THE UNIVERSITY OF CHICAGO

THE CLASSIFICATION OF FIVE-DIMENSIONAL GEOMETRIES

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To my sister June, who is working on the other half of our MD-PhD.

Four-dimensional hypergeometric eigenschlumpkins!

— Alar Toomre

Wait, is this going to be on the test?

— Arka Dhar

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ABSTRACT

We classify the 5-dimensional homogeneous geometries in the sense of Thurston. The 5-dimensional geometries with irreducible isotropy are the irreducible Riemannian symmetric spaces, while those with trivial isotropy are simply-connected solvable Lie groups of the form $\mathbb{R}^3 \rtimes \mathbb{R}^2$ or $N \rtimes \mathbb{R}$ where N is nilpotent. The geometries with nontrivial reducible isotropy are mostly products, but a number of interesting examples arise. These include a countably infinite family $L(a;1) \times_{S^1} L(b;1)$ of inequivalent geometries diffeomorphic to $S^3 \times S^2$, an uncountable family $\widetilde{\mathrm{SL}_2} \times_\alpha S^3$ in which only a countable subfamily admits compact quotients, and the non-maximal geometry $\mathrm{SO}(4)/\mathrm{SO}(2)$ realized by two distinct maximal geometries.

PART I

SUMMARY

CHAPTER 1

INTRODUCTION

By the classification of closed surfaces (see e.g. [Mat02, Thm. 5.11]), every closed surface is diffeomorphic to a quotient of $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$ by a discrete group of isometries. It is a classical result that in dimension 2, these three are the only connected, simply-connected, complete Riemannian manifolds with transitive isometry group (see e.g. [Thu97, Thm. 3.8.2]).

The quest for the 3-dimensional generalization that became Thurston's Geometrization Conjecture led to a version of the following definition. (The equivalence to older definitions is outlined in Part II, Prop. 6.5.)

Definition 1.1 (**Geometries**, following [Thu97, Defn. 3.8.1] and [Fil83, §1.1])**.**

(i) A *geometry* is a connected, simply-connected homogeneous space $M = G/G_p$ where G is a connected Lie group acting faithfully with compact point stabilizers G_p.

(ii) M is a *model geometry* if there is some lattice $\Gamma \subset G$ that acts freely on M. Then the manifold $\Gamma\backslash G/G_p$ is said to be *modeled on M*.

(iii) M is *maximal* if it is not G-equivariantly diffeomorphic to any other geometry G'/G'_p with $G \subsetneq G'$. Any such G'/G'_p is said to *subsume* G/G_p.

Then a closed 3-manifold is a quotient of at most one maximal model geometry, which can be determined from the fundamental group [Thu97, Thm. 4.7.8] or from the existence of certain bundle structures (usually Seifert bundles) and some topological data (usually two Euler numbers) [Sco83, Thm. 5.3]. Thurston classified the 3-dimensional maximal model geometries and found eight (see [Thu97, Thm. 3.8.4]).

In 4 dimensions, Filipkiewicz classified the maximal model geometries in [Fil83]. Though 4-manifolds without geometric decompositions [Hil02, §13.3 #3] indicate there is less hope for a straightforward generalization of geometrization, Filipkiewicz's classification highlights a few interesting firsts. The list comprises 18 geometries and—for the first time—a countably infinite family, named $\mathrm{Sol}^4_{m,n}$. (See e.g. [Hil02, §7.1] or [Wal86, §1, Table 1] for the names currently in use.) One of the eighteen is $\mathbb{F}^4 = \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}) / \mathrm{SO}(2)$, the first geometry to admit finite-volume quotients but no compact quotients.

The direction to take should now seem straightforward. One seeks a classification of maximal model geometries in all dimensions; but a handful of obstacles stand in the way of such a classification:

1. Existing classifcations, including now the present paper, rely on tools that may become unusable with increasing dimension. For example, the case of discrete point stabilizers ([Thu97, Thm. 3.8.4(c)] in dimension 3, [Fil83, Ch. 6] in dimension 4, and (Thm. 5.1(ii)) in Part II) relies on a classification of solvable Lie algebras over $\mathbb{R}$, which is incomplete in dimensions 7 and up. (See e.g. [ŠW12, Introduction] for a summary of known progress, and [BFNT13] for a wider survey.)

2. The aforementioned aspects of the 4-dimensional classification suggest that new phenomena may continue to appear for a few more dimensions. A workable approach to a general classification may not be evident without knowledge of such features.

An optimistic interpretation of these obstacles is that the 5-dimensional case is both tractable and potentially illustrative. Having carried out the classification, the new phenomena are summarized in Section 2; the main result is the following list.

Theorem 1.2 (**Classification of 5-dimensional geometries**). *The maximal model geometries of dimension* 5 *are:*

1. *The geometries with constant curvature:*

$$\mathbb{E}^5 = \mathbb{R}^5 \rtimes \mathrm{SO}(5) / \mathrm{SO}(5) \qquad S^5 = \mathrm{SO}(6) / \mathrm{SO}(5) \qquad \mathbb{H}^5 = \mathrm{SO}(5,1) / \mathrm{SO}(5);$$

2. *The irreducible Riemannian symmetric spaces* $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ *and* $\mathrm{SU}(3)/\mathrm{SO}(3)$;

3. *The unit tangent bundles or universal covers of circle bundles:*

$$T^1(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})/\mathrm{SO}(2) \quad T^1(\mathbb{E}^{1,2}) = \mathbb{R}^3 \rtimes \mathrm{SO}(1,2)^0/\mathrm{SO}(2) \quad \mathrm{U}(\widetilde{2,1)/\mathrm{U}}(2);$$

4. *The associated bundles (see e.g. [Sha00, §1.3 Vector Bundles] for the notation):*

$$\begin{aligned}
\mathrm{Heis}_3 \times_{\mathbb{R}} S^3 &= (\mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SO}(2)}) \times S^3/\{(0,0,s),\gamma(t),e^{\pi i s}\}_{s,t\in\mathbb{R}}\\
\mathrm{Heis}_3 \times_{\mathbb{R}} \widetilde{\mathrm{SL}_2} &= (\mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SO}(2)}) \times \widetilde{\mathrm{SL}_2}/\{(0,0,s),\gamma(t),\gamma(s)\}_{s,t\in\mathbb{R}}\\
\widetilde{\mathrm{SL}_2} \times_\alpha S^3 &= \widetilde{\mathrm{SL}_2} \times S^3 \times \mathbb{R}/\{\gamma(s),e^{\pi i t},\alpha s+t\}_{s,t\in\mathbb{R}}, \quad 0<\alpha<\infty\\
\widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2} &= \widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}/\{\gamma(s),\gamma(t),\alpha s+t\}_{s,t\in\mathbb{R}}, \quad 0<\alpha\le 1\\
L(a;1) \times_{S^1} L(b;1) &= S^3 \times S^3 \times \mathbb{R}/\{e^{\pi i s},e^{\pi i t},as+bt\}_{s,t\in\mathbb{R}}, \quad 0<a\le b \text{ coprime in } \mathbb{Z},
\end{aligned}$$

where the Heisenberg group Heis_3 *is* $\mathbb{R}^3$ *with the multiplication law*

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+xy'-x'y),$$

on which $\mathrm{SO}(2)$ *acts through the action of* $\mathrm{SL}(2,\mathbb{R})$ *on the* x,y *plane, and* $t \mapsto e^{\pi i t} \in S^3$ *and* $\gamma : \mathbb{R} \to \widetilde{\mathrm{SO}(2)} \subset \widetilde{\mathrm{SL}_2}$ *are* 1*-parameter subgroups sending* $\mathbb{Z}$ *to the center;*

5. *The three principal* $\mathbb{R}$*-bundles with non-flat connections over the* $\mathbb{F}^4$ *geometry, distinguished from each other by their curvatures:*

$$\begin{aligned}
\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2} &\cong (\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}) \rtimes \mathrm{SO}(2)/\mathrm{SO}(2)\\
\mathbb{F}^5_a &= \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SL}_2}/\{(0,0,at),\gamma(t)\}_{t\in\mathbb{R}}, \quad a=0 \text{ or } 1;
\end{aligned}$$

6. *The six simply-connected indecomposable nilpotent Lie groups, named by their Lie al-*

gebras as in [PSWZ76, Table II], in which the point stabilizer of the identity element is a maximal compact group of automorphisms (specified in Table 3.2):

$$A_{5,1} = \mathbb{R}^4 \underset{x^2,\, x^2}{\rtimes} \mathbb{R} \qquad A_{5,2} = \mathbb{R}^4 \underset{x^4}{\rtimes} \mathbb{R} \qquad A_{5,3} = (\mathbb{R} \times \mathrm{Heis}_3) \underset{x_3 \to x_2 \to y}{\rtimes} \mathbb{R}$$

$$A_{5,4} = \mathrm{Heis}_5 \qquad A_{5,5} = \mathrm{Nil}^4 \underset{3 \to 1}{\rtimes} \mathbb{R} \qquad A_{5,6} = \mathrm{Nil}^4 \underset{4 \to 3 \to 1}{\rtimes} \mathbb{R};$$

7. *The simply-connected indecomposable non-nilpotent solvable Lie groups, specified the same way:*

$$A_{5,7}^{a,b,-1-a-b} = \mathbb{R}^4 \underset{\text{4 distinct real roots}}{\rtimes} \mathbb{R} \qquad A_{5,7}^{1,-1-a,-1+a} = \mathbb{R}^4 \underset{\text{2 complex, 2 distinct real}}{\rtimes} \mathbb{R}$$

$$A_{5,7}^{1,-1,-1} = \mathbb{R}^4 \underset{x-1,\, x-1,\, x+1,\, x+1}{\rtimes} \mathbb{R} \qquad A_{5,8}^{-1} = \mathbb{R}^4 \underset{x^2,\, x-1,\, x+1}{\rtimes} \mathbb{R}$$

$$A_{5,9}^{-1,-1} = \mathbb{R}^4 \underset{(x-1)^2,\, x+1,\, x+1}{\rtimes} \mathbb{R} \qquad A_{5,15}^{-1} = \mathbb{R}^4 \underset{(x-1)^2,\, (x+1)^2}{\rtimes} \mathbb{R}$$

$$A_{5,20}^{0} = (\mathbb{R} \times \mathrm{Heis}_3) \underset{\text{Lorentz},\, y \to x_1}{\rtimes} \mathbb{R} \qquad A_{5,33}^{-1,-1} = \mathbb{R}^3 \rtimes \{xyz = 1\}^0;$$

8. *and all twenty-nine products of lower-dimensional geometries involving no more than one Euclidean factor, named as in [Wal86, Table 1].*

   (a) 4-*by*-1*:*

$$S^4 \times \mathbb{E} \qquad \mathbb{H}^4 \times \mathbb{E} \qquad \mathbb{CP}^2 \times \mathbb{E} \qquad \mathbb{CH}^2 \times \mathbb{E} \qquad \mathbb{F}^4 \times \mathbb{E}$$

$$\mathrm{Nil}^4 \times \mathbb{E} \qquad \mathrm{Sol}_0^4 \times \mathbb{E} \qquad \mathrm{Sol}_1^4 \times \mathbb{E} \qquad \mathrm{Sol}_{m,n}^4 \times \mathbb{E}$$

(b) 3-by-2:

	$\mathbb{E}^3 \times S^2$	$\mathbb{E}^3 \times \mathbb{H}^2$
$S^3 \times \mathbb{E}^2$	$S^3 \times S^2$	$S^3 \times \mathbb{H}^2$
$\mathbb{H}^3 \times \mathbb{E}^2$	$\mathbb{H}^3 \times S^2$	$\mathbb{H}^3 \times \mathbb{H}^2$
$\mathrm{Heis}_3 \times \mathbb{E}^2$	$\mathrm{Heis}_3 \times S^2$	$\mathrm{Heis}_3 \times \mathbb{H}^2$
$\mathrm{Sol}^3 \times \mathbb{E}^2$	$\mathrm{Sol}^3 \times S^2$	$\mathrm{Sol}^3 \times \mathbb{H}^2$
$\widetilde{\mathrm{SL}_2} \times \mathbb{E}^2$	$\widetilde{\mathrm{SL}_2} \times S^2$	$\widetilde{\mathrm{SL}_2} \times \mathbb{H}^2$

(c) 2-by-2-by-1:

$$S^2 \times S^2 \times \mathbb{E} \qquad S^2 \times \mathbb{H}^2 \times \mathbb{E} \qquad \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{E}$$

More explicit instructions for constructing these geometries—such as the solvable Lie groups and their automorphism groups—are delegated to where they occur in the classification in Parts II and III [Gen16a, Gen16b].

