complex $L_{Y/X}$. □

Proposition 3.4. Given a locally free sheaf $\mathscr{F}$ in a smooth scheme X, there are equivalences

$$\mathrm{Perf}(\mathrm{Vect}_X(\mathscr{F}[1])) \simeq \mathrm{Perf}_X(\mathrm{Vect}_X(\mathscr{F}^\vee[-2]))$$

and

$$\mathrm{Coh}(\mathrm{Vect}_X(\mathscr{F}[1])) \simeq \mathrm{Perf}(\mathrm{Vect}_X(\mathscr{F}^\vee[-2]))$$

which interchanges convolution with the usual tensor product.

Proof. The trick is that $\mathrm{Coh}(\mathrm{Vect}_X(\mathscr{F}[1]))$ is generated by $\mathcal{O}_X$ as a $\mathrm{Perf}(X)-$linear category. This is because any coherent sheaf is generated by sheaves scheme-theoretically supported on the zero section and such sheaves are generated by $\mathcal{O}_X$ because X is smooth. □

Definition 3.5. Given a map $Y \to X$, its convolution groupoid $Y \times_X Y$ is its Cech Nerve.

The convolution unit in $\mathrm{QCoh}(Y \times_X Y)$ is $\mathcal{O}_Y$. If the map $Y \to X$ is proper and finite tor-dimension, convolution is also defined for coherent and perfect sheaves. If we interpret this categories via 2.4 and 2.6, convolution is composition of functors by 2.2 and we get

$$\begin{array}{ccccc}
\mathrm{Perf}(Y \times_X Y) & \overset{\Psi_L}{\dashrightarrow} & \mathrm{Coh}(Y \times_X Y) & \xrightarrow{\Psi} & \mathrm{QCoh}(Y \times_X Y) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Perf}(Y) \otimes_X \mathrm{Perf}(Y) & \dashrightarrow & \mathrm{Fun}^{ex}_X(\mathrm{Perf}(Y), \mathrm{Perf}(Y)) & \longrightarrow & \mathrm{Fun}^L_X(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y)).
\end{array}$$

Remark 3.6. The above functors are fully faithful and compatible with convolution.

Proposition 3.7. If Y is a complete intersection in X, then $Y \times_X Y$ and $N_{Y/X}[-1]$ coincide as groupoids schemes over Y. In particular, $Y \times_X Y$ is a group and if Y is smooth:

$$(\mathrm{Coh}(Y \times_X Y), *) \simeq (\mathrm{Coh}(N_{Y/X}[-1]), *) \simeq (\mathrm{Perf}(N^*_{Y/X}[2]), \otimes).$$

Proof. The last sentence is a direct consequence of the first and 3.4. For the first observe that the formation of convolution groupoids and normal bundles of a morphism commutes with pullbacks on the base. This easily reduce us to the case of the origin on affine space.

The derived self intersection $0 \times_{\mathbb{A}^n} 0$ can be computed via an explicit Koszul resolution which shows the convolution groupoid and the shifted normal bundle agree as schemes.

To finish the proof, we need to show that in $0 \times_{\mathbb{A}^n} 0$, the convolution product, which is composition of loops, and the normal bundle product, which is pointwise addition using the vector space sum of $\mathbb{A}^n$, agree. The two products are compatible in the sense of the argument of Eckmann-Hilton and hence they agree. The case $n = 1$ is in [Pre, 3.1.1.1]. □

Proposition 3.8. Let Y be a complete intersection in X. Then, in $\mathrm{QCoh}(Y \times_X Y)$ the convolution unit $\mathcal{O}_Y$, is a filtered colimit of finite colimits of shifts of the tensor unit $\mathcal{O}_{Y \times_X Y}$.

Proof. The hyphotesis of complete intersection implies that via pullback we can reduce to the case of the origin on $\mathbb{A}^n$. This is because pullbacks are symmetric monoidal for convolution and tensor product. In this case, the convolution groupoid $0 \times_{\mathbb{A}^n} 0$ is affine and hence the statement is trivial because everything is a colimit of the tensor unit. □

Example 3.9. If the map $Y \to X$ is the inclusion of the origin on $\mathbb{A}^1$ the above results can be made explicit. The convolution groupoid $0 \times_{\mathbb{A}^1} 0$ is the loop group $\Omega_0\mathbb{A}^1$ which can be described as the affine group scheme Spec $k[\lambda]$ with λ of degree -1.

In 3.7, the cotangent fiber $T_0^*\mathbb{A}^1[2]$ is given by Spec $k[\beta]$ with β of degree 2 and the equivalence is the Koszul duality

$$(\mathrm{Coh}(\Omega_0\mathbb{A}^1), *) \simeq (\mathrm{Coh}(k[\lambda]), *) \simeq (\mathrm{Perf}(k[\beta]), \otimes) \simeq (\mathrm{Perf}(T_0^*\mathbb{A}^1[2]), \otimes).$$

In 3.8, the convolution unit $\mathcal{O}_0$ is k and the tensor unit $\mathcal{O}_{\Omega_0\mathbb{A}^1}$ is $k[\lambda]$. The resolution of the convolution unit $\mathcal{O}_0$ is given by

$$\cdots \to \mathcal{O}_{\Omega_0\mathbb{A}^1}[2] \to \mathcal{O}_{\Omega_0\mathbb{A}^1}[1] \to \mathcal{O}_{\Omega_0\mathbb{A}^1} \;\text{ in }\; \mathrm{QCoh}(\Omega_0\mathbb{A}^1)$$

which is the usual Koszul resolution of k as a $k[\lambda]-$module. □

CHAPTER 4

PRELIMINARIES III: QUASI-COHERENT SHEAVES OF CATEGORIES

In this section we review the general formalism of quasi-coherent sheaves of categories on schemes following the expositions [Gai], [Lur5] and [Lur6].

Definition 4.1. The functor $2-\mathrm{QCoh} : \mathrm{PreStacks}_k \to \widehat{\mathrm{Cat}}_\infty$ is the right Kan extension of the functor that to any affine scheme Spec A associates the category DGCat_A.

For every prestack X, there is a category $2-\mathrm{QCoh}(X)$ which can be seen to have all limits and colimits. The objects can be described as quasi-coherent sheaves of categories, that is, for every $A-$point we have an $A-$linear big stable category with the usual compatibilities.

Definition 4.2. We say that a quasi-coherent sheaf of categories is compactly generated if for every $A-$point, the associated $A-$linear big stable category is compactly generated.

The category $2-\mathrm{QCoh}(X)$ is closed symmetric monoidal. In particular, a compactly generated quasi-coherent sheaf of categories $\mathcal{C}$ is dualizable with $\mathcal{C}^\vee \simeq \mathrm{Ind}((\mathcal{C}^c)^{op})$ and moreover the internal Hom satisfies

$$\mathrm{Fun}^L_X(\mathcal{C}, \mathcal{D}) \simeq \mathcal{C}^\vee \otimes \mathcal{D}.$$

Remark 4.3. We recall that since $2-\mathrm{QCoh}(X)$ contains all colimits, given an algebra $\mathcal{A}$, a right module $\mathcal{C}$ and a left module $\mathcal{D}$, the relative tensor product $\mathcal{C} \otimes_\mathcal{A} \mathcal{D}$ can be computed via the cobar construction as the colimit of

$$\cdots \mathcal{C} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{D} \rightrightarrows \mathcal{C} \otimes \mathcal{A} \otimes \mathcal{D} \rightrightarrows \mathcal{C} \otimes \mathcal{D}.$$

For every map $f : X \to Y$ there is a symmetric monoidal pullback f^*. It commutes with limits and colimits and, for schemes, its right adjoint f_* commutes with limits and colimits.

Remark 4.4. For a scheme X, the lax-monoidal functor of global sections gives an equivalence between the category $2-\mathrm{QCoh}(X)$ and that of $\mathrm{QCoh}(X)-$Modules in DGCat_k [Gai, 2.1.1].

Remark 4.5. The functor $A \mapsto \mathrm{DGCat}_A$ is an etale sheaf. This implies quasi-coherent sheaves of categories are fppf sheaves [Gai, 1.5.4] and $2-\mathrm{QCoh}$ is a fppf sheaf [Gai, 1.5.7].

Theorem 4.6. For schemes, the base change formula holds [Lur5, 1.5.3]. The same is also true for the projection formula [Lur5, 2.6.6].

Recall that an $A-$linear big category is proper, respectively smooth, if it is compactly generated and the evaluation, respectively the coevaluation, preserves compact objects.