Roadmap The present paper (Part I) summarizes the classification; Section 2 picks out illustrative examples, Section 3 outlines the strategy, and Section 4 briefly surveys related classifications. Part II classifies the point stabilizer subgroups G_p and classifies the geometries where G_p acts irreducibly or trivially on tangent spaces. Part III classifies the remaining geometries after showing that they all admit invariant fiber bundle structures (hence the name "fibering geometries").

CHAPTER 2

SALIENT EXAMPLES

2.1 New phenomena

Much of our interest in this classification is in the search for phenomena that occur for the first time in dimension 5, in hopes of finding a pattern that continues in higher dimensions. See also Section 3 for a discussion of new tools.

An uncountable family of geometries. The associated bundles $\widetilde{\mathrm{SL}_2} \times_\alpha S^3$ $(0 < \alpha < \infty)$ form an *uncountable* family of maximal model geometries. (Taking $\widetilde{\mathrm{SL}_2} \times_\alpha S^3$ as a circle bundle over $S^2 \times \mathbb{H}^2$ with an invariant connnection, the parameter α is a ratio of curvatures in the S^2 and $\mathbb{H}^2$ directions.) This and $\widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2}$ are the first occurrences of uncountable families. Since every lattice in a Lie group is finitely presented [OV00, Thm. I.1.3.1], π_1 of the quotient manifolds will not determine the geometries. Details are in Part III, (Prop. 15.35).

An infinite family without compact quotients. In fact, $\widetilde{\mathrm{SL}_2} \times_\alpha S^3$ admits compact quotients if and only if α is rational (Prop. 15.36). (Recall that beginning with $\mathbb{H}^2$, geometries can have noncompact quotients of finite volume; and beginning with $\mathbb{F}^4 = \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, model geometries might have no compact quotients.)

Non-unique maximality. The geometry $T^1 S^3 = \mathrm{SO}(4)/\mathrm{SO}(2)$ is a non-maximal form of $S^3 \times S^2$ and $L(1;1) \times_{S^1} L(1;1)$—both of which are maximal (Rmk. 15.40). This contrasts with the positive results for unique maximality listed in the discussion after [Fil83, Prop. 1.1.2].

Inequivalent compact geometries with the same diffeomorphism type. Using Barden's diffeomorphism classification [Bar65] of simply-connected 5-manifolds by second homology and second Stiefel-Whitney class, one can prove that the associated bundles of lens spaces $L(a;1) \times_{S^1} L(b;1)$ are all diffeomorphic to $S^3 \times S^2$ [Ott09, Cor. 3.3.2].

More broadly one can attempt to give the classification up to diffeomorphism, following previous results such as [Mos50, Cor. p. 624], [Gor77], [Ish55], and [Ott09, Thm. 1.0.3]. Most of the geometries are products of $\mathbb{R}^k$ and some spheres; the two exceptions are $\mathbb{CP}^2 \times \mathbb{E}$ and the rational homology sphere $\mathrm{SU}(3)/\mathrm{SO}(3)$, named X_{-1} in Barden's classification [BG02, Introduction].

Note that while the correct diffeomorphism type may be obvious enough to guess, it is not as obviously correct. The problem is proving that the space is a direct product of $\mathbb{R}^k$ and the product of spheres onto which it deformation retracts—such a claim is false for any nontrivial vector bundle (such as TS^2) and for a homogeneous example by Samelson discussed in [Mos55, §5 Example 4]. Instead one has to use either an explicit description of the diffeomorphism type from [Mos62a, Thm. A] or the fact that sufficiently nice bundles over contractible spaces are trivial [Hus94, Cor. 10.3].

Isotropy irreducible spaces. The geometries $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SU}(3)/\mathrm{SO}(3)$ have point stabilizers $\mathrm{SO}(3)$ acting irreducibly on the (5-dimensional) tangent spaces. This is the first occurrence of such an action being irreducible and not the standard representation for a group of the same isomorphism type. These two geometries are still symmetric spaces, but in higher dimensions there exist homogeneous spaces with irreducibly-acting point stabilizers that are not symmetric spaces (See e.g. [HZ96, Introduction]).

2.2 Examples that highlight tools

The classification of geometries requires an increasingly wide range of tools as the dimension increases. These are a handful of examples where either unexpected tools appeared, or familiar tools exhibit behavior that is not completely obvious at first glance.

Model geometries via Galois theory and Dirichlet's unit theorem. Some Galois theory is needed to answer questions of lattice existence, such as to prove that $\mathbb{R} \rtimes$

$\mathrm{Conf}^+ \mathbb{E}^3 / \mathrm{SO}(3)$ (where the action of $\mathrm{Conf}^+ \mathbb{E}^3$ on $\mathbb{R}$ is chosen to make the semidirect product unimodular) is not a model geometry (Prop. 14.1(iv)).

Dirichlet's unit theorem makes an appearance when we construct a lattice in $\mathbb{R}^3 \rtimes \{xyz = 1\}^0$ by taking a finite index subgroup of $\mathcal{O}_K \rtimes \mathcal{O}_K^\times$ where K is a totally real cubic field extension of $\mathbb{Q}$ (Prop. 9.16).

Point stabilizers not realized. The classification of geometries starts by classifying subgroups of SO(5) in order to classify point stabilizers—but not every subgroup is realized by a maximal model geometry. For example, SO(3) in its standard representation is one such subgroup, though the non-model geometry $\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3 / \mathrm{SO}(3)$ mentioned above suggests this can be thought of as a near miss. The non-occurrence of SU(2) is another example, a feature shared by the 4-dimensional classification of geometries. Other subgroups—namely SO(4) and $\mathrm{SO}(3) \times \mathrm{SO}(2)$—are point stabilizers only of product geometries. A listing of (non-product) geometries by point stabilizer is given below in Table 3.2 after Figure 3.1 names the subgroups.

Geometries in higher dimensions with reducible isotropy and no fibering. When point stabilizers act reducibly on tangent spaces, our strategy breaks down the problem by showing the existence of an invariant fiber bundle structure. That this is possible is is a convenient accident of low dimensions; higher dimensions introduce isotropy-reducible geometries that admit no fibering.

For example, in dimension 18, there is $\mathrm{Sp}(3)/\mathrm{Sp}(1)$, where the embedding $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(3)$ is given by the irreducible representation of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ on $\mathbb{C}^6$. This has two isotropy summands but admits no nontrivial fibering since Sp(1) is maximal (so no larger group can be a point stabilizer of the base space) [DK08, Example V.10]. A strategy that continues to break the problem down using invariant fiber bundle structures may have to account for these exceptions, likely using Dynkin's work on classifying maximal subgroups of semisimple

Lie groups in [Dyn00a, Dyn00b].

Non-geometries as base spaces of fiber bundles. Even when invariant fiber bundle structures exist, a number of compliactions prevent the classification from having a straightforward recursive solution. Filipkiewicz warns in [Fil83, Prop. 2.1.3] that the base space of an invariant fiber bundle structure may fail to be a geometry due to noncompact point stabilizers—e.g. the action of $\mathrm{PSL}(2,\mathbb{C})$ on $S^2 \cong \mathbb{CP}^1$ makes $T^1\mathbb{H}^3 = \mathrm{PSL}(2,\mathbb{C})/\mathrm{PSO}(2)$ a fiber bundle over S^2. Even when point stabilizers are compact, the base may fail to be maximal (e.g. Heis_5 fibers over $\mathbb{E}^4$ with $\mathrm{U}(2)$ point stabilizers) or a model geometry (e.g. Sol^3 over $\mathrm{Aff}^+ \mathbb{R}$, which cannot admit a lattice since it is not unimodular).

2.3 Examples that clarify how the classification is organized

The eight categories of Theorem 1.2 and the grouping of geometries into parametrized families involved some arbitrary choices. This section discusses the chosen method of organization and some variations.

The omission of some spaces that one might have guessed. Some of the categories in Thm. 1.2 are conspicuously missing geometries that happen to be non-model or non-maximal.

3. The tautological unit circle bundle $\mathrm{U}(3)/\mathrm{U}(2)$ over $\mathbb{CP}^2$ is non-maximal, being equivariantly diffeomorphic to S^5. The two other unit tangent bundles of 3-dimensional spaces of constant curvature are also non-maximal.

$$T^1(S^3) = \mathrm{SO}(4)/\mathrm{SO}(2) \cong S^2 \times S^3 \qquad T^1(\mathbb{E}^3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2 \times \mathbb{E}^3$$

7. Many of the solvable Lie groups arising from the list in [PSWZ76, Table II] are not unimodular and hence do not admit lattices.

8. Every product geometry with multiple Euclidean factors is non-maximal—but all other products are maximal, usually as a consequence of the de Rham decomposition theorem. (See (Prop. 12.12) in Part III.)

Counting the geometries and families. The list given in Thm. 1.2 includes 53 individual geometries and the following 6 infinite families of geometries.

$$\begin{aligned}
L(a;1) \times_{S^1} L(b;1), &\quad a \leq b \text{ coprime positive integers} \\
\widetilde{\mathrm{SL}_2} \times_\alpha S^3, &\quad 0 < \alpha < \infty \\
\widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2}, &\quad 0 < \alpha \leq 1 \\
\mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}, &\quad e^A \text{ semisimple integer matrix with 4 real eigenvalues} \\
\mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}, &\quad e^A \text{ semisimple integer matrix with 2 real eigenvalues} \\
\mathrm{Sol}^4_{m,n} \times \mathbb{E}, &\quad m, n \in \mathbb{Z}
\end{aligned}$$

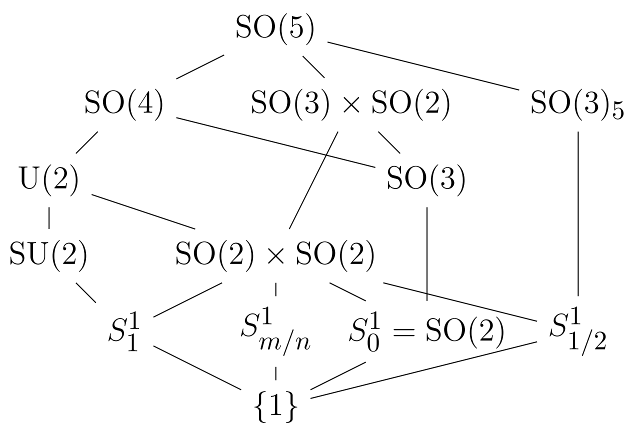
To some extent, this count depends on interpretation. The first three families could be expanded to include products of spheres and hyperbolic spaces, while the last three could be unified with $\mathbb{R}^4 \rtimes_{x-1,\, x-1,\, x+1,\, x+1} \mathbb{R}$ to form one large family with a name like $\mathrm{Sol}^5_{m,n,p}$ (where m, n, and p are the middle coefficients of the characteristic polynomial of e^A). Indeed, [PSWZ76, Table II] suggests this latter unification by listing all semidirect products $\mathbb{R}^4 \rtimes \mathbb{R}$ with diagonalizable action under the family $A_{5,7}$. We keep the subfamilies of $\mathrm{Sol}^5_{m,n,p}$ separate since their point stabilizers have different dimensions.

CHAPTER 3

OVERVIEW OF METHOD

The classification of 5-dimensional geometries $M = G/G_p$ begins, following Thurston [Thu97, §3.8] and [Fil83, §1.2], by using the representation theory of compact groups to list the subgroups $G_p \subseteq \mathrm{SO}(T_pM)$ that could be point stabilizers (Figure 3.1).

Figure 3.1: Closed connected subgroups of SO(5), with inclusions. $\mathrm{SO}(3)_5$ denotes SO(3) acting on its 5-dimensional irreducible representation; and $S^1_{m/n}$ acts as on the direct sum $V_m \oplus V_n \oplus \mathbb{R}$ where S^1 acts irreducibly on V_m with kernel of order m. See Part II, Prop. 7.1 for the proof.