Definition 4.7. A quasi-coherent sheaf of categories is proper, respectively smooth, if for every $A-$point, the associated $A-$linear big stable category is proper, respectively smooth.

It is proved that for a scheme, a quasi-coherent sheaf of categories is compactly generated, respectively proper, respectively smooth if and only if its global sections are compactly generated [Lur5, 3.2.1], respectively proper [Lur6, 1.3.3], respectively smooth [Lur6, 3.4.7].

Definition 4.8. We define $2-\mathrm{Perf}(X)$ as the full subcategory of $2-\mathrm{QCoh}(X)$ of proper and smooth quasi-coherent sheaves of categories i.e. perfect sheaves of categories.

It is easy to see that being proper or smooth is preserved by pullbacks. The pushforward preserves being proper if and only if the map is proper and finite tor-dimension [Lur6, 1.4.1] and preserves smoothness if and only if the map is smooth [Lur6, 3.6.1].

Remark 4.9. In the formalism described above it seems hopeless to have $f^!$ because f^* is both, the left and the right adjoint of f_*. It is instructive to explain why.

The category $2-\mathrm{QCoh}(X)$ is a 2-category and the unit and counit of the adjunction that presents f^* as a left adjoint to f_* are right adjointable. This adjoint natural transformations can be showed to work as counit and unit to present f^* as the right adjoint to f_*.

The above has another consequence which is a general fact for such an adjunction between symmetric monoidal $2-$category. If we restrict to the subcategory of dualizable objects with only right adjointable morphism, the functor f^* is still the left adjoint to f_*.

CHAPTER 5

THE SMALL VERSION OF $2 - \mathbf{QCOH}(X)$

Now we present a small modification of the formalism of the previous section. Later on, this will be used to get $f^!$. The idea, following remark 4.9, is to switch from $2 - \mathrm{QCoh}(X)$ to its subcategory of compactly generated sheaves and right adjointable morphisms.

Definition 5.1. We define $2 - \mathrm{QCoh}^{sm}(X)$ as the subcategory of $2 - \mathrm{QCoh}(X)$ with objects compactly generated quasi-coherent sheaves and morphisms compact-preserving functors.

The above is the formal definition but because of 2.7, we will think of it in a slightly different way. Lets recall that compactly generated big stable categories corresponds to small stable categories. Moreover, right adjointable functors between compactly generated big stable categories corresponds to functors between their small counterparts.

From now on, we think of objects of $2 - \mathrm{QCoh}^{sm}(X)$ as an assigment of an $A-$linear small stable category to every $A-$point of X with suitable compatibilities.

Proposition 5.2. The category $2 - \mathrm{QCoh}^{sm}(X)$ is presentable, in particular it has all small limits and all small colimits. Moreover the inclusion in $2 - \mathrm{QCoh}(X)$ preserves colimits.

Proof. This reduces to the special case $X = \mathrm{Spec}\ k$. To see it has all limits and filtered colimits check [Lur4, 1.1.4.4] and [Lur4, 1.1.4.6]. The existence of colimits and that they are preserved by the inclusion into presentable categories is [Lur1, 5.5.3.13]. □

Proposition 5.3. For a scheme X, the global sections functor gives an equivalence between the category $2 - \mathrm{QCoh}^{sm}(X)$ and the category of $\mathrm{Perf}(X)-$Modules in DGCat_k.

Proof. This is Gaitsgory's theorem from 4.4 and the fact that since $\mathrm{QCoh}(X)$ is rigid there is no difference between an action of $\mathrm{QCoh}(X)$ on a compactly generated big stable category and an action of $\mathrm{Perf}(X)$ on its compact objects [Gai, 5.1.7, D.2.2]. □

Remark 5.4. For the rest of the text we will use the same notation for a quasi-coherent sheaf on a scheme X and for its global sections as $\mathrm{Perf}(X)-$linear small stable category.

The category $2-\mathrm{QCoh}^{sm}(X)$ has a tensor product: the compact objects of the tensor product of their ind-completions. There is also an internal Hom: their functor category.

Proposition 5.5. For a scheme X, the category $2-\mathrm{QCoh}^{sm}(X)$ is closed symmetric monoidal and its tensor product commutes with colimits in each variable. Moreover, the inclusion into $\mathrm{QCoh}(X)$ is a symmetric monoidal functor.

Proof. The last sentence is clear. To check that the tensor product and the internal Hom defined above satisfies the right adjunction property we need to prove that

$$\mathrm{Fun}_X(\mathcal{C}\otimes\mathcal{D},\mathcal{E})=\mathrm{Fun}_X(\mathcal{C},\mathrm{Fun}_X(\mathcal{D},\mathcal{E})).$$

This follows from the same identity between their ind-completed versions, which is easy, and the fact that the cartesian product $\mathcal{C}\times\mathcal{D}$ generates the compact objects of $\mathrm{Ind}(\mathcal{C})\otimes\mathrm{Ind}(\mathcal{D})$.

The fact that the tensor product commutes with colimits in each variable can be checked with big categories, in which case is clear, because the inclusion commutes with colimits. □

In particular, $2-\mathrm{QCoh}^{sm}(X)$ has a relative tensor product, the compact objects of the relative tensor product of their ind-completions, and a relative Hom, the compact preserving functors of the relative Hom between their ind-completions.

Example 5.6. Given an algebra $\mathcal{A}$ in $2-\mathrm{QCoh}^{sm}(X)$ we consider an $\mathcal{A}-$module $\mathcal{M}$, together with an $\mathcal{A}-$algebra $\mathcal{B}$ and a $\mathcal{B}-$module $\mathcal{N}$. Then $\mathrm{Fun}_{\mathcal{A}}(\mathcal{M},\mathcal{N})=\mathrm{Fun}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{M},\mathcal{N})$.

This amounts to show that the functor $\mathrm{Ind}(\mathcal{M})\to\mathrm{Ind}(\mathcal{N})$ preserves compact objects if and only if the functor $\mathrm{Ind}(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{M})\to\mathrm{Ind}(\mathcal{N})$ does. For this, its enough to check that the unit and counit of the extensions of scalars adjunction,

$$\mathrm{Ind}(\mathcal{M})\to\mathrm{Ind}(\mathcal{B})\otimes_{\mathrm{Ind}(\mathcal{A})}\mathrm{Ind}(\mathcal{M})\quad\text{and}\quad\mathrm{Ind}(\mathcal{B})\otimes_{\mathrm{Ind}(\mathcal{A})}\mathrm{Ind}(\mathcal{N})\to\mathrm{Ind}(\mathcal{N}),$$

preserve compact objects. This is true because the functor $\mathrm{Ind}(\mathcal{A})\to\mathrm{Ind}(\mathcal{B})$ does and the above unit and counit are easily constructed from it. □

Definition 5.7. A quasi-coherent sheaf in $2-\mathrm{QCoh}^{sm}(X)$ is proper, respectively smooth, if for every $A-$point, the corresponding $A-$linear category is proper, respectively smooth.

A quasi-coherent sheaf in $2-\mathrm{QCoh}^{sm}(X)$, is proper or smooth if and only if its global sections are. Moreover, proper and smooth is equivalent to being dualizable [Lur6, 4.0.2] or, equivalently, to being fully dualizable in the bigger $2-$category $2-\mathrm{QCoh}(X)$.

Proposition 5.8. For a map $f : X \to Y$ between schemes, we have that the enriched adjoint pair of functors f^* and f_* between the big versions of quasi-coherent sheaves of categories restricts to an enriched adjoint pair between the small versions. Moreover f^* is symmetric monodial and base change and projection formula holds.

Proof. The functors f^* and f_* restricts to the small version because they preserve compactly generated sheaves and compact preserving functors. The claim about the enriched adjunction is 5.6 and the last sentence can be checked after the inclusion in the big version. □

Lemma 5.9. Given a map $f : X \to Y$, an algebra $\mathcal{A}$ in $2-\mathrm{QCoh}^{sm}(Y)$, a left $\mathcal{A}-$module $\mathcal{D}$ and a right $f^*(\mathcal{A})-$module $\mathcal{C}$, there is an isomorphism

$$f_*(\mathcal{C} \otimes_{f^*(\mathcal{A})} f^*(\mathcal{D})) \simeq f_*(\mathcal{C}) \otimes_{\mathcal{A}} \mathcal{D}.$$

Proof. This is 4.3, the fact that f_* commutes with colimits and the projection formula. □

Proposition 5.10. The functor f^* preserves being proper or smooth. Moreover f_* preserves properness if f is proper and finite tor-dimension and smoothness if f is smooth.

Proof. This follows from the similar property but for big categories. □

Finally we introduce a finiteness condition on quasi-coherent sheaves which will be used in the last section. It is milder than perfect but still is enough for some things.

Definition 5.11. A quasi-coherent sheaf of categories in $2-\mathrm{QCoh}^{sm}(X)$ is "coherent" if it is proper relative to the scheme X and it is smooth relative to Spec k.

CHAPTER 6

THE EXCEPTIONAL PULLBACK AND MATRIX FACTORIZATIONS

We introduce the exceptional pullback $p^!$ for p proper and finite tor-dimension and we compare it with the usual pullback p^*. It can be defined as the right adjoint of p_* or by an explicit formula inspired by [BZNP, 1.1.5]. We also define the matrix factorization functor.

Definition 6.1. For a proper and finite tor-dimension map $p : Y \to X$, we define the exceptional pullback functor $p^!(-)$ by the formula $\mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), -)$.

The formula is "what it should be". That is, if we assume there is a right adjoint, it should be given by the formula above. Lets check directly the enriched adjunction. For this, we need to show that for every quasi-coherent sheaf $\mathcal{D}$ in Y we have

$$\mathrm{Fun}_X^{ex}(\mathcal{D}, \mathcal{C}) = \mathrm{Fun}_Y^{ex}(\mathcal{D}, \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathcal{C}))$$

as full subcategories of

$$\mathrm{Fun}_X^L(\mathrm{Ind}(\mathcal{D}), \mathrm{Ind}(\mathcal{C})) = \mathrm{Fun}_Y^L(\mathrm{Ind}(\mathcal{D}), \mathrm{Fun}_X^L(\mathrm{QCoh}(Y), \mathrm{Ind}(\mathcal{C}))).$$

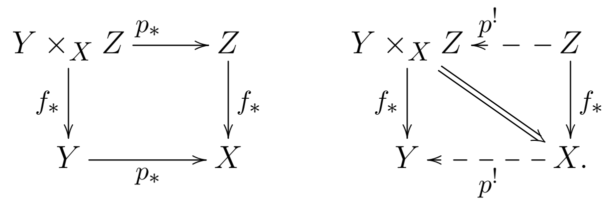
On the left we have the full subcategory of functors that send $\mathcal{D}$ to $\mathcal{C}$. Similarly, on the right, we have the full subcategory of functors that send objects of $\mathcal{D}$ to functors that send compact objects of $\mathrm{QCoh}(Y)$ to $\mathcal{C}$. But because of the $\mathrm{QCoh}(Y)$-linearity of the functors, to check this last condition it is enough to do it for $\mathcal{O}_Y$. This shows both sides agree.

Example 6.2. [BZNP, 1.1.3] For a proper map $p : Y \to X$, by 2.6, for any Z over X we have that

$$p^!(\mathrm{Coh}(Z)) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathrm{Coh}(Z)) = \mathrm{Coh}(Y \times_X Z),$$

and in particular $p^!(\mathrm{Coh}(X)) = \mathrm{Coh}(Y)$ i.e. $\mathrm{Coh}(X)$ behaves as the dualizing sheaf. □

Proposition 6.3. If $p : Y \to X$ is proper and finite tor-dimension and $f : Z \to X$ is an arbitrary morphism, then the natural transformation $f_* p^! \to p^! f_*$ is an equivalence.



Proof. This is a formal consequence of base change between f^* and f_*: in a 2−category a square is horizontally right adjointable if and only if it is vertically left adjointable. We can also give a direct proof. For every $\mathcal{C}$ on Z we have,

$$f_* p^!(\mathcal{C}) = f_*(\mathrm{Fun}_Z^{ex}(\mathrm{Perf}(Y \times_X Z), \mathcal{C})) = f_*(\mathrm{Fun}_Z^{ex}(\mathrm{Perf}(Y) \otimes_{\mathrm{Perf}(X)} \mathrm{Perf}(Z), \mathcal{C}))$$

where the second equality is the formula for Perf on a fiber product, and hence

$$f_* p^!(\mathcal{C}) = f_*(\mathrm{Fun}_Z^{ex}(f^*(\mathrm{Perf}(Y)), \mathcal{C})) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), f_*(\mathcal{C})) = p^! f_*(\mathcal{C})$$

where the second equality is the enriched adjunction between f^* and f_*. $\square$.

Now lets fix a proper and finite tor-dimension map $p : Y \to X$ and a quasi-coherent sheaf of categories $\mathcal{C}$ and lets compare $p^*(\mathcal{C})$ and $p^!(\mathcal{C})$. The key observation is that both small categories fully-faithfull embeds into the big functor category

$$\mathrm{QCoh}(Y) \otimes_X \mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}_X^L(\mathrm{QCoh}(Y), \mathrm{Ind}(\mathcal{C}))$$

the first as its compact objects and the second as the functors that preserve compact objects.

Since p is proper and finite tor-dimension, p_* preserves perfect complexes and 2.1 implies that the Fourier-Mukai transform is defined for small categories. In particular, there is a functor $\Psi_L(-) : p^*(-) \to p^!(-)$ that sends $M \otimes_X C$ to the functor $p_*((-) \otimes_Y M) \otimes_X C$.

The above can be summarized in the following picture:

$$\begin{array}{ccc} p^*(\mathcal{C}) & \overset{\Psi_L}{\dashrightarrow} & p^!(\mathcal{C}) \\ & \searrow & \downarrow{\scriptstyle \Psi} \\ & & \operatorname{Ind}(p^*(\mathcal{C})) \end{array} \qquad \begin{array}{ccc} \operatorname{Perf}(Y) \otimes_X \mathcal{C} & \overset{\Psi_L}{\dashrightarrow} & \operatorname{Fun}_X^{ex}(\operatorname{Perf}(Y), \mathcal{C}) \\ & \searrow & \downarrow{\scriptstyle \Psi} \\ & & \operatorname{Fun}_X^{L}(\operatorname{QCoh}(Y), \operatorname{Ind}(\mathcal{C})). \end{array}$$

The two vertical functors are fully faithful and hence also the horizontal one. The notation is because the functor $\operatorname{Ind}(\Psi_L)$ is left adjoint to the ind-extension of Ψ.

Example 6.4. If $p : Y \to X$ is proper and smooth, then $\operatorname{Perf}(Y) \otimes_X \mathcal{C} = \operatorname{Fun}_X^{ex}(\operatorname{Perf}(Y), \mathcal{C})$ because $\operatorname{Perf}(Y)$ is dualizable over $\operatorname{Perf}(X)$. In this case Ψ_L is an equivalence. □

Example 6.5. If $p : Y \to X$ is proper and $\mathcal{C}$ dualizable, $\operatorname{Coh}(Y) \otimes_X \mathcal{C} = \operatorname{Fun}_X^{ex}(\operatorname{Perf}(Y), \mathcal{C})$ because $\operatorname{Coh}(Y) = \operatorname{Fun}_X^{ex}(\operatorname{Perf}(Y), \operatorname{Perf}(X))$. If Y is smooth, then Ψ_L is an equivalence. □

In the above examples, the fully faithful embedding between the two pullbacks is an equivalence but in general it is not. This inspires the following definition.

Definition 6.6. For $p : Y \to X$ proper of finite tor-dimension and $\mathcal{C}$ a quasi-coherent sheaf of categories over X we define $\operatorname{MF}_p(\mathcal{C})$ as the cofiber of $p^*(\mathcal{C}) \to p^!(\mathcal{C})$.

We will refer to $\operatorname{MF}_p(\mathcal{C})$ as the matrix factorizations of $\mathcal{C}$ with respect to p. This is arguably the most important definition in the text and it comes with a short exact sequence

$$p^*(\mathcal{C}) \to p^!(\mathcal{C}) \to \operatorname{MF}_p(\mathcal{C}). \tag{6.1}$$

The examples shows $\operatorname{MF}_p(\mathcal{C})$, as a sheaf over Y, is trivial wherever the morphism p is smooth or the category $\mathcal{C}$ is dualizable. In this sense, matrix factorizations is reminiscent of vanishing cycles because it is supported in the singularities of the map p and of the sheaf $\mathcal{C}$.