The problem divides into cases by the action of G_p on the tangent space T_pM (the "linear isotropy representation")—more specifically, by the highest dimension of an irreducible subrepresentation V.

At the extremes, one can appeal to existing classifications—the classification of strongly isotropy irreducible homogeneous spaces by Manturov [Man61a, Man61b, Man66, Man98], Wolf [Wol68, Wol84], and Krämer [Krä75] when $G_p \curvearrowright T_pM$ is irreducible ($\dim V = 5$); and the classification of low-dimensional solvable real Lie algebras by Mubarakzyanov [Mub63] and Dozias [Doz63] if $G_p \curvearrowright T_pM$ is trivial ($\dim V = 1$). These cases are handled in Part II, including the production of an identification key for trivial-isotropy geometries (Fig. 3.4).

Table 3.2: Using the classification, non-product geometries can be listed by point stabilizer.

Stabilizer	Geometries
$\mathrm{SO}(5)$	$\mathbb{E}^5$, S^5, $\mathbb{H}^5$
$\mathrm{U}(2)$	Heis_5 and $\widetilde{\mathrm{U}(2,1)}/\,\mathrm{U}(2)$
$\mathrm{SO}(3)_5$	$\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ and $\mathrm{SU}(3)/\,\mathrm{SO}(3)$
$\mathrm{SO}(2) \times \mathrm{SO}(2)$	$\underset{x-1,\,x-1,\,x+1,\,x+1}{\mathbb{R}^4 \rtimes \mathbb{R}}$ and the associated bundles (Thm. 1.2(4))
$\mathrm{SO}(2)$	$\underset{\text{2 real roots}}{\mathbb{R}^4 \rtimes \mathbb{R}}$ and $\underset{(x-1)^2,\,x+1,\,x+1}{\mathbb{R}^4 \rtimes \mathbb{R}}$
$S^1_{1/2}$	All line bundles over $\mathbb{F}^4$ (Thm. 1.2(5))
S^1_1	The two unit tangent bundles (Thm. 1.2(3)), $\underset{x^2,\,x^2}{\mathbb{R}^4 \rtimes \mathbb{R}}$, and $\underset{x_3 \to x_2 \to y}{(\mathbb{R} \times \mathrm{Heis}_3) \rtimes \mathbb{R}}$
$\{1\}$	The remaining solvable Lie groups

Figure 3.3: Flowchart of the classification. Let V be an irrep in $G_p \curvearrowright T_pM$ of maximal dimension.

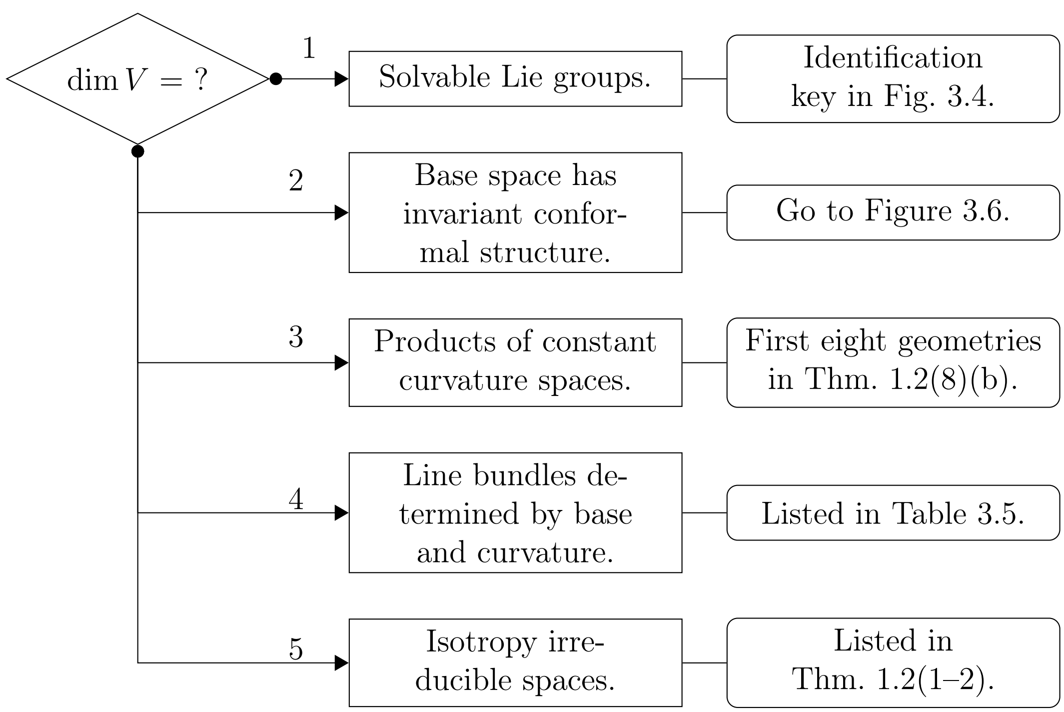
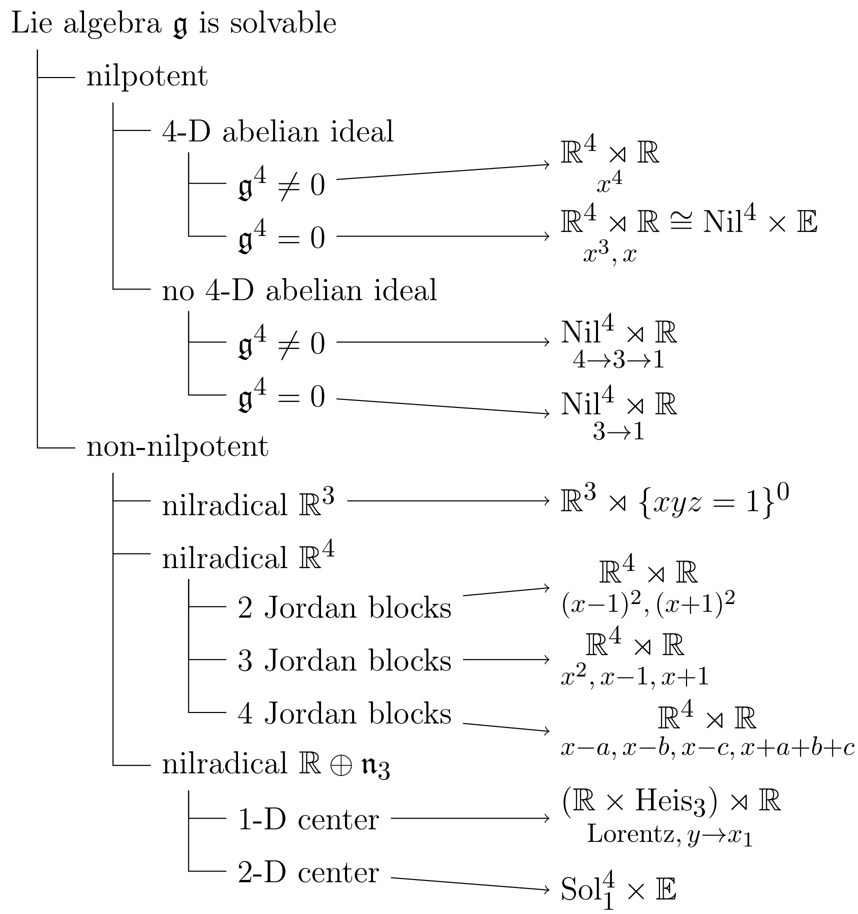


Figure 3.4: Identification key for solvable geometries $G = G/\{1\}$.



Otherwise $G_p \curvearrowright T_pM$ is nontrivial and reducible ($2 \leq \dim V \leq 4$). We classify these—the "fibering geometries"—in Part III, starting by proving the existence of a G-invariant fiber bundle structure on M (Prop. 12.3). The Uniformization Theorem and version of a theorem by Obata and Lelong-Ferrand [Oba73, Lemma 1] imply the base space has an invariant conformal structure. Beyond this common behavior, the properties of the fibering and the relevant tools vary with the dimension of the subrepresentation V, naturally suggesting the cases in Figure 3.3.

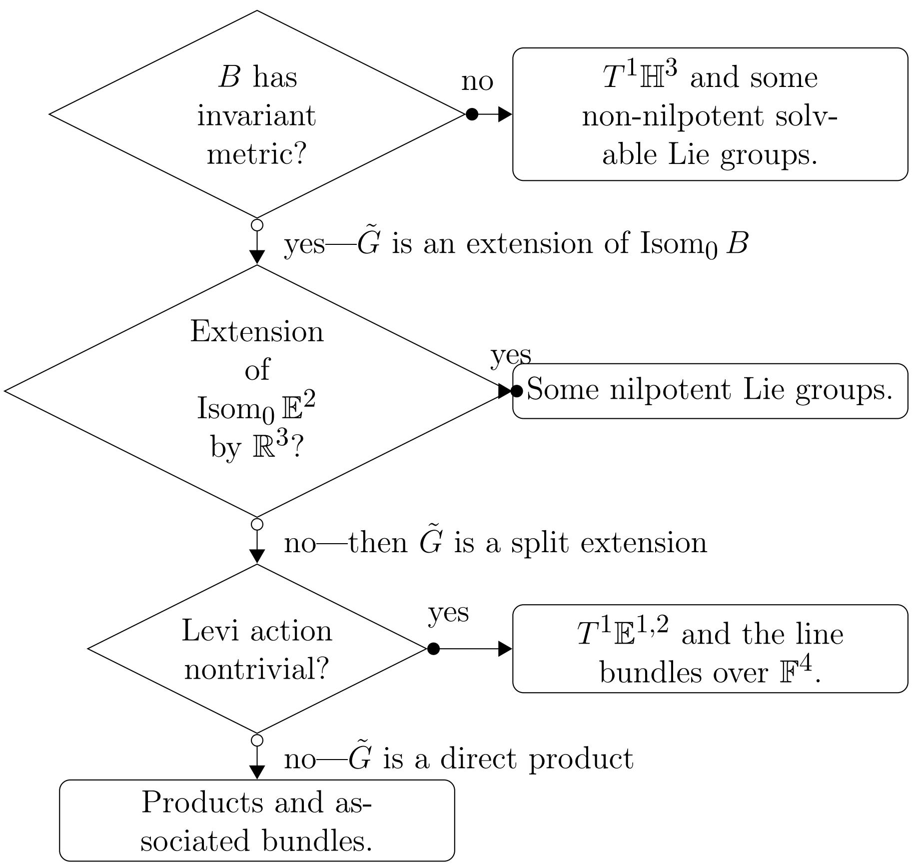
When $\dim V = 4$, the geometries are determined by curvature and base, in a fashion closely resembling Thurston's treatment of $\dim V = 2$ and $\dim M = 3$ in [Thu97, Thm. 3.8.4(b)]; Table 3.5 lists the results.

Table 3.5: Geometries with irreducible 4-dimensional isotropy summand

Base	Flat (product)	Curved
S^4	$S^4 \times \mathbb{E}$	
$\mathbb{E}^4$	non-maximal $\mathbb{E}^5$	
$\mathbb{H}^4$	$\mathbb{H}^4 \times \mathbb{E}$	
$\mathbb{CP}^2$	$\mathbb{CP}^2 \times \mathbb{E}$	non-maximal S^5
$\mathbb{C}^2$	non-maximal $\mathbb{E}^5$	Heis_5
$\mathbb{CH}^2$	$\mathbb{CH}^2 \times \mathbb{E}$	$\widetilde{\mathrm{U}(2,1)}/\mathrm{U}(2)$

Otherwise, we work systematically with G-invariant fiber bundle structures by recasting the problem as an extension problem for the Lie algebra of G and solving it with the help of Lie algebra cohomology. Over 3-dimensional base spaces there happen to be only products; but over 2-dimensional base spaces a daunting array of possibilities requires some attempt to organize the problem, summarized in Figure 3.6.

Figure 3.6: Classification strategy for geometries $M = G/G_p$ fibering over 2-D spaces B.



CHAPTER 4

RELATED WORK

Classification of compact homogeneous spaces. Gorbatsevich has produced classification results for *compact* homogeneous spaces M by using a fiber bundle described in [GOV93, §II.5.3.2] whose fibers have compact transformation group, whose base is aspherical, and whose total space is a finite cover of M. The classification is complete in dimension up to 5 in general, in dimension 6 up to finite covers, and in dimension 7 in the aspherical case [Gor12].