Example 6.7. If $p : Y \to X$ is proper and of finite tor-dimension and $\mathcal{C} = \operatorname{Coh}(Z)$ for some map $Z \to X$, then (6.1) is $\operatorname{Perf}(Y \times_X Z) \to \operatorname{Coh}(Y \times_X Z) \to \operatorname{Sing}(Y \times_X Z)$. □

CHAPTER 7

THE MONODROMY

We define, for every proper and finite tor-dimension p, the monodromy action on the exceptional pullback $p^!$. The main result is 7.3, which gives a relation between p^* and $p^!$.

For $p : Y \to X$ proper and finite tor-dimension, $\mathrm{QCoh}(Y \times_X Y)$ has a convolution product, defined in 3.5, which restricts to a convolution on coherent and also on perfect complexes. Under the Fourier-Mukai equivalence 2.4, it agrees with composition of functors.

Now, let $\mathcal{C}$ be a quasi-coherent sheaf of categories on X. The point is that the convolution groupoid naturally acts on the different versions of the pullbacks by p. For example, there is a natural monoidal action by precomposition of

$$\mathrm{Coh}(Y \times_X Y) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathrm{Perf}(Y)) \quad \text{on} \quad p^!(\mathcal{C}) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathcal{C}),$$

and similarly, of $\mathrm{Perf}(Y \times_X Y)$ and $\mathrm{QCoh}(Y \times_X Y)$ on $p^*(\mathcal{C})$ and $\mathrm{Ind}(p^*(\mathcal{C}))$ respectively.

We will refer to the above as the monodromy action of the convolution groupoid on the different versions of the pullbacks. This can be summarized in the following picture where the monoidal categories on the left acts on the categories on the right

$$\begin{array}{ccc} \mathrm{Perf}(Y \times_X Y) & \overset{\Psi_L}{\dashrightarrow} & \mathrm{Coh}(Y \times_X Y) \\ & \searrow & \downarrow \Psi \\ & & \mathrm{QCoh}(Y \times_X Y) \end{array} \qquad \begin{array}{ccc} p^*(\mathcal{C}) & \overset{\Psi_L}{\dashrightarrow} & p^!(\mathcal{C}) \\ & \searrow & \downarrow \Psi \\ & & \mathrm{Ind}(p^*(\mathcal{C})). \end{array}$$

Example 7.1. The monodromy of $\mathcal{O}_{Y \times_X Y}$ on a functor F in $p^!(\mathcal{C}) = \mathrm{Fun}_X^{ex}(\mathrm{Perf}(Y), \mathcal{C})$ can be described explicitly by the formula

$$(\mathcal{O}_{Y \times_X Y} * F)(-) = F(p^*(p_*(-))) = p_*(-) \otimes_X F(\mathcal{O}_Y) = \Psi_L(\mathcal{O}_Y \otimes_X F(\mathcal{O}_Y))(-),$$

with the first equality by 2.3, the second by F being $\mathrm{Perf}(X)-$linear and the last by (2.1). $\square$

Lemma 7.2. Let $p : Y \to X$ be proper and finite tor-dimension and $\mathcal{C}$ a quasi-coherent sheaf of categories over X. The functor $P^* : \mathcal{C} \to \mathrm{Fun}_X(\mathrm{Perf}(Y), \mathcal{C})$ with $P^*(c)(-) = p_*(-) \otimes c$ has a right adjoint P_* with $P_*(F) = F(\mathcal{O}_Y)$. Moreover $P^*P_*(F) = \mathcal{O}_{Y\times_X Y} * F$.

Proof. The proof is a straightforward computation. It follows from the definitions and the previous example, that the two compositions are

$$P_*P^*(c) = p_*(\mathcal{O}_Y) \otimes c \quad \text{and} \quad P^*P_*(F) = p_*(-) \otimes F(\mathcal{O}_Y) = \mathcal{O}_{Y\times_X Y} * F.$$

The unit morphism comes from $\mathcal{O}_X \to p_*(\mathcal{O}_Y)$ and the counit from $\mathcal{O}_{Y\times_X Y} \to \mathcal{O}_Y$. □

For the next proposition observe that, since the tensor product of categories preserves short exact sequences, from (1.4) we have a fully faithful embedding

$$\Phi : \mathrm{Perf}(Y \times_X Y) \otimes_{\mathrm{Coh}(Y\times_X Y)} p^!(\mathcal{C}) \to p^!(\mathcal{C}). \tag{7.1}$$

Proposition 7.3. Let X be a smooth scheme and $p : Y \to X$ a proper complete intersection morphism. Then, there is a canonical equivalence

$$\mathrm{Perf}(Y \times_X Y) \otimes_{\mathrm{Coh}(Y\times_X Y)} p^!(\mathcal{C}) \to p^*(\mathcal{C})$$

for every quasi-coherent sheaf of categories $\mathcal{C}$ over X.

Proof. The functor we are going to construct will be such that its composition with the canonical $p^*(\mathcal{C}) \to p^!(\mathcal{C})$ is the functor Φ from (7.1). This rigidifies the problem and it means that what we are trying to prove is that the image of Φ is the full subcategory $p^*(\mathcal{C})$.

First lets assume $Y \to X$ is a complete intersection closed immersion. The convolution groupoid $Y \times_X Y$ is an affine group over Y by 3.7. This implies $\mathrm{Perf}(Y \times_X Y)$ is generated as an $\mathrm{Perf}(Y)-$linear category by $\mathcal{O}_{Y\times_X Y}$ and the image of Φ is contained in $p^*(\mathcal{C})$ by 7.1.

The above shows there is a functor as in the statement which moreover is fully faithful because it is the corestriction of the fully faithful functor Φ along $\Psi_L : p^*(\mathcal{C}) \to p^!(\mathcal{C})$.

To prove surjectivity we use 3.8 to write $\mathcal{O}_Y$ as $colim_n M_n$ with the colimit filtered and each M_n a finite colimit of shifts of $\mathcal{O}_{Y\times_X Y}$. In particular, we get

$$c = \mathcal{O}_Y * c = colim_n M_n * c \quad \text{for every } c \text{ in } \mathrm{Ind}(p^*(\mathcal{C})).$$

If c is in $p^*(\mathcal{C})$, then it is compact and the identity $c \to c$ factors through some term of the above filtered colimit. This implies c is a retract of some $M_n * c$, which are finite colimits of shifts of $\mathcal{O}_{Y\times_X Y} * c$. This objects are in the image of our functor and hence c is.

This proves the proposition for complete intersection closed immersions. For proper and smooth maps it is obvious: the convolution groupoid is smooth and p^* and $p^!$ agree by 6.5.

To finish we show that if the statement is true for $q : Z \to Y$ and for $p : Y \to X$ then it is also true for the composition $p \circ q : Z \to X$. Indeed, since the functor q^* is symmetric monoidal, the data of two equivalences

$$\mathrm{Perf}(Y \times_X Y) \otimes_{\mathrm{Coh}(Y\times_X Y)} p^!(-) \to p^*(-) \text{ and } \mathrm{Perf}(Z \times_Y Z) \otimes_{\mathrm{Coh}(Z\times_Y Z)} q^!(-) \to q^*(-)$$

naturally produce a third:

$$q^*(\mathrm{Perf}(Y \times_X Y)) \otimes_{q^*(\mathrm{Coh}(Y\times_X Y))} \mathrm{Perf}(Z \times_Y Z) \otimes_{\mathrm{Coh}(Z\times_Y Z)} q^!(p^!(\mathcal{C})) \to q^*(p^*(\mathcal{C})).$$

This gives the claim in the statement for $Z \to X$ using that

$$q^*(\mathrm{Perf}(Y\times_X Y))\otimes_{q^*(\mathrm{Coh}(Y\times_X Y))}(\mathrm{Perf}(Z\times_Y Z)\otimes_{\mathrm{Coh}(Z\times_Y Z)}\mathrm{Coh}(Z\times_X Z)) \simeq \mathrm{Perf}(Z\times_X Z),$$

which follows from $\mathrm{Coh}(Z \times_X Z) = (p \circ q)^!(\mathrm{Coh}(Z))$ and the claim for p and q. □

Remark 7.4. The condition on p of being a proper complete intersection morphism is too restrictive. For example, the statement is still true for the diagonal embedding of a smooth scheme because $X \times_{X\times X} X$ is a group and is affine over X by [AC] and 3.8 still holds.

Finally, we want to prove $p^!(\mathcal{C})$ is proper over its monodromy and for this we need to understand its mapping spaces. We compute them in $\mathrm{Ind}(p^*(\mathcal{C}))$ and then restrict to $p^!(\mathcal{C})$.