Another approach would be to group the problem by the number of isotropy summands. The *Riemannian* homogeneous spaces with irreducible isotropy were classified by Manturov [Man61a, Man61b, Man66, Man98], Wolf [Wol68, Wol84], and Krämer [Krä75]; and the *compact Riemannian* homogeneous spaces with two isotropy summands were classified by Dickinson and Kerr in [DK08].

Classification of naturally reductive spaces. The *naturally reductive* Riemannian homogeneous spaces G/G_p—those whose geodesics through p are orbits of 1-parameter subgroups tangent to the representation complementary to $T_{\mathbf{1}}G_p$ in $G_p \curvearrowright T_{\mathbf{1}}G$ (see e.g. [KN69, §X.3])—have been classified in dimension 6 by Agricola, Ferreira, and Friedrich [AFF15]; and in lower dimensions by work of Kowalski and Vanhecke (see [KPV96, §6] for a summary).

The case of dimension 5, in [KV85, Thm. 2.1],[1] shares the following features with the classification of geometries.

1. Everything with SU(2) isotropy is realized by a homogeneous space with U(2) isotropy [KV91, main result (b)].

1. The changes in the corrected version [KPV96, 6.4] appear to amount to (1) changing "symmetric" and "decomposable" to "locally symmetric" and "locally decomposable" and (2) changing the rational parameter to a real parameter in the Type II family (Heisenberg-group-like bundles).

2. The associated bundle geometries appear as indecomposable naturally reductive spaces.

The differences between geometries and naturally reductive spaces bear mentioning as well:

1. Naturally reductive spaces need not be maximal as geometries, as demonstrated by non-maximal realizations of $S^3 \cong S^3 \rtimes S^1/S^1$ and $S^5 \cong \mathrm{SU}(3)/\mathrm{SU}(2)$.

2. Some geometries—particularly those with trivial isotropy—may not be realizable by naturally reductive spaces. In 3 dimensions, there is just Sol^3; and in 4 dimensions, there are $\mathbb{F}^4$ and the four solvable Lie group geometries other than $\mathrm{Heis}_3 \times \mathbb{E}$ and $\mathbb{E}^4$. In 5 dimensions, there are the unit tangent bundles $T^1\mathbb{H}^3$ and $T^1\mathbb{E}^{1,2}$, the line bundles over $\mathbb{F}^4$, the products involving Sol^3, and any solvable Lie group geometries that are not $\mathbb{E}^5$, Heis_5, or a product involving Heis_3.

A chart in [KPV96, 5.1] summarizes the relations between several other classes of spaces.

Other geometric structures. One can replace the assumption of an invariant Riemannian metric (compact point stabilizers) with other structures. A number of difficulties may result from this: geodesic completeness may no longer coincide with other notions of completeness (e.g. in [DZ10, Thm. 2.1]); isotropy representations may fail to be semisimple or faithful; In spite of these challenges, there do exist substantial results.

- Using conformal structures without relaxing other assumptions yields only S^n and $\mathbb{E}^n$: a manifold whose conformal automorphism group acts transitively with the identity component preserving no Riemannian metric is conformally equivalent to one of the two, by theorems of Obata [Oba73, Lemma 1] and Lafontaine [Laf88, Thm. D.1].

- The interaction of complex structures and 4-dimensional geometries was investigated by Wall in [Wal86]; and almost-complex structures on homogeneous spaces up to dimension 6 are classified by Winther in [Win12] with the additional assumption that point stabilizers are semisimple.

- The pseudo-Riemannian geometries were classified in dimension 3 by Dumitrescu and Zeghib in [DZ10]; and the pseudo-Riemannian naturally reductive spaces were classified in dimension 4 by Batat, López, and María [BLM15].

PART II

THE NON-FIBERED GEOMETRIES

CHAPTER 5

OVERVIEW

Riemannian homogeneous spaces appeared in Thurston's Geometrization Conjecture as local models for pieces in a decomposition of 3-manifolds. Those with compact quotients and maximal symmetry, the eight *geometries* (see Defn. 6.1 for details), are classified by Thurston in [Thu97, Thm. 3.8.4].

Filipkiewicz classified the 4-dimensional geometries in [Fil83], retaining the conditions that make a homogeneous space a geometry in the sense of Thurston. Imitating this approach in the 5-dimensional case, the problem divides into cases for each representation by which the point stabilizer G_p of $p \in M = G/G_p$ acts on the tangent space T_pM (the "linear isotropy representation").

If T_pM decomposes in a nice way into lower-dimensional irreducible sub-representations, then this decomposition can be exploited to classify the resulting 5-dimensional geometries, via a G-invariant fiber bundle structure on M. Such an approach is carried out in Part III. The present paper concerns itself with the classification when T_pM is irreducible or trivial; in this case, T_pM has no proper characteristic summands as an G_p-representation, which makes its decomposition less useful.

Instead, we appeal to existing classifications: the classification of strongly isotropy irreducible homogeneous spaces by Manturov [Man61a, Man61b, Man66, Man98], Wolf [Wol68, Wol84], and Krämer [Krä75] when T_pM is irreducible; and the method of classifying unimodular solvable real Lie algebras by nilradical, used by Mubarakzyanov in [Mub63] and Filipkiewicz in [Fil83, Ch. 6], when T_pM is trivial. Adapting these classifications to the setting of Thurston's geometries requires answering questions about lattices and maximal isometry groups, which is done in this paper. The result is as follows.

Theorem 5.1 (**Classification of 5-dimensional maximal model geometries with irreducible or trivial isotropy**). *Let $M = G/G_p$ be a 5-dimensional maximal model* *geometry.*

(i) *If G_p acts irreducibly on the tangent space T_pM, then M is one of the classical spaces $\mathbb{E}^5$, S^5, and $\mathbb{H}^5$ with its usual isometry group, or one of the other irreducible Riemannian symmetric spaces $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SU}(3)/\mathrm{SO}(3)$. (Prop. 8.1)*

(ii) *If G_p acts trivially on T_pM, then $M \cong G$ is one of the following connected, simply-connected, solvable Lie groups.*

(a) $\mathbb{R}^3 \rtimes \{xyz = 1\}^0$, *the semidirect product $\mathbb{R}^3 \rtimes \mathbb{R}^2$ where $\mathbb{R}^2$ acts by diagonal matrices with positive entries and determinant 1. (Prop. 9.12)*

(b) $\mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}$, *where the characteristic polynomials of the Jordan blocks of A are one of the following lists. These are abbreviated as $\underset{\textit{polynomials}}{\mathbb{R}^4 \rtimes \mathbb{R}}$. (Prop. 9.19)*

- $x - a$, $x - b$, $x - c$, $x + a + b + c$, *where*
  1. $a \neq b \neq c \neq a$, *and*
  2. e^{tA} *has integer characteristic polynomial for some t.*

  (This is a countably infinite family of geometries.)
- x^2, $x - 1$, $x + 1$
- $(x-1)^2$, $(x+1)^2$
- x^3, x
- x^4

(c) $\underset{3\to 1}{\mathrm{Nil}^4} \rtimes \mathbb{R}$ *and* $\underset{4\to 3\to 1}{\mathrm{Nil}^4} \rtimes \mathbb{R}$, *whose Lie algebras have basis $x_1, \ldots, x_5$ and the following nonzero brackets. (Prop. 9.22)*

$$[x_4, x_3] = x_2 \qquad [x_5, x_3] = x_1$$

$$[x_4, x_2] = x_1 \qquad [x_5, x_4] = 0 \text{ or } x_3 \text{ respectively.}$$

(d) $\mathbb{R} \times \mathrm{Sol}_1^4$ *and* $(\mathbb{R} \times \mathrm{Heis}_3) \underset{Lorentz,\, y \to x_1}{\rtimes} \mathbb{R}$*, whose Lie algebras have basis* y, x_1, x_2, x_3, z *and the following nonzero brackets. (Prop. 9.25)*

$$[x_3, x_2] = x_1 \quad [z, x_2] = x_2 \quad [z, x_3] = -x_3 \quad [z, y] \mapsto 0 \text{ or } x_1 \text{ respectively.}$$

Moreover, all of the above spaces are maximal model geometries.

Remark 5.2. Properties of the geometries are not explored in depth here. For the symmetric spaces in Thm. 5.1(i), one can see e.g. [Wol11, §9.6] for a discussion of $\mathrm{SU}(3)/\mathrm{SO}(3)$ and [BGS85, Appendix 5] for a discussion of $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. To make sense of the array of solvable Lie groups in Thm. 5.1(ii), one can construct an identification key to distinguish them, such as Figure 5.3 (proven in Prop. 9.31).

Remark 5.4. When the group is a direct product with $\mathbb{R}$, the geometry is written as a product with $\mathbb{E}$, following Thurston's convention in [Thu97, Thm. 3.8.4] to highlight that the $\mathbb{R}$ factor behaves as a 1-dimensional Euclidean space.

Some of the $\mathbb{R}^4 \rtimes \mathbb{R}$ geometries can also be named as products of lower-dimensional geometries, using names from [Wal86, Table 1]. If the polynomials are x^3 and x, then the Lie group is isomorphic to $\mathrm{Nil}^4 \times \mathbb{R}$. If there are 4 polynomials and one of them is x, then the Lie group is isomorphic to $\mathrm{Sol}^4_{m,n} \times \mathbb{R}$.

An alternative naming scheme is given in [PSWZ76, Table II], from Mubarakzyanov's classification [Mub63] of solvable Lie algebras; Table 5.5 shows the concordance.

Chapter 6 collects some basic facts about geometries and homogeneous spaces. Chapter 7 classifies the isotropy representations (point stabilizers). Sections 8 and 9 carry out the classification of geometries when the isotropy is irreducible or trivial, respectively.

Figure 5.3: Identification key for geometries $G = G/\{1\}$.

Lie algebra $\mathfrak{g}$ is solvable

- nilpotent
  - 4-D abelian ideal
    - $\mathfrak{g}^4 \neq 0$ → $\mathbb{R}^4 \rtimes_{x^4} \mathbb{R}$
    - $\mathfrak{g}^4 = 0$ → $\mathbb{R}^4 \rtimes_{x^3, x} \mathbb{R} \cong \mathrm{Nil}^4 \times \mathbb{E}$
  - no 4-D abelian ideal
    - $\mathfrak{g}^4 \neq 0$ → $\mathrm{Nil}^4 \rtimes_{4 \to 3 \to 1} \mathbb{R}$
    - $\mathfrak{g}^4 = 0$ → $\mathrm{Nil}^4 \rtimes_{3 \to 1} \mathbb{R}$
- non-nilpotent
  - nilradical $\mathbb{R}^3$ → $\mathbb{R}^3 \rtimes \{xyz = 1\}^0$
  - nilradical $\mathbb{R}^4$
    - 2 Jordan blocks → $\mathbb{R}^4 \rtimes_{(x-1)^2, (x+1)^2} \mathbb{R}$
    - 3 Jordan blocks → $\mathbb{R}^4 \rtimes_{x^2, x-1, x+1} \mathbb{R}$
    - 4 Jordan blocks → $\mathbb{R}^4 \rtimes_{x-a, x-b, x-c, x+a+b+c} \mathbb{R}$
  - nilradical $\mathbb{R} \oplus \mathfrak{n}_3$
    - 1-D center → $(\mathbb{R} \times \mathrm{Heis}_3) \rtimes_{\mathrm{Lorentz}, y \to x_1} \mathbb{R}$
    - 2-D center → $\mathrm{Sol}^4_1 \times \mathbb{E}$

Table 5.5: Lie algebra names from [PSWZ76, Tables I, II] for geometries with trivial isotropy.

Geometry	Lie algebra
$\mathrm{Nil}^4 \times \mathbb{E}$	$A_{4,1} \oplus \mathbb{R}$
$\mathrm{Sol}^4_1 \times \mathbb{E}$	$A_{4,8} \oplus \mathbb{R}$
$\underset{x^4}{\mathbb{R}^4 \rtimes \mathbb{R}}$	$A_{5,2}$
$\underset{3\to 1}{\mathrm{Nil}^4 \rtimes \mathbb{R}}$	$A_{5,5}$
$\underset{4\to 3\to 1}{\mathrm{Nil}^4 \rtimes \mathbb{R}}$	$A_{5,6}$
$\underset{x-a,\, x-b,\, x-c,\, x+a+b+c}{\mathbb{R}^4 \rtimes \mathbb{R}}$	$A_{5,7}^{a,b,c}$ $(a+b+c=-1)$
$\underset{x^2,\, x-1,\, x+1}{\mathbb{R}^4 \rtimes \mathbb{R}}$	$A_{5,8}^{-1}$
$\underset{(x-1)^2,\, (x+1)^2}{\mathbb{R}^4 \rtimes \mathbb{R}}$	$A_{5,15}^{-1}$
$\underset{\text{Lorentz},\, y\to x_1}{(\mathbb{R} \times \mathrm{Heis}_3) \rtimes \mathbb{R}}$	$A_{5,20}^{0}$
$\mathbb{R}^3 \rtimes \{xyz=1\}^0$	$A_{5,33}^{-1,-1}$

CHAPTER 6

GENERAL BACKGROUND: GEOMETRIES

This section recalls terms and basic facts about geometries, including their interpretations as Riemannian manifolds or homogeneous spaces. This starts with the definition of a geometry, following Thurston in [Thu97, Defn. 3.8.1] and Filipkiewicz in [Fil83, §1.1].

Definition 6.1 (**Geometries**).

(i) A *geometry* is a pair $(M, G \subseteq \operatorname{Diff} M)$ where M is a connected, simply-connected smooth manifold and G is a connected Lie group acting transitively, smoothly, and with compact point stabilizers.

(ii) (M, G) is a *model geometry* if there is a finite-volume complete Riemannian manifold N locally isometric to M with some G-invariant metric. Such an N is said to be *modeled on* (M, G).

(iii) (M, G) is *maximal* if there is no geometry (M, H) with $G \subsetneq H$. Any such (M, H) is said to *subsume* (M, G).

As Thurston remarks in [Thu97, §3.8], M can be thought of as a Riemannian manifold as long as one is willing to change the metric. Since the tools for proving maximality are stated in the language of Riemannian geometry, here is the explicit relationship with Riemannian manifolds.

Proposition 6.2 (**Existence of invariant metric**; see e.g. [Thu97, Prop. 3.4.11])**.** *Suppose a Lie group G acts transitively, smoothly, and with compact point stabilizers on a smooth manifold M. Then M has a G-invariant Riemannian metric.*

This makes G a subgroup of the isometry group. The next Proposition asserts this inclusion is an equality for maximal geometries, which makes it possible to approach questions of maximality by leveraging existing results describing isometry groups of various spaces.

Proposition 6.3 (**Description of maximality**, [Fil83, Prop. 1.1.2])**.**

(i) Every geometry is maximal or subsumed by a maximal geometry.

(ii) If (M, G) is a maximal geometry, then G is the identity component $(\operatorname{Isom} M)^0$ *of* $\operatorname{Isom} M$ *in any G-invariant metric on M.*

Remark 6.4. By connectedness, if (M, G) is realized by (M, H) with $G \subsetneq H$, then $\dim G < \dim H$ and $\dim G_p < \dim H_p$. Hence to verify maximality of a geometry (M, G), it suffices to distinguish (M, G) from the geometries whose point stabilizers contain G_p. Knowing inclusions between possible point stabilizers (Figure 7.2) will facilitate this.

The interpretation of geometries as homogeneous spaces is already integral to the 4-dimensional classification by Filipkiewicz, and some geometries are most concisely expressed as homogeneous spaces. So the dictionary is provided explicitly below, with an outline of the proof and prerequisites in case the reader requires details.

Proposition 6.5 (**Geometries described as homogeneous spaces**)**.**

(i) Geometries (M, G) correspond one-to-one with simply-connected homogeneous spaces G/G_p where G is a connected Lie group and G_p is compact and contains no nontrivial normal subgroups of G. The correspondence is $G/G_p \leftrightarrow (G/G_p, G)$.

(ii) G/G_p is a model geometry if and only if some lattice $\Gamma \subset G$ intersects no conjugate of G_p nontrivially.

(iii) G/G_p is a maximal geometry if and only if it is not G-equivariantly diffeomorphic to a geometry G'/G'_p with $G \subsetneq G'$.

Proof. Part (i) follows from two standard facts:

1. If a Lie group G acts transitively on a smooth manifold M with subgroup G_p stabilizing $p \in M$, then G/G_p with its natural smooth structure is G-equivariantly diffeomorphic

to M [Fil83, p. 1]. (See also [Hel78, Thm. II.3.2, Prop. II.4.3(a)], [GOV93, Thm. II.1.1.2], or [Wol11, 1.5.9].)

2. A group G acts faithfully on the coset space G/G_p (i.e. is a subgroup of $\mathrm{Diff}(G/G_p)$, as opposed to merely surjecting onto one) if and only if G_p contains no nontrivial normal subgroups of G. (Proof: The kernel of $G \to \mathrm{Diff}(G/G_p)$ is normal and contained in G_p since it fixes the identity coset. Conversely, if $N \subseteq G_p$ is normal in G, then $G \to \mathrm{Diff}(G/G_p)$ factors through G/N.)

Part (ii) is the following argument distilled from [Thu97, §3.3–3.4]. Fix any G-invariant Riemannian metric on $M = G/G_p$. Discrete subgroups Γ correspond to orbifolds $N = \Gamma\backslash M$ covered by M. Such N naturally inherits a Riemannian metric (i.e. is a manifold) if and only if Γ acts freely on M, i.e. Γ intersects no point stabilizer nontrivially. Since G acts transitively on M with compact point stabilizers,

1. every manifold quotient $\Gamma\backslash M$ is complete [Thu97, Cor. 3.5.12 and Prop. 3.4.15];

2. every complete Riemannian manifold N locally isometric to M is isometric to a quotient space $\Gamma\backslash M$ [Thu97, Prop. 3.4.5 and 3.4.15]; and

3. $\Gamma\backslash M$ has finite volume if and only if $\Gamma\backslash G$ does (i.e. Γ is a lattice). (Proof: Let ω be a left-invariant volume form on G. Cosets of G_p are compact, so one may integrate along them, recovering the invariant volume form on M up to a scale factor. Then $\mathrm{vol}(\Gamma\backslash M)$ is a multiple of $\mathrm{vol}(\Gamma\backslash G)$.)

Part (iii) is merely the original definition (6.1(iii)) rephrased in terms of the correspondence from part (i). □

CHAPTER 7

CLASSIFICATION OF ISOTROPY REPRESENTATIONS

This section presents the first step in the strategy of Thurston and Filipkiewicz, which is to classify the linear isotropy representations—representations of point stabilizers G_p on tangent spaces T_pM.

Since an isometry of a connected Riemannian manifold is determined by its value and derivative at a point (see e.g. [BP92, Prop. A.2.1]), any such action is faithful and preserves the Riemannian metric on T_pM. Since G_p is connected by a homotopy exact sequence calculation [Fil83, Prop. 1.1.1] and compact, classifying representations $G_p \curvearrowright T_pM$ is equivalent to classifying closed connected subgroups $G_p \subseteq \mathrm{SO}(5)$, up to conjugacy in $\mathrm{GL}(5,\mathbb{R})$. The classification is as follows, summarized in Figure 7.2.[1]

Proposition 7.1 (Classification of isotropy representations). *The closed connected subgroups of* $\mathrm{SO}(5)$ *are, up to conjugacy in* $\mathrm{GL}(5,\mathbb{R})$,

- *one of the following groups, acting by its standard representation over* $\mathbb{R}$ *on a subspace of* $\mathbb{R}^5$ *and trivially on the orthogonal complement;*

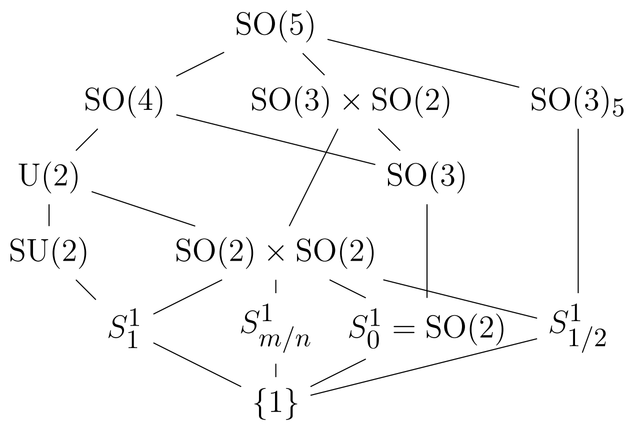
$$\{1\} \qquad \mathrm{SO}(2) \qquad \mathrm{SO}(3) \qquad \mathrm{SO}(4) \qquad \mathrm{SO}(5) \qquad \mathrm{SO}(2)\times\mathrm{SO}(2) \qquad \mathrm{SO}(2)\times\mathrm{SO}(3)$$

- *one of* $\mathrm{SU}(2)$ *or* $\mathrm{U}(2)$ *acting by its standard representation on* $\mathbb{C}^2 \cong \mathbb{R}^4 \subset \mathbb{R}^5$*;*
- $\mathrm{SO}(3)_5$*, a copy of* $\mathrm{SO}(3)$ *acting irreducibly on* $\mathbb{R}^5$*; or*
- $S^1_{m/n}$ $(0 \le \frac{m}{n} \le 1)$*, the 1-parameter subgroup of* $\mathrm{SO}(2)\times\mathrm{SO}(2)$ *defined by*

$$S^1 \ni z \mapsto (z^m, z^n) \in S^1 \times S^1 \cong \mathrm{SO}(2)\times\mathrm{SO}(2).$$

1. Proofs of the inclusions are omitted; all are readily guessed except for $S^1_{1/2} \subset \mathrm{SO}(3)_5$. This last inclusion can be seen from the description of $\mathrm{SO}(3)_5$ as $\mathrm{SO}(3)$ acting by conjugation on traceless symmetric bilinear forms—a 90 degree rotation in the first two coordinates has order 4 but acts with order 2 on the diagonal matrix $d(1,-1,0)$.

Figure 7.2: Closed connected subgroups of SO(5), with inclusions.



7.1 Proof of the classification of isotropy representations

The proof presented below is similar in spirit to that of [Fil83, §1.2], albeit with somewhat more representation-theoretic language. In higher dimensions, it would be advantageous to use an alternative strategy that provided more structure and fewer opportunities for human error. One such approach, to list maximal subgroups recursively using results by Dynkin, is outlined in [KK13, Tables 1, 3] for several groups including SO(5) and SO(6).

In dimension 5, the groups involved are small enough that only representations of S^1 and SU(2) really need to be understood; this permits a slightly more elementary approach, using some standard facts about representations of low-dimensional groups. These will be stated where used, with proofs deferred to Section 7.2 so that they are available but not a source of clutter.

The first step is the classification up to finite covers by using the classification of compact simple Lie groups.

Proposition 7.3. *Any closed connected proper subgroup $K \subset \mathrm{SO}(5)$ is finitely covered by a product of at most two factors, each of which is S^1 or* SU(2).

Proof. Every compact connected Lie group is finitely covered by a product of circles S^1 and compact simply-connected simple Lie groups [BD85, Thm. V.8.1]. By the classification of compact connected simple Lie groups [Hel78, Ch. X, §6 (p. 516)], the simple factors could be

- $\mathrm{Sp}(n)$, $n \geq 1$ (dim. $n(2n+1)$)
- $\mathrm{SU}(n)$, $n \geq 3$ (dim. $n^2 - 1$)
- $\mathrm{Spin}(n)$, $n \geq 7$ (dim. $\binom{n}{2}$)
- G_2 (dim. 14), F_4 (dim. 52), E_6 (dim 78), E_7 (dim. 133), or E_8 (dim. 248).

The dimension of SO(5) is 10; so only $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ and SU(3) could be simple factors of a cover of a proper subgroup of SO(5). Moreover, SU(3) is not a factor since it has no nontrivial 5-dimensional representations (Prop. 7.4).

Hence K is finitely covered by a product whose factors are S^1 or SU(2). Since SO(5) has rank 2, such a product has at most two factors. □

At this stage, the full classification (Prop. 7.1) reduces to listing low-dimensional representations of S^1 and SU(2).