Example 7.5. [Pre, 3.1.2.1] Consider a smooth scheme X, an hypersurface $i : X_0 \to X$ and a quasi-coherent complex M on X_0. In loc. cit. it is shown that there is an $\mathbb{S}^1-$action on the complex $i^* i_* M$, whose $\mathbb{S}^1-$invariants equals to M. In particular, given quasi-coherent sheaves M_1 and M_2, using the fact that $\mathrm{Hom}(-, M_2)$ commutes with colimits we get

$$\mathrm{Hom}_{\mathrm{Coh}(X_0)}(M_1, M_2) = \mathrm{Hom}_{\mathrm{Coh}(X_0)}(i^* i_* M_1, M_2)^{\mathbb{S}^1} = \mathrm{Hom}_{\mathrm{Coh}(X)}(i_* M_1, i_* M_2)^{\mathbb{S}^1}.$$

This is the formula we will try to imitate. Here the map $p : Y \to X$ is the inclusion of the origin on $\mathbb{A}^1$ and $\mathcal{C}$ is the proper quasi-coherent sheaf of categories $\mathrm{Perf}(X)$ corresponding to a proper map $\phi : X \to \mathbb{A}^1$. We have $p^*(\mathcal{C}) = \mathrm{Perf}(X_0)$ and $p^!(\mathcal{C}) = \mathrm{Coh}(X_0)$. □

Lets fix $p : Y \to X$ proper and finite tor-dimension and a quasi-coherent sheaf of categories $\mathcal{C}$ over X. For every F_1 and F_2 in $\mathrm{Ind}(p^*(\mathcal{C}))$ and M in $\mathrm{QCoh}(Y \times_X Y)$ lets observe that the complex

$$\mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(M * F_1, F_2) \quad \text{in} \quad \mathrm{QCoh}(Y)$$

gets an action of $\mathrm{End}_{Y \times_X Y}(M)$ via the first coordinate.

Lemma 7.6. If M can be written as a colimit of shifts of the tensor unit $\mathcal{O}_{Y \times_X Y}$ in the category $\mathrm{QCoh}(Y \times_X Y)$, then we have the following equality

$$\mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(M * F_1, F_2) = \mathrm{Hom}_{\mathrm{QCoh}(Y \times_X Y)}(M, \mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(\mathcal{O}_{Y \times_X Y} * F_1, F_2))$$

in the category of $\mathrm{End}_{Y \times_X Y}(M)-$modules in $\mathrm{QCoh}(Y)$.

Proof. As functors of M, from $\mathrm{QCoh}(Y \times_X Y)$ to $\mathrm{QCoh}(Y)$, both sides commutes with colimits and agree on $\mathcal{O}_{Y \times_X Y}$ as well as on its endomorphisms. This proves the claim and moreover the equality can be improved to an equality in $\mathrm{End}(M)-$modules in $\mathrm{QCoh}(Y)$. □

We recall the Koszul duality from 3.7, if $p : Y \to X$ is a complete intersection closed embedding, the functor $\mathrm{Hom}_{Y\times_X Y}(\mathcal{O}_Y, -)$ gives an equivalence between coherent complexes on the convolution grupoid $Y \times_X Y$ and perfect complexes in the conormal bundle $N^*_Y X[2]$.

Theorem 7.7. Let $p : Y \to X$ be a locally complete intersection in a smooth X. For every proper quasi-coherent sheaf $\mathcal{C}$ over X, we have $p^!(\mathcal{C})$ is proper over $(\mathrm{Coh}(Y \times_X Y), *)$.

Proof. We can easily assume that p is a complete intersection. Then by 3.8, for any two objects F_1 and F_2 in the category $p^!(\mathcal{C})$, the lemma for $M = \mathcal{O}_Y$ says that:

$$\mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(F_1, F_2) = \mathrm{Hom}_{\mathrm{QCoh}(Y\times_X Y)}(\mathcal{O}_Y, \mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(\mathcal{O}_{Y\times_X Y} * F_1, F_2))$$

in $\mathrm{QCoh}(N^*_Y X[2])$ which is the same as $\mathrm{End}_{Y\times_X Y}(\mathcal{O}_Y)-$modules in $\mathrm{QCoh}(Y)$.

This can be rewritten, using the fully faithful embedding $p^!(\mathcal{C}) \to \mathrm{Ind}(p^*(\mathcal{C}))$ on the left together with the equality $\mathcal{O}_{Y\times_X Y} * (-) = P^* P_*(-)$ on the right, and we get

$$\mathrm{Hom}_{p^!(\mathcal{C})}(F_1, F_2) = \mathrm{Hom}_{\mathrm{QCoh}(Y\times_X Y)}(\mathcal{O}_Y, \mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(P^* P_*(F_1), F_2))$$

in the category $\mathrm{QCoh}(N^*_Y X[2])$.

The proposition asserts the term on the left is perfect. If we use Koszul duality from 3.7 and the adjunction from 7.2, it is enough to show that the following is coherent

$$\mathrm{Hom}_{\mathrm{Ind}(p^*(\mathcal{C}))}(P^* P_*(F_1), F_2) = \mathrm{Hom}_{\mathcal{C}}(F_1(\mathcal{O}_Y), F_2(\mathcal{O}_Y)) \text{ in } \mathrm{QCoh}(Y \times_X Y).$$

This is perfect over X because, by hyphotesis, $\mathcal{C}$ is proper over X. To finish we use that, since $Y \times_X Y \to X$ is proper and affine, something is coherent over $Y \times_X Y$ if and only if its pushforward to X is coherent. This is true because by assumption, the quasi-coherent sheaf of categories $\mathcal{C}$ is proper over X and hence its mapping spaces are perfect over X. □

CHAPTER 8

THE FUNDAMENTAL DRINFELD-VERDIER SEQUENCE

Lets fix a proper and finite tor-dimension map $p : Y \to X$. We explain the main result about (6.1), its says that the whole short exact sequence can be recovered from the exceptional pullback $p^!(\mathcal{C})$ and its monodromy by the convolution groupoid $Y \times_X Y$.

First, lets observe that for a morphism $p : Y \to X$ as above, we have the following short exact sequence of monoidal categories

$$\mathrm{Perf}(Y \times_X Y) \to \mathrm{Coh}(Y \times_X Y) \to \mathrm{Sing}(Y \times_X Y). \tag{8.1}$$

Before we state the theorem, we need the following lemma. We recall that a small monoidal stable category is rigid if every object admits a left and a right dual.

Lemma 8.1. [Pre, 3.1.4.2] Suppose that $\mathcal{A}$ is a rigid monoidal stable category and the following is a short exact sequence in $\mathcal{A}-$RMod:

$$\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$$

then, for every $\mathcal{D}$ in $\mathcal{A}-$LMod, the sequence

$$\mathcal{C}_1 \otimes_{\mathcal{A}} \mathcal{D} \to \mathcal{C}_2 \otimes_{\mathcal{A}} \mathcal{D} \to \mathcal{C}_3 \otimes_{\mathcal{A}} \mathcal{D}$$

is again a short exact sequence.

Proof. For the proof check the reference. There it is stated for symmetric monoidal categories but the proof works in general. The main point is that $\mathcal{A}$ being a rigid monoidal stable category, for any $\mathcal{A}-$linear category $\mathcal{D}$, we have that $\mathrm{Ind}(\mathcal{D})$ is dualizable as an $\mathrm{Ind}(\mathcal{A})-$linear category. This easily implies that $(-) \otimes_{\mathrm{Ind}(\mathcal{A})} \mathrm{Ind}(\mathcal{D})$ preserves colimits diagrams. □

The next theorem is the combination of the results of the previous section. The first part of the statement is 7.7 and the second is 7.3 with the previous lemma.

Theorem 8.2. Let $p : Y \to X$ be a complete intersection in a smooth X. For every proper quasi-coherent sheaf of categories $\mathcal{C}$ over X, we have that $p^!(\mathcal{C})$ is proper over $\mathrm{Coh}(Y \times_X Y)$ and the short exact sequence

$$p^*(\mathcal{C}) \to p^!(\mathcal{C}) \to \mathrm{MF}_p(\mathcal{C})$$

is obtained from (8.1) via $- \otimes_{\mathrm{Coh}(Y \times_X Y)} p^!(\mathcal{C})$.