Proof of Prop. 7.1 (Classification of closed connected subgroups $K \subseteq \mathrm{SO}(5)$). By Prop. 7.3 above, a closed connected subgroup $K \subseteq \mathrm{SO}(5)$ either is SO(5) or has a finite cover $\tilde{K} \cong K_1 \times K_2$, where K_1 and K_2 are S^1 or SU(2). When $\tilde{K}$ is a product, there are the following cases.

Case 1: $\tilde{K} \cong \{1\}$. Then $K = \{1\}$.

Case 2: $\tilde{K} \cong S^1$. Then K is the image of some homomorphism $f : \mathbb{R} \to \mathrm{SO}(5)$. The eigenvalues of $f(t) \in \mathrm{SO}(5)$ are 1 and two pairs of complex conjugates of norm 1, with homomorphic dependence on t; so there is some decomposition of $\mathbb{R}^5$ as $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ in which

$$f(t) = (e^{xt}, e^{yt}, 1) \in \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(1).$$

Since the image is closed, x and y must be rationally dependent; so $K = S^1_{m/n}$ where either $\frac{x}{y}$ or $\frac{y}{x}$ is equal to $\frac{m}{n}$.

Case 3: $\tilde{K} \cong S^1 \times S^1$**.** Since the two S^1 factors commute, they share eigenspaces. Using $\mathbb{R}^5 \cong \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ as above, K must be a 2-dimensional subgroup of $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(1) \cong \mathrm{SO}(2) \times \mathrm{SO}(2)$.

Case 4: $\tilde{K} \cong \mathrm{SU}(2)$**.** The classification of irreducible representations of SU(2) (Prop. 7.5) implies that K is one of SU(2) acting on $\mathbb{C}^2$, SO(3) acting on $\mathbb{R}^3$, and SO(3) acting irreducibly on $\mathbb{R}^5$.

Case 5: $\tilde{K} \cong \mathrm{SU}(2) \times S^1$**.** This SU(2) cannot act irreducibly on $\mathbb{R}^5$ since that action has trivial centralizer (Cor. 7.8). Hence the SU(2) factor acts as either SO(3) on $\mathbb{R}^3$ or SU(2) on $\mathbb{C}^2$.

- If SU(2) acts as SO(3) on some $\mathbb{R}^3 \subset \mathbb{R}^5$, then the S^1 factor preserves the decomposition $\mathbb{R}^5 \cong_{\mathrm{SO}(3)} \mathbb{R}^3 \oplus 2\mathbb{R}$ since it commutes with SO(3). So $K \subseteq \mathrm{S}(\mathrm{O}(3) \times \mathrm{O}(2))$, and by connectivity $K \subseteq \mathrm{SO}(3) \times \mathrm{SO}(2)$. Equality holds since the dimensions match.

- If SU(2) acts as SU(2) on some $\mathbb{C}^2 \subset \mathbb{R}^5$, then as above $K \subseteq \mathrm{SO}(4) \times \mathrm{SO}(1) \cong \mathrm{SO}(4)$. Since SO(4) is covered by $\mathrm{SU}(2) \times \mathrm{SU}(2)$, the S^1 factor can be taken to lie in the second SU(2) as a 1-parameter subgroup—all of which are conjugate. Thus there is only one action of $\mathrm{SU}(2) \times S^1$ on $\mathbb{R}^4$ up to conjugacy. It can be realized as the action by SU(2) on $\mathbb{C}^2$ along with the action of the unit-norm scalars, which amounts to U(2).

Case 6: $\tilde{K} \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$**.** As in Case 5 above, $K \subseteq \mathrm{SO}(4)$. Since the dimensions match, $K = \mathrm{SO}(4)$. □

7.2 Standard facts about representations of lower-dimensional groups

This subsection collects some basic facts about representations of SU(2), SU(3), and SO(3). These are only needed for the above classification of isotropy representations (Prop. 7.1), so a reader familiar with these groups may wish to skip this subsection.

Proposition 7.4. *Representations of* SU(3) *of dimension less than* 6 *over* $\mathbb{R}$ *are trivial.*

Proof. The complex irreducible representations of $\mathfrak{su}_3\mathbb{C} \cong \mathfrak{sl}_3\mathbb{C}$ [2] exponentiate to those of the simply-connected group SU(3). Listing highest weights as done in [FH91, Ch. 12] shows that they have dimensions 1, 3, 6, and higher. Over $\mathbb{R}$, an irreducible representation over $\mathbb{C}$ either remains irreducible or splits into two isomorphic irreducible summands [BD85, Prop. II.6.6(vii-ix)]; so any real irreducible representation has dimension 1, 2, 3, 6, or higher.

Then choosing an invariant inner product, an irreducible action of SU(3) on $\mathbb{R}^k$ ($k < 6$) factors through SO(3). Since SU(3) is simple and of higher dimension than SO(3), its image in SO(3) is trivial. □

Proposition 7.5. *Let V be the standard representation of* SU(2) *over* $\mathbb{C}$*. The finite-dimensional irreducible representations of* SU(2) *over* $\mathbb{R}$ *are*

(i) for even $n \geq 0$, an invariant real subspace of $\mathrm{Sym}^n V$ *(dimension $n+1$); and*

(ii) for odd $n \geq 1$, the representation $\mathrm{Sym}^n V$ *taken as a real vector space (dimension $2(n+1)$).*

2. For the isomorphism, write $\mathfrak{sl}_n\mathbb{R}$ as the sum of its skew-symmetric part $\mathfrak{k}$ and symmetric part $\mathfrak{p}$. Then since $\mathfrak{su}_n$ consists of the matrices A which satisfy $A + A^* = 0$ and are traceless, $\mathfrak{su}_n = \mathfrak{k} + i\mathfrak{p}$.

Proof. The irreducible representations of SU(2) over $\mathbb{C}$ are the symmetric powers of V [BD85, Prop. II.5.1, II.5.3] (see also [FH91, 11.8]). The action can be written explicitly as

$$\begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix} f(x,y) = f(ax+by, -\bar{b}x+\bar{a}y)$$

where $f \in \operatorname{Sym}^n V$ is homogeneous of degree n in $\mathbb{C}[x,y]$.

Define $\phi : \operatorname{Sym}^n V \to \operatorname{Sym}^n V$ by

$$(\phi f)(x,y) = \overline{f(\overline{y}, -\overline{x})}.$$

One can verify from this formula that ϕ is SU(2)-equivariant and satisfies $\phi i = -i\phi$ and $\phi^2 = (-1)^n$.

(i) If n is even, then ϕ is a real structure on $\operatorname{Sym}^n V$. An irreducible representation over $\mathbb{C}$ with a real structure is the direct sum of two copies of an irreducible representation over $\mathbb{R}$ [BD85, Prop. II.6.6(vii)]. A summand can be recovered as $(\phi + \mathrm{Id})(\operatorname{Sym}^n V)$, consisting of the polynomials $f(x,y) = \sum_m f_m x^m y^{n-m}$ where $f_m = (-1)^m \overline{f_{n-m}}$.

(ii) If n is odd, then ϕ is a quaternionic structure on $\operatorname{Sym}^n V$. An irreducible representation over $\mathbb{C}$ with a quaternionic structure is irreducible over $\mathbb{R}$ [BD85, Prop. II.6.6(ix)]. □

Remark 7.6 ([BD85, II.5.4]). If and only if n is even, the scalar $-1 \in \mathrm{SU}(2)$ acts trivially on $\operatorname{Sym}^n V$, allowing the action of SU(2) to descend to an action of $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm 1\} = \mathrm{SU}(2)/Z(\mathrm{SU}(2))$.

Proposition 7.7. $\operatorname{End} W \cong \mathbb{R}$ *for any irreducible representation* W *of* SO(3) *over* $\mathbb{R}$.

Proof. Every endomorphism of W extends to an endomorphism of $W \otimes \mathbb{C}$, which is irreducible over $\mathbb{C}$. By Schur's lemma, $\phi \otimes \mathbb{C}$ is multiplication by a scalar. Since $\dim W$ is odd (Rmk. 7.6), the only scalars that preserve W are the reals. □

Corollary 7.8. *The centralizer of* $\mathrm{SO}(3)_5$ *in* $\mathrm{SO}(5)$ *is trivial.*

CHAPTER 8

THE CASE OF IRREDUCIBLE ISOTROPY

When $G_p \curvearrowright T_pM$ is irreducible, the classification of isotropy irreducible homogeneous spaces and the classification of irreducible Riemannian symmetric spaces produce a list (Prop. 8.1) of homogeneous spaces as candidates. Then it only remains to check that each is a maximal model geometry with irreducible isotropy (Prop. 8.4–8.5). Taken together, these results prove Thm. 5.1(i).

8.1 The list of candidates

The first step of this classification is to obtain, from the classification of isotropy irreducible homogeneous spaces, the following explicit list of candidate geometries.

Proposition 8.1. *Let $M = G/G_p$ be a 5-dimensional maximal geometry for which $G_p \curvearrowright T_pM$ is irreducible. Then M is one of the following Riemannian symmetric spaces.*

$$\begin{array}{ll} \mathbb{E}^5 = \mathbb{R}^5 \rtimes \mathrm{SO}(5)/\,\mathrm{SO}(5) & \\ S^5 = \mathrm{SO}(6)/\,\mathrm{SO}(5) & \mathrm{SU}(3)/\,\mathrm{SO}(3) \\ \mathbb{H}^5 = \mathrm{SO}(5,1)/\,\mathrm{SO}(5) & \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) \end{array}$$

Aside from consulting existing classifications, only the following standard fact is needed to produce the above list.

Lemma 8.2. *All copies of* SO(3) *in* SU(3) *are conjugate.*

Proof. The irreducible representations of SO(3) over $\mathbb{C}$ of dimension 3 or lower are the trivial and the standard representation; so any two embeddings $\mathrm{SO}(3) \hookrightarrow \mathrm{SU}(3)$ are conjugate by some representation isomorphism $h \in \mathrm{GL}(3,\mathbb{C})$.

Isomorphic unitary representations are unitarily isomorphic (see e.g. [BD85, Exercise II.1.8], or a proof in [BR86, Prop. 5.2.1] using polar decomposition). Then h can be taken to lie in SU(3), so the two embeddings of SO(3) are conjujgate in SU(3). □

Proof of Prop. 8.1. Assume $M = G/G_p$ is a 5-dimensional maximal geometry such that $G_p \curvearrowright T_pM$ is irreducible. It is a theorem of Wolf that if a homogeneous space G/G_p has compact, connected, irreducibly-acting isotropy G_p, then either G/G_p is a Riemannian symmetric space or G is a compact simple Lie group [Wol68, Thm. 1.1].[1] From the classification of isotropy representations $G_p \curvearrowright T_pM$ (Prop. 7.1), G_p is either SO(5) or $\mathrm{SO}(3)_5$.

Case 1: $G_p = \mathrm{SO}(5)$. Since SO(5) acts transitively on 2-planes through the origin in $\mathbb{R}^5$, M has constant sectional curvature—and is therefore exactly one of $\mathbb{E}^5$, S^5, and $\mathbb{H}^5$ by the Killing-Hopf theorem [Wol11, Cor. 2.4.10].

Case 2: $G_p = \mathrm{SO}(3)_5$. In this case, G is 8-dimensional. If M is an irreducible symmetric space, then M is Euclidean, $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, or $\mathrm{SU}(3)/\mathrm{SO}(3)$ by the classification of irreducible symmetric spaces (see [Hel78, X.6 Table V and p.515–518]).

Otherwise, G is an 8-dimensional compact simple Lie group. By the classification of compact simple Lie groups [Hel78, X.6 Table IV], $G \cong \mathrm{SU}(3)$. Since all copies of SO(3) in SU(3) are conjugate (Lemma 8.2), X is the symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$. □

Remark 8.3. We have not explicitly verified that the candidate spaces have irreducible isotropy. That this holds for every irreducible Riemannian symmetric space (which seems to be well known; see e.g. [Wol68, Ch. 1, condition (v)]) is a side effect of proving the usual

1. In [Wol68, Table, p. 107–110], Wolf gave a more explicit classification of strongly isotropy irreducible spaces. Wang and Ziller remark in [WZ91, p. 2] that this classification has an omission but do not say whether the erratum in [Wol84] completes it. Instead they refer the reader to two other classifications, by Manturov in [Man61a, Man61b, Man66] (earlier, also with omissions; see also [Man98]) and by Krämer in [Krä75] (believed complete). A slightly weaker version of the result used here—omitting the claim that G is simple—was known to Matsushima, with proof first given by Nagano in [Nag59, Appendix].

decomposition theorem for Riemannian symmetric spaces by using orthogonal involutive Lie algebras. Such an approach can be found in [Wol11, Thm. 8.3.8].