Example 8.3. Consider a complete intersection X_0 inside a smooth X which is obtained as the special fiber of a proper map $X \to \mathbb{A}^n$. If p is the inclusion of the origin in $\mathbb{A}^n$ and we think of $\mathcal{C} = \mathrm{Coh}(X)$ as coherent sheaf of categories over $\mathbb{A}^n$, then the exact sequence of the statement is

$$\mathrm{Perf}(X_0) \to \mathrm{Coh}(X_0) \to \mathrm{Sing}(X_0).$$

In this case, the convolution groupoid $0 \times_{\mathbb{A}^n} 0$ is the loop group $\Omega_0\mathbb{A}^n$. Its action on X_0 produce an action on $\mathrm{Coh}(X_0)$ by cohomological operations. This is the monodromy.

In particular, 8.2 shows $\mathrm{Coh}(X_0)$ is proper over $\mathrm{Coh}(\Omega_0\mathbb{A}^n)$. This is a reformulation of the fact that for complete intersection schemes, the Ext algebras of coherent sheaves are finitely generated over its cohomological operations. This is in [Gul] and in [AG, App D].

In [AG, 4.2.6] it is proved that for complete intersections, cohomological operations are nilpotent exactly on the subcategory of perfect complexes. This is a reformulation of the fact that

$$\mathrm{Perf}(X_0) \simeq \mathrm{Perf}(\Omega_0\mathbb{A}^n) \otimes_{\mathrm{Coh}(\Omega_0\mathbb{A}^n)} \mathrm{Coh}(X_0).$$

The singularity category $\mathrm{Sing}(X_0)$ coincide with $\mathrm{Sing}(\Omega_0\mathbb{A}^n) \otimes_{\mathrm{Coh}(\Omega_0\mathbb{A}^n)} \mathrm{Coh}(X_0)$. □

CHAPTER 9

SUPPORT FOR STABLE CATEGORIES

Fix an E_2-algebra $\mathcal{A}$ and a noetherian graded algebra $A \to \bigoplus H^{2k}(\mathcal{A})$. We explain how to localize $\mathcal{A}-$linear categories over Spec A. This will define its support as a closed conical subset. We work with compactly generated big stable categories. This is in [AG] [BIK].

Definition 9.1. Given a stable category $\mathcal{C}$, an object c and a morphism $\phi : c \to c[k]$, we define the localization $Loc_\phi(c)$ as the colimit

$$colim(c \to c[k] \to c[2k] \to \cdots \to c[nk] \to \cdots) \quad \text{in} \;\; \mathcal{C}.$$

In the case that ϕ comes from a cohomological endomorphism of the identity of $\mathcal{C}$, this defines a functor $Loc_\phi : \mathcal{C} \to \mathcal{C}$. We will refer to this as localization. Observe that this functor is a filtered colimit and hence it commutes with functors that commutes with them.

Definition 9.2. Given a stable category $\mathcal{C}$, an object c and a morphism $\phi : c \to c[k]$, we say that c is $\phi-$torsion if there is a positive integer n such that $\phi^n : c \to c[nk]$ is nullhomotopic and we say that it is locally $\phi-$torsion if $Loc_\phi(c) = 0$.

It is easy to see that if c is torsion, then it is locally torsion. In turn, if c is locally torsion, then $c \simeq \text{cone}(c \to Loc_\phi(c)) \simeq \text{colim cone}(c \to c[nk])$ and hence we can write c as a filtered colimit of torsion objects using that $\text{cone}(c \to c[nk])$ is torsion.

For some of the proofs that follows below, it will be very helpful to have a complementary functor to the localization $Loc_\phi(c)$. This will be the colocalization functor $coLoc_\phi(c)$.

Definition 9.3. Given a stable category $\mathcal{C}$, an object c and a morphism $\phi : c \to c[k]$, we define the colocalization functor $coLoc_\phi(c)$ as the fiber of the canonical map $c \to Loc_\phi(c)$.

It follows from the definitions that for homogeneous a and a' in A, the localization and colocalization functors $Loc_a, coLoc_a, Loc_b$ and $coLoc_b$ commutes with each other and moreover, we have that $coLoc_a \circ Loc_a = Loc_a \circ coLoc_a = 0$.

The next lemma summarizes the main equivalences for being torsion or locally torsion with respect to cohomological operations. We sketch the proof, the details can be checked in the references. For compact objects we can be more precise.

Lemma 9.4. [AG, 3.4.4] [BIK, 5.3] Fix A and $\mathcal{A}$ as before. For an $\mathcal{A}-$linear compactly generated stable category $\mathcal{C}$ and an homogeneous a in A, the following conditions on an object c in $\mathcal{C}$ are equivalent to each other

- the object c is locally $a-$torsion in $\mathcal{C}$,
- the mapping space $\mathrm{Hom}_{\mathcal{C}}(c', c)$ is locally $a-$torsion in $\mathcal{A}-\mathrm{Mod}$ for all compact c' in $\mathcal{C}$,
- the graded module $\mathrm{Ext}_{\mathcal{C}}(c', c)$ is locally $a-$torsion in $A-\mathrm{Mod}^{gr}$ for all compact c' in $\mathcal{C}$,

and moreover, if c is compact, we can add the following

- the graded module $\mathrm{Ext}_{\mathcal{C}}(c, c)$ is locally $a-$torsion in $A-\mathrm{Mod}^{gr}$,
- the graded module $\mathrm{Ext}_{\mathcal{C}}(c, c)$ is $a-$torsion in $A-\mathrm{Mod}^{gr}$,
- the object c is $a-$torsion in $\mathcal{C}$.

Proof. First, $Loc_a(c) = 0$ if and only if $\mathrm{Hom}(c', Loc_a(c)) = 0$ for all compact c' which is equivalent to the second condition because the functor that represent c' commutes with filtered colimits and hence also commutes with the localization functor.

The second and the third are equivalent because cohomology commutes with filtered colimits and an object of $\mathcal{A}-\mathrm{Mod}$ is trivial if its cohomology is trivial.

If c is compact, the third condition implies the fourth. The fifth implies 1 in $\mathrm{Ext}_{\mathcal{C}}(c, c)$ is killed by some power of a and hence the sixth condition. The sixth easily implies the first.

We need to prove the fourth condition implies the fifth. If $\mathrm{Ext}_{\mathcal{C}}(c, c)$ is locally $a-$torsion, it can be written as a filtered colimit of $a-$torsion modules M_n. Now, because A is compact as a graded module, the unit $A \to \mathrm{Ext}_{\mathcal{C}}(c, c)$ factors through some $A \to M_n$. This implies that the unit 1 in the A-algebra $\mathrm{Ext}_{\mathcal{C}}(c, c)$ is $a-$torsion and hence the fifth condition. □

We are ready to give the key definition. If we denote Spec A by X, the graduation on the graded A defines a $\mathbb{G}_m-$action on X. For every closed conical Y of X, we have the homogeneous ideal $I(Y)$ which is finitely generated by the noetherian assumption on A.

Definition 9.5. For an $\mathcal{A}-$linear category $\mathcal{C}$, the support of an object c is the minimal closed conical subset Y of X, such that c is locally $a-$torsion for all homogeneous a in $I(Y)$.

It follows from the previous proposition that for a compact object c, its support coincides with the support of $\mathrm{Hom}(c, c)$ in $\mathcal{A}-$mod and with the support of $\mathrm{Ext}(c, c)$ in $A - \mathrm{Mod}^{gr}$.

Now we explain how to form the subcategory of objects supported on a closed conical subset and how to restrict to its complement. We can also localize on closed conical subsets.

Definition 9.6. For an $\mathcal{A}-$linear category $\mathcal{C}$ and a closed conical Y of X, let $\mathcal{C}_Y$ denote the full subcategory of objects with support contained in Y and let $\mathcal{C}_{X-Y}$ be its right orthogonal.

For an $\mathcal{A}-$linear category $\mathcal{C}$, the colocalization and localization functors from above let us decompose every object into a part in $\mathcal{C}_Y$, its local cohomology, and a part in $\mathcal{C}_{X-Y}$, its restriction. In particular, the fully faithful embedding $\mathcal{C}_Y \to \mathcal{C}$ has a right adjoint.

Proposition 9.7. [AG, 3.3.7] For an $\mathcal{A}-$linear category $\mathcal{C}$ and closed conical subset Y of X we have a short exact sequence of categories

$$\mathcal{C}_Y \to \mathcal{C} \to \mathcal{C}_{X-Y}.$$

Proof. First assume Y is cut out by an homogeneous a in A. In this case, the colocalization functor $coLoc_a$ is the right adjoint of the fully faithful $\mathcal{C}_Y \to \mathcal{C}$ and the localization Loc_a is the left adjoint of the fully faithful $\mathcal{C}_{X-Y} \to \mathcal{C}$. This proves it in this particular case.