Alternatively, it follows in the case of SO(5) from SO(5) having a transitive action on S^4; and in the case of $\mathrm{SO}(3)_5$ from observing that SO(3) acts irreducibly on the space V of traceless symmetric 3×3 matrices, and writing

$$\mathfrak{sl}_3\mathbb{R} = \mathfrak{so}_3\mathbb{R} + V \qquad\qquad \mathfrak{su}_3\mathbb{R} = \mathfrak{so}_3\mathbb{R} + iV.$$

8.2 Maximality and the existence of compact quotients

It happens that all non-Euclidean isotropy irreducible spaces—with two exceptions, neither of which has dimension 5—are already known to be maximal model geometries. So to prove that the 5-dimensional isotropy irreducible geometries (i.e. those produced in Prop. 8.1) are maximal model geometries, it suffices to collect some existing theorems.

Proposition 8.4. *Any geometry with irreducible isotropy is a model geometry (i.e. admits a compact manifold quotient).*

Proof. As part of the classification of strongly isotropy irreducible spaces, such a geometry is either already compact or Riemannian symmetric [Wol68, Thm. 1.1]. Borel proved in [Bor63, Thm. A] that every simply-connected Riemannian symmetric space G/K admits a compact manifold quotient $\Gamma \backslash G/K$. □

Since all of the 5-dimensional isotropy irreducible geometries are Riemannian symmetric spaces, it suffices to know when a Riemannian symmetric space is maximal. The geometry $\mathbb{E}^5$ is maximal since its isotropy SO(5) is maximal (Rmk. 6.4); and the other candidates G/K have semisimple G, so the following result proves them maximal.

Proposition 8.5. *Suppose G/K is a Riemannian symmetric space—i.e. suppose K is an open subgroup of the fixed set in G of some order 2 element of $\operatorname{Aut} G$. Suppose further that*

G is semisimple and acts faithfully on G/K. Then G/K is a maximal geometry.

Proof. If G/K is a Riemannian symmetric space with G semisimple and acting faithfully, then $G = (\operatorname{Isom} G/K)^0$ in every G-invariant metric on G/K [Hel78, Thm. V.4.1(i)]. At least one of these invariant metrics has an isometry group whose identity component is the transformation group G' of a maximal geometry G'/K' realizing G/K (Prop. 6.3). Then $G' = G$, so G/K is maximal. □

Remark 8.6. Wolf also proved maximality in the non-symmetric case in [Wol68, Thm. 17.1]. That is, except for $G_2/\operatorname{SU}(3) \cong S^6$ and $\operatorname{Spin}(7)/G_2 \cong S^7$, a simply-connected isotropy irreducible Riemannian homogeneous space G/K is maximal if G is semisimple (i.e. G/K is not Euclidean) and acts faithfully. The same theorem also includes a description of the full isometry group—not just the identity component.

Remark 8.7. One could instead verify maximality by checking that the listed spaces are not extended by any geometries with larger isotropy groups. The constant-curvature geometries have maximal isotropy; $\operatorname{SL}(3,\mathbb{R})/\operatorname{SO}(3)$ is distinguished from the constant-curvature geometries by having rank 2 [BGS85, Appendix 5 §2, p. 242]; and $\operatorname{SU}(3)/\operatorname{SO}(3)$ is distinguished by having nonzero π_2, which can be calculated using the homotopy exact sequence.

CHAPTER 9

THE CASE OF TRIVIAL ISOTROPY: SOLVABLE LIE GROUPS

This section proves Thm. 5.1(ii), the classification of 5-dimensional maximal model geometries $M = G/G_p$ for which $G_p \curvearrowright T_pM$ is trivial. (The reader may wish to consult the identification key in Figure 5.3 for a reminder of the results.)

Overview (see also "Roadmap" below). Our strategy, following that of Filipkiewicz in [Fil83, §6], begins by invoking Filipkiewicz's reduction to a classification of simply-connected solvable groups.

Proposition 9.1 ([Fil83, Prop. 6.1.3])**.** *If $M = G/G_p$ is a maximal model geometry with 0-dimensional point stabilizers, then $M \cong G$ is a connected, simply-connected, unimodular solvable Lie group, and $\operatorname{Aut}(G)$ is solvable and simply-connected.*

The classification proceeds by expressing G as an extension of an abelian group by a nilpotent group (such as the nilradical). Conveniently, only split extensions are needed in order to produce the maximal geometries. That is, Section 9.2 will prove that

Proposition 9.2. *If $G = G/\{1\}$ is a maximal model geometry of dimension 5, then either*

(i) $G \cong \mathbb{R}^3 \rtimes \mathbb{R}^2$ where $\mathbb{R}^2$ acts on $\mathbb{R}^3$ as the diagonal matrices with positive entries and determinant 1; or

(ii) $G \cong N \rtimes \mathbb{R}$ where N is nilpotent, connected, and simply-connected.

The problem then reduces to classifying semidirect products and checking lattice existence (for model geometries) and maximality. We perform this classification in the language of Lie algebras, using the correspondence between Lie algebras and connected, simply-connected Lie groups (see e.g. [GOV93, Thm. I.2.2.10–11] and [GOV94, Thm. 1.4.2]).

Roadmap. After Section 9.1 lists some notation, Section 9.2 proves the above proposition using some Lie algebra cohomology. Details on the $\mathbb{R}^3 \rtimes \mathbb{R}^2$ geometry (in (i)) are in Section 9.3, including an application of Dirichlet's unit theorem (Prop. 9.16). For each of the three groups that can occur as N in (ii), a subsection of Section 9.4 lists semidirect products, omitting any that are easily shown not to produce a maximal model geometry; and questions of lattice existence determine the model geometries. Section 9.5 proves maximality (Prop. 9.30) using a general theorem by Gordon and Wilson, and lists features distinguishing the geometries from each other (Prop. 9.31). Taken together, these results prove Thm. 5.1(ii).

Remark 9.3. The 5-dimensional solvable Lie algebras over $\mathbb{R}$ having already been classified by Mubarakzyanov in [Mub63] using a largely similar approach,[1] much of this case could be reduced to consulting a table such as [PSWZ76, Table II]. It is instead presented explicitly here, since the method stands on its own and is illustrative—exposing tools and calculations that will be reused in classifying the fibered geometries (particularly a partial classification of extensions of $\mathbb{R}^2$ by $\mathbb{R}^3$ in Lemmas 9.13 and 9.15). Only a fraction of the classification of solvable Lie algebras over $\mathbb{R}$ is duplicated, since only those which admit lattices and are tangent to maximal geometries are of interest.

9.1 Notations

The strategy outlined above for classifying solvable Lie groups involves nilpotent subalgebras of their Lie algebras. The following two definitions will aid in naming such subalgebras.

Definition 9.4. $\mathfrak{n}_k$ is the semidirect sum $\mathbb{R}^{k-1} \niplus \mathbb{R}$ where some $x_k \in \mathbb{R}$ acts on $\mathbb{R}^{k-1}$ in its standard basis $\{x_1, \ldots, x_{k-1}\}$ by a single Jordan block with eigenvalue 0.

Example 9.5. $\mathfrak{n}_3$ is the 3-dimensional Heisenberg Lie algebra, and $\mathfrak{n}_4$ is the unique 4-dimensional indecomposable nilpotent Lie algebra (see [PSWZ76, Table I] or Prop. 9.18

1. The classification is complete in dimensions ≤ 6 and is known in limited cases for higher dimensions; see [ŠW12, Introduction] for a survey.

below).

Definition 9.6 (**Nilradical**, see e.g. [GOV94, §2.5])**.** The *nilradical* $\mathrm{nil}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the unique maximal nilpotent ideal of $\mathfrak{g}$.

Lie algebra extensions that split are semidirect sums $\mathfrak{h} \ni \mathfrak{g}$, which are classified by the action $\mathfrak{g} \to \mathrm{der}(\mathfrak{h}) / \mathrm{ad}(\mathfrak{h}) = \mathrm{out}(\mathfrak{h})$. As the actions encountered are usually traceless due to unimodularity considerations (such as in Lemma 9.9 below), we also make the following definition.

Definition 9.7. If $\mathfrak{h}$ is a unimodular Lie algebra (so that $\mathrm{ad}\,\mathfrak{h}$ acts tracelessly), $\mathrm{sout}\,\mathfrak{h} \subseteq \mathrm{out}\,\mathfrak{h}$ denotes the subalgebra consisting of traceless outer derivations.

9.2 Reduction to semidirect products

This section proves Prop. 9.2, which asserts that a maximal model geometry $G = G/\{1\}$ of dimension 5 is one of two forms of semidirect product. Since such a claim bears a close resemblance to [Fil83, Prop. 6.1.4] from the 4-dimensional case, an n-dimensional generalization such as the following may be of interest.

Proposition 9.8 (G is an extension of an abelian group by a nilpotent group)**.** *Suppose* $\mathfrak{g}$ *is a unimodular solvable Lie algebra of dimension* $n > 1$.

(i) If $\mathfrak{g}$ *is not nilpotent then* $\mathfrak{g}$ *is an extension*

$$0 \to \mathrm{nil}\,\mathfrak{g} \to \mathfrak{g} \to \mathbb{R}^k \to 0$$

for some $0 < k < n$*, and* $\mathfrak{g}$ *acts tracelessly on* $\mathrm{nil}\,\mathfrak{g}$.

(ii) If $\mathfrak{g}$ *is nilpotent, then it is a semidirect sum of a nilpotent ideal* $\mathfrak{n}$ *and* $\mathbb{R}$*, with* $\mathbb{R}$ *acting tracelessly.*

The following outline summarizes the gap between the above extension problems and the two semidirect products described in Prop. 9.2.

Proof outline of Prop. 9.2. The Lie algebra $\mathfrak{g}$ of a model geometry $G = G/\{1\}$ is unimodular and solvable (Prop. 9.1). The nilradical $\operatorname{nil}\mathfrak{g}$ of a 5-dimensional unimodular solvable Lie algebra $\mathfrak{g}$ is either $\mathbb{R}^3$ or of dimension at least 4 (Prop. 9.10). The two cases of Prop. 9.2 are proven as follows.

(i) If $\operatorname{nil}\mathfrak{g} \cong \mathbb{R}^3$ and G is a maximal geometry, then the extension in Prop. 9.8(i) above is split and $\mathbb{R}^2$ acts on $\mathbb{R}^3$ by traceless diagonal matrices (Prop. 9.12).

(ii) If $\operatorname{nil}\mathfrak{g}$ has dimension at least 4, then $\mathfrak{g}$ is an extension of $\mathbb{R}$ by a nilpotent algebra by Prop. 9.8 above; and the extension splits since any linear map from $\mathbb{R}$ is a homomorphism.

The requirement that geometries are simply-connected then allows these Lie algebra results to apply to the corresponding Lie groups. □

To complete the proof, the following subsections each prove one component—the n-dimensional extension problem (Prop. 9.8), the restriction on $\operatorname{nil}\mathfrak{g}$ (Prop. 9.10), and the case when $\operatorname{nil}\mathfrak{g} = \mathbb{R}^3$ (Prop. 9.12).

9.2.1 The general extension problem

This section proves Prop. 9.8—the description of unimodular solvable Lie algebras as extensions of abelian algebras by nilpotent algebras. The claims about actions being traceless will be proven using the following observation.