In general, if $I(Y) = \langle a_1, \dots, a_n \rangle$ then $coLoc_{a_1} \circ coLoc_{a_2} \circ \cdots \circ coLoc_{a_n}$ is the continuous right adjoint to $\mathcal{C}_Y \to \mathcal{C}$. It is then a consequence of the following general fact: if the embedding of a full subcategory has a continuous right adjoint then the embedding of its right orthogonal admits a left adjoint and they fit into an exact sequence. □

Observe that since we can restrict $\mathcal{C}$ to the complement of any closed conical subset, we can also define the localization around a closed conical subset as an appropiate colimit.

Definition 9.8. For an $\mathcal{A}-$linear category $\mathcal{C}$ and a closed conical subset Y of X, the localization $\mathcal{C}_{(Y)}$ is the filtered colimit of $\mathcal{C}_{X-Z}$ over all possible Z that does not contain Y.

It is easy to see that for every pair of closed conical subsets Y and Z of X, there is an equivalence between $(\mathcal{C}_Y)_{X-Z}$ and $(\mathcal{C}_{X-Z})_Y$ as full subcategories of $\mathcal{C}$ and hence, after passing to appropiate filtered colimits, between $(\mathcal{C}_Y)_{(Y)}$ and $(\mathcal{C}_{(Y)})_Y$.

Definition 9.9. For an $\mathcal{A}-$linear category $\mathcal{C}$ and a closed conical subset Y of X we define the generic fibre at Y as $(\mathcal{C}_Y)_{(Y)} \simeq (\mathcal{C}_{(Y)})_Y$ and denote it by $\mathcal{C}_{Y_\eta}$.

For a closed conical subset Y of X, we define the following functors $\mathcal{C} \to \mathcal{C}$ which play the role of local cohomology around Y, restriction to the complement of Y and localization around Y respectively:

1. $coLoc_Y : \mathcal{C} \to \mathcal{C}$ is the composition $\mathcal{C} \to \mathcal{C}_Y \to \mathcal{C}$,
2. $Loc_{X-Y} : \mathcal{C} \to \mathcal{C}$ is the composition $\mathcal{C} \to \mathcal{C}_{X-Y} \to \mathcal{C}$,
3. $Loc_{(Y)} : \mathcal{C} \to \mathcal{C}$ is the composition $\mathcal{C} \to \mathcal{C}_{(Y)} \to \mathcal{C}$.

Proposition 9.10. [BIK, 5.5] For an $\mathcal{A}-$linear category $\mathcal{C}$ and a compact object c with the graded group $\mathrm{Ext}_{\mathcal{C}}(c, c)$ finitely generated as a graded module over A, a closed conical subset Z is contained in the support of $\mathrm{Ext}_{\mathcal{C}}(c, c)$ if and only if $Loc_{(Z)} \circ coLoc_Z(c) \neq 0$.

The special case of $\mathcal{C}$ equal to $\mathcal{A} - \mathrm{Mod}$ plays an important role. In particular, it can be proved that the general case can be constructed from this one.

Proposition 9.11. [AG, 3.5.5] For an $\mathcal{A}-$linear category $\mathcal{C}$ and a given closed conical subset Y of X, there are natural equivalences of categories

$$\mathcal{C}_Y \simeq \mathcal{A} - \mathrm{Mod}_Y \otimes_{\mathcal{A}-\mathrm{Mod}} \mathcal{C} \quad \text{and} \quad \mathcal{C}_{X-Y} \simeq \mathcal{A} - \mathrm{Mod}_{X-Y} \otimes_{\mathcal{A}-\mathrm{Mod}} \mathcal{C}.$$

The next corollary says that for every closed conical subset Y, the restriction $\mathcal{C}_{X-Y}$ can be obtained from $\mathcal{C}$ by restricting the mapping spaces to $X - Y$.

Corollary 9.12. For every $\mathcal{A}-$linear category $\mathcal{C}$ and closed conical subset Y of X, there is, for every pair of objects c_1 and c_2, a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}_{X-Y}}(c_1, c_2) = Loc_{X-Y}(\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)) \text{ in } \mathcal{A} - \mathrm{Mod}.$$

There are analogous versions of the proposition and the corollary for the localization around a closed conical subset. We wont use any of them so we omit proofs.

Definition 9.13. For an $\mathcal{A}-$linear category $\mathcal{C}$, its support Y in X, is the minimal closed conical subset of X such that $\mathcal{C}_{X-Y}$ vanishes or, equivalently, $\mathcal{C}_Y \to \mathcal{C}$ is an equivalence.

In commutative algebra we can check if a point is in the support of a finitely generated module by checking its fiber at that point. At this level of generality this wont make sense for support of categories. Nevertheless we have a similar criteria using local cohomology.

Proposition 9.14. Consider an $\mathcal{A}-$linear category $\mathcal{C}$ with a compact generator and such that the mapping spaces are finitely generated as graded modules over A. For a closed conical subset Z of $X = \mathrm{Spec}\ A$, the following are equivalent

1. The closed conical subset Z is contained in the support of $\mathcal{C}$.

2. The full subcategory $\mathcal{C}_{Z_\eta}$ of $\mathcal{C}$ is non-empty.

Proof. Lets denote the compact generator by c. From 9.11, we see that $Loc_{X-Y}(c)$ generates the category $\mathcal{C}_{X-Y}$ and hence the support of c coincides with the support of $\mathcal{C}$. In particular, the first condition is equivalent to Z being contained in the support of $\mathrm{Ext}_{\mathcal{C}}(c, c)$.

In turn, by compactness and the properness assumption on the statement and 9.10, this is equivalent to $Loc_{(Z)} \circ coLoc_Z(c)$ being non-trivial i.e. $(\mathcal{C}_Z)_{(Z)} \simeq \mathcal{C}_{Z_\eta}$ is non-empty. $\square$

CHAPTER 10

PUNCTUAL SINGULAR SUPPORT

Now we come back to context of sections 6 and 7 in the special case of the closed embedding of a point in a smooth scheme. In this case, we will see that the monodromy defines a closed conical subset of the cotangent space of the point.

Lets fix the closed embedding of point $i_x : x \to X$ and an isomorphism of the formal completion of $\mathcal{O}_{X,x}$ with $k[\![t_1, \ldots, t_n]\!]$. The convolution groupoid is the loop group $\Omega_x X$.

Definition 10.1. Given a quasi-coherent sheaf of categories $\mathcal{C}$ over a scheme X, the fiber and cofiber at x are the pullbacks $i_x^*(\mathcal{C})$ and $i_x^!(\mathcal{C})$ together with its monodromies by $\mathrm{Coh}(\Omega_x X)$.

The loop group $\Omega_x X$ is Koszul dual to the shifted cotangent space $T_x^* X[2]$. To be more explicit we use the $k-$algebra $k[\beta_1, \ldots, \beta_n]$ with all β_i of degree 2. By, 3.7 we have

$$\mathrm{Perf}(\Omega_x X) \simeq \mathrm{Perf}_0(k[\beta_1, \ldots, \beta_n]) \quad \text{and} \quad \mathrm{Coh}(\Omega_x X) \simeq \mathrm{Perf}(k[\beta_1, \ldots, \beta_n]) \tag{10.1}$$

and hence monodromy can be interpreted as a $k[\beta_1, \ldots, \beta_n]-$linear structure on the cofiber.

Definition 10.2. Given a point x in a smooth scheme X and a proper quasi-coherent sheaf of categories $\mathcal{C}$ over it, we define $SS_x(\mathcal{C})$ as the $k[\beta_1, \ldots, \beta_n]-$linear support of $i_x^!(\mathcal{C})$.

The above defines the punctual singular support $SS_x(\mathcal{C})$ of a proper quasi-coherent sheaf of categories $\mathcal{C}$ as a closed conical subset of the cotangent fiber $T_x^* X$. For proper and smooth sheaves the punctual singular support is trivial i.e. is contained in the zero section.

Definition 10.3. A proper $k-$linear small stable category $\mathcal{D}$ is said to be right saturated, as a $k-$linear category, if every exact functor $\mathcal{D} \to \mathrm{Perf}(k)$ is representable.

For a proper dg-algebra A, the category of perfect $A-$modules is right saturated if and only if the forgetful map $A - \mathrm{Mod} \to k - \mathrm{Mod}$ detects compact objects. For proper k-linear categories, smooth implies regular and regular implies right saturated [BvdB]. I dont know in characteristic 0 an example of a right saturated category which is not smooth.

Proposition 10.4. Given a coherent sheaf of categories $\mathcal{C}$ over a smooth scheme X, the following conditions are all equivalent:

- The punctual singular support $SS_x(\mathcal{C})$ is contained in the zero section.
- The functor $i_x^*(\mathcal{C}) \to i_x^!(\mathcal{C})$ is an equivalence.
- The $k-$linear category $i_x^*(\mathcal{C})$ is right saturated.

Proof. It follows from 9.11 and 9.13 that the first condition is equivalent to the natural fully faithful functor

$$\mathrm{Perf}_0(k[\beta_1, \ldots, \beta_n]) \otimes_{\mathrm{Perf}(k[\beta_1,\ldots,\beta_n])} i_x^!(\mathcal{C}) \to i_x^!(\mathcal{C})$$

being an equivalence. If we use 7.3, the above identifies with $i_x^*(\mathcal{C}) \to i_x^!(\mathcal{C})$ and hence it is the same as the second condition. This proves the first two conditions are equivalent.

Lets prove that the second and the third conditions are equivalent. We can assume that the scheme X corresponds to a $k-$algebra A, the point x to an augmentation $A \to k$ and the coherent sheaf of categories $\mathcal{C}$ to $B-$mod with B a proper A-algebra, smooth as a $k-$algebra.

In the above, $\mathrm{Ind}(i^*(\mathcal{C}))$ corresponds to $k \otimes_A B-$Mod, the fiber $i^*(\mathcal{C})$ to perfect modules and the cofiber $i^!(\mathcal{C})$ to modules that are perfect as $B-$modules.

$$\begin{array}{ccccc} k \otimes_A B - \mathrm{Mod} & \xrightarrow{\alpha} & B - \mathrm{Mod} & & \\ \downarrow \beta & & \downarrow \beta' & & \\ k - \mathrm{Mod} & \xrightarrow{\alpha'} & A - \mathrm{Mod} & \xrightarrow{\pi} & k - \mathrm{Mod} \end{array}$$

The second condition on the statement is equivalent to α detecting compact objects, that is, a bimodule is compact if it gives a functor which evaluated at k is compact and the third is equivalent to β detecting compact objects, that is, $\mathrm{Perf}(k \otimes_A B)$ being right saturated.

In conclusion, we need to prove that α detects compact objects if and only if β does. Recall that if an $R-$algebra S is proper or smooth then the forgetful $S - \mathrm{Mod} \to R - \mathrm{Mod}$ preserves or detects being compact respectively. [Lur4, 4.6.4] [Lur3, 4.7.5]

Assume α detects compact objects. Then $\beta(M)$ compact implies $\pi \circ \alpha'(\beta(M))$ compact and hence $\pi \circ \beta'(\alpha(M))$ is compact. But, by the previous paragraph, B smooth as a $k-$algebra implies $\alpha(M)$ is compact and, since α detects compact objects, M is compact.

Assume β detects compact objects. Then $\alpha(M)$ compact implies $\beta'(\alpha(M))$ is compact because B is proper over A. This implies $\alpha'(\beta(M))$ is compact. This is a skyscraper and it can be compact only if $\beta(M)$ is compact. By hyphotesis, this implies M is compact. □

For the following, given a function $\phi : X \to \mathbb{A}^1$, we denote by $i_\phi : X_0 \to X$ the closed immersion of the special fiber. Moreover, for every point x in X_0 let i_{x,X_0} and $i_{x,X}$ denote its closed immersions into X_0 and X respectively. The next lemma computes the fiber at a given point x of MF_{i_ϕ} in terms of the cofiber $i^!_x$. For the notation in the lemma check 9.9.

Lemma 10.5. Consider a smooth scheme X, with an arbitrary function $\phi : X \to \mathbb{A}^1$ and a quasi-coherent sheaf of categories $\mathcal{C}$. If ℓ denotes the line generated by $d_x\phi$ in $T^*_x X$, we have an equivalence of categories

$$i^*_{x,X_0}(\mathrm{MF}_{i_\phi}(\mathcal{C})) \simeq i^!_{x,X}(\mathcal{C})_{\ell_\eta}.$$

Proof. This is a straightforward computation. From (6.1) we get the following short exact sequence of categories:

$$i^*_{x,X_0} i^*_\phi(\mathcal{C}) \to i^*_{x,X_0} i^!_\phi(\mathcal{C}) \to i^*_{x,X_0}\mathrm{MF}_{i_\phi}(\mathcal{C}), \tag{10.2}$$

and moreover, from 7.3 for i_{x,X_0} and $i_{x,X}$, it follows that

$$i^*_{x,X_0} i^*_\phi(\mathcal{C}) = \mathrm{Perf}(\Omega_x X) \otimes_{\mathrm{Coh}(\Omega_x X)} i^!_{x,X}(\mathcal{C})$$

and

$$i^*_{x,X_0} i^!_\phi(\mathcal{C}) = \mathrm{Perf}(\Omega_x X_0) \otimes_{\mathrm{Coh}(\Omega_x X_0)} i^!_{x,X}(\mathcal{C}).$$

This means that the first two terms of (10.2) corresponds to the part of $i^!_{x,X}(\mathcal{C})$ supported at 0 and ℓ respectively. In particular, they correspond to the first two terms of

$$i^!_{x,X}(\mathcal{C})_0 \to i^!_{x,X}(\mathcal{C})_\ell \to (i^!_{x,X}(\mathcal{C})_\ell)_{(\ell)} \simeq i^!_{x,X}(\mathcal{C})_{\ell_\eta},$$

and hence the third term of each exact sequence identify as desired. □

Remark 10.6. This can be interpreted as follows: the fiber of $\mathrm{MF}_{i_\phi}(\mathcal{C})$ at a given point x is the generic fiber at $d_x\phi$ of the microlocalization of $i^!_{x,X}(\mathcal{C})$ along the cotangent T^*_xX.

Now, we give criteria to decide when a covector (x, v^*) is in the punctual singular support of a coherent sheaf of categories. It is analogous to the vanishing cycle test.

Proposition 10.7. Consider a smooth scheme X with a function $\phi : X \to \mathbb{A}^1$ and a coherent sheaf of categories $\mathcal{C}$. Given a point x, if the fiber $i^*_{x,X_0}(\mathrm{MF}_{i_\phi}(\mathcal{C}))$ is non-trivial then the covector $(x, d_x\phi)$ is in $SS_x(\mathcal{C})$. If $i^!_{x,X}(\mathcal{C})$ has a compact generator the converse holds.

Proof. If we use the lemma, the proposition reduce to the statement of 9.14. The hyphotesis there are satisfied because by 7.7, the $k[\beta_1, \ldots, \beta_n]-$linear category $i^!_{x,X}(\mathcal{C})$ is proper. □

Remark 10.8. The cofiber $i^!_{x,X}(\mathcal{C})$ is expected to be a smooth category and in particular it should have a compact generator. If $\mathcal{C}$ is $\mathrm{Coh}(Y)$ with the map $Y \to X$ proper and the scheme Y smooth, the cofiber $i^!_{x,X}(\mathcal{C})$ is $\mathrm{Coh}(Y_x)$ which is smooth by a theorem of Efimov.

To finish we give a deformation theory interpretation. Recall that for a $k-$linear category, its curved deformations over an Artinian E_2-algebra A are classified by $\mathrm{KD}_2(A)-$linear structures on it, with $\mathrm{KD}_2(A)$ the E_2-Koszul dual of A. The unobstructed objects are those on which the augmentation ideal of $\mathrm{KD}_2(A)$ acts trivially. This is in [Lur2].

The point is that the $k[\beta_1, \ldots, \beta_n]-$linear structure on the cofiber $i^!_x(\mathcal{C})$ is encoding a deformation over Spf $\widehat{\mathcal{O}}_{X,x}$. The unobstructed objects are those on which $\beta_1, \ldots, \beta_n$ acts trivially and the proof of 7.3 shows that the full subcategory generated by them is $i^*_x(\mathcal{C})$.

In particular, 10.4 says that the punctual singular support is contained in the zero section if and only if, the whole cofiber is generated by unobstructed objects. Moreover, in any curved deformation of a right saturated category, every object is a retract of a finite colimit of unobstructed object. The criteria in 10.7 says that when (x, v^*) is not in the punctual singular support $SS_x(\mathcal{C})$, if an object in $i_x^!(\mathcal{C})$ is unobstructed along X_0, then it is unobstructed.

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