Lemma 9.9. *If $\mathfrak{a}$ is an ideal in a unimodular solvable Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{a}$ is unimodular, then $\mathfrak{g}$ acts tracelessly on $\mathfrak{a}$.*

Proof. Suppose $g \in \mathfrak{g}$. Since $\mathfrak{g}$ is unimodular, $\operatorname{tr}\operatorname{ad}_{\mathfrak{g}} g = 0$. Since $\mathfrak{g}/\mathfrak{a}$ is unimodular, $\operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{a}} g = 0$. The conclusion follows from

$$\operatorname{tr}\operatorname{ad}_{\mathfrak{g}} = \operatorname{tr}\operatorname{ad}_{\mathfrak{g}/\mathfrak{a}} + \operatorname{tr}\operatorname{ad}_{\mathfrak{a}}.$$

□

Armed with this, let $\mathfrak{g}$ be a unimodular solvable Lie algebra that we hope to write as an extension; the proof of Prop. 9.8 divides into the following two almost-independent cases.

Proof of Prop. 9.8(i) (the non-nilpotent case). By a theorem of Chevalley (see e.g. [Jac62, II.7 Thm. 13]), the derived algebra of a finite-dimensional solvable Lie algebra is contained in the nilradical;[2] so $\mathfrak{g}/\operatorname{nil}(\mathfrak{g})$ is abelian, and thus unimodular. Then $\mathfrak{g}/\operatorname{nil}(\mathfrak{g})$ is some $\mathbb{R}^k$ acting tracelessly on $\operatorname{nil}\mathfrak{g}$ (Lemma 9.9). Since $k = 0$ or $k = n$ would make $\mathfrak{g}$ nilpotent, $0 < k < n$. □

Proof of Prop. 9.8(ii) (the nilpotent case). Since $\mathfrak{g}$ is nilpotent, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Then any proper vector subspace $\mathfrak{n}$ of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent ideal. Taking $\mathfrak{n}$ to be of codimension 1 makes $\mathfrak{g}/\mathfrak{n} \cong \mathbb{R}$, so $\mathfrak{g}$ is an extension

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathbb{R} \to 0.$$

As an extension of $\mathbb{R}$, this splits (any section of $\mathfrak{g} \to \mathbb{R}$ as a linear map is immediately a homomorphism); so $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathbb{R}$. This action is traceless by Lemma 9.9. □

9.2.2 Nilradicals

This section proves the following restriction on nilradicals of $\mathfrak{g}$.

2. Alternatively, one could use Lie's theorem that $\mathfrak{g}$ has a faithful representation as upper-triangular matrices.

Proposition 9.10. *The nilradical of a 5-dimensional unimodular solvable Lie algebra is either* $\mathbb{R}^3$ *or of dimension at least* 4.

Nilradicals of dimension 4 are classified later, in Prop. 9.18. The upcoming proof makes use of one technical lemma—that the action of $\mathfrak{g}/\operatorname{nil}(\mathfrak{g})$ on $\operatorname{nil}(\mathfrak{g})$ is somehow morally as good as faithful.

Lemma 9.11. *Suppose* $\mathfrak{g}$ *is a finite-dimensional solvable Lie algebra with* $\mathfrak{g}/\operatorname{nil}(\mathfrak{g}) \cong \mathbb{R}^k$. *If* $f : \mathbb{R}^k \to \mathfrak{g}$ *is any section of the quotient map as a map of vector spaces, then*

$$\operatorname{ad}|_{\operatorname{nil}(\mathfrak{g})} \circ f : \mathbb{R}^k \to \operatorname{der}(\operatorname{nil}(\mathfrak{g}))$$

is injective.

Proof. Let $\mathfrak{k}$ be the kernel of this map, and let $\mathfrak{h}$ be the vector subspace $\operatorname{nil}(\mathfrak{g}) + f(\mathfrak{k})$ of $\mathfrak{g}$. Since $\mathfrak{g}/\operatorname{nil}(\mathfrak{g})$ is abelian,

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{nil}\mathfrak{g} \subseteq \mathfrak{h};$$

so $\mathfrak{h}$ is an ideal. This inclusion also implies $[\mathfrak{h}, \mathfrak{h}] \subseteq \operatorname{nil}\mathfrak{g}$, which is the base case for the following induction that shows $\mathfrak{h}$ is nilpotent.

$$\mathfrak{h}^{i+1} = [\mathfrak{h}, \mathfrak{h}^i] \subseteq [\operatorname{nil}(\mathfrak{g}) + f(\mathfrak{k}), \operatorname{nil}(\mathfrak{g})^{i-1}] \subseteq \operatorname{nil}(\mathfrak{g})^i + 0$$

Then $\mathfrak{h} = \operatorname{nil}\mathfrak{g}$ by the definition of the nilradical, so $\mathfrak{k} = 0$. □

Lemma 9.11 will also be used in a later section for the classification in the case $\operatorname{nil}\mathfrak{g} = \mathbb{R}^3$. For now, its role is to provide constraints on dimension in the proof of Prop. 9.10.

Proof of Prop. 9.10. The two nilpotent Lie algebras of dimension 3 are $\mathbb{R}^3$ and $\mathfrak{n}_3$ (see e.g. [FH91, Lec. 10], [PSWZ76, Table I], or [Mac99, Table 21.3]); so it will suffice to show that $\operatorname{nil}\mathfrak{g}$ has dimension at least 3 and is not $\mathfrak{n}_3$.

Step 1: $\operatorname{nil}\mathfrak{g}$ has dimension at least 3. This follows from a bound by Mubarakzyanov on the dimension of the nilradical (see e.g. [GOV94, Thm. 2.5.2]) but can also be proven directly, as follows.

First, $\operatorname{nil}(\mathfrak{g}) \neq 0$ since that would imply $\mathfrak{g} = \mathfrak{g}/\operatorname{nil}(\mathfrak{g}) = \mathbb{R}^5$, which has nilradical $\mathbb{R}^5$. Also, $\operatorname{nil}\mathfrak{g} \ncong \mathbb{R}$ since Lemma 9.11 would then require some injective linear map

$$\mathfrak{g}/\operatorname{nil}(\mathfrak{g}) \cong \mathbb{R}^4 \to \operatorname{der}\mathbb{R} \cong \mathfrak{gl}_1\mathbb{R} \cong \mathbb{R}.$$

Similarly, if $\dim\operatorname{nil}\mathfrak{g} = 2$, then any linear section of $\mathfrak{g} \to \mathfrak{g}/\operatorname{nil}(\mathfrak{g}) \cong \mathbb{R}^3$ would induce an injective linear map $\mathbb{R}^3 \to \mathfrak{gl}_2\mathbb{R}$. In fact this map would have to land in $\mathfrak{sl}_2\mathbb{R}$ since unimodularity of $\mathfrak{g}$ and $\mathfrak{g}/\operatorname{nil}(\mathfrak{g}) \cong \mathbb{R}^k$ requires $\mathfrak{g}$ to act tracelessly (Lemma 9.9). Then $\mathfrak{sl}_2\mathbb{R}$ would occur as a subalgebra of $\mathfrak{g}$—in which case $\mathfrak{g}$ would not be solvable since every term of its derived series would contain $\mathfrak{sl}_2\mathbb{R}$. Hence $\dim\operatorname{nil}\mathfrak{g} \neq 2$.

Step 2: The traceless outer derivation algebra $\operatorname{sout}(\mathfrak{n}_3)$ is the isomorphic image of some $\mathfrak{sl}_2\mathbb{R} \subset \operatorname{der}(\mathfrak{n}_3)$. Let $\mathfrak{n}_3$ have basis x_1, x_2, x_3 where x_1 is central and $[x_3, x_2] = x_1$.

A derivation $D : \mathfrak{n}_3 \to \mathfrak{n}_3$ induces a linear map $\mathbb{R}x_2 + \mathbb{R}x_3 \to \mathfrak{n}_3/(\mathbb{R}x_2 + \mathbb{R}x_3) \cong \mathbb{R}x_1$. These are in bijection with the inner derivations, so up to subtracting an inner derivation $D(\mathbb{R}x_2 + \mathbb{R}x_3) \subset \mathbb{R}x_2 + \mathbb{R}x_3$. Then

$$Dx_1 = D[x_3, x_2] = [x_3, Dx_2] + [Dx_3, x_2],$$

so relative to the basis $\{x_1, x_2, x_3\}$, the matrix of D is

$$\begin{pmatrix} a+d & & \\ & a & b \\ & c & d \end{pmatrix}.$$

If D is traceless, then $a+d=0$. Then $\mathrm{sout}(\mathfrak{n}_3) \cong \mathfrak{sl}_2\mathbb{R}$, and a section of $\mathrm{sder}(\mathfrak{n}_3) \to \mathrm{sout}(\mathfrak{n}_3)$ is given by the above matrix.

Step 3: $\mathfrak{n}_3$ is not $\mathrm{nil}(\mathfrak{g})$. An extension

$$0 \to \mathfrak{n}_3 \to \mathfrak{g} \to \mathbb{R}^2 \to 0$$

defines a map $\mathbb{R}^2 \to \mathrm{sout}(\mathfrak{n}_3)$ by lifting to $\mathfrak{g}$ and taking brackets. Composition with the section from Step 2 produces a homomorphism $\phi : \mathbb{R}^2 \to \mathfrak{sl}_2\mathbb{R}$. Since $\mathfrak{sl}_2\mathbb{R}$ admits no 2-dimensional abelian subalgebras (any such would make $[\cdot,\cdot] : \Lambda^2\mathfrak{sl}_2\mathbb{R} \to \mathfrak{sl}_2\mathbb{R}$ fail to be surjective), ϕ has nonzero kernel. Then $\mathfrak{n}_3$ cannot be the nilradical of $\mathfrak{g}$—since if it were, Lemma 9.11 would require ϕ to be injective. □

9.2.3 The elimination of non-split extensions

The last main ingredient in Prop. 9.2 is the elimination of non-split extensions of $\mathbb{R}^2$ by $\mathbb{R}^3$ and extensions with actions other than the one specified, as follows.

Proposition 9.12. *Suppose $G = G/\{1\}$ is a maximal model geometry of dimension 5 and $\mathfrak{g}$ is its Lie algebra. If $\mathrm{nil}\,\mathfrak{g} \cong \mathbb{R}^3$, then $\mathfrak{g}$ is the semidirect sum $\mathbb{R}^3 \niplus \mathbb{R}^2$ where $\mathbb{R}^2$ acts by traceless diagonal matrices.*

The proof makes use of two main computations: the classification of faithful actions $\mathbb{R}^2 \hookrightarrow \mathfrak{sl}_3\mathbb{R}$ (Lemma 9.13) and the classification of extensions using Lie algebra cohomology (Lemma 9.15). These are carried out below, followed by the proof of Prop. 9.12.

Lemma 9.13. *Any embedding $\phi : \mathbb{R}^2 \to \mathfrak{sl}_3\mathbb{R}$ is conjugate under linear changes of coordi-*

nates in $\mathbb{R}^3$ *and* $\mathbb{R}^2$ *to one of the following maps. Blank entries are zero.*

$$\phi_1(x,y) = \begin{pmatrix} 0 & x & y \\ & 0 & x \\ & & 0 \end{pmatrix} \qquad \phi_2(x,y) = \begin{pmatrix} 0 & & y \\ & 0 & x \\ & & 0 \end{pmatrix} \qquad \phi_3(x,y) = \begin{pmatrix} 0 & x & y \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$\phi_4(x,y) = \begin{pmatrix} x & y & \\ & x & \\ & & -2x \end{pmatrix} \qquad \phi_5(x,y) = \begin{pmatrix} x & y & \\ -y & x & \\ & & -2x \end{pmatrix} \qquad \phi_6(x,y) = \begin{pmatrix} x & & \\ & y & \\ & & -x-y \end{pmatrix}$$

Proof. Name the above embeddings ϕ_1 through ϕ_6, and let $\{e_1, e_2\}$ be a basis for $\mathbb{R}^2$. Suppose $\phi : \mathbb{R}^2 \to \mathfrak{sl}_3\mathbb{R}$ is an embedding. The strategy is to find the Jordan form for $\phi(e_1)$ and determine what matrices commute with it in $\mathfrak{sl}_3\mathbb{R}$.

Case 1: $\phi(\mathbb{R}^2)$ contains no matrices with nonzero eigenvalues. Changing coordinates to put $\phi(e_1)$ in Jordan form (or something like it), and then computing centralizers, either

$$\left\{ \begin{array}{lcl} \phi(e_1) & = & \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \\ \phi(e_2) & = & \begin{pmatrix} a & b & c \\ & a & b \\ & & a \end{pmatrix} \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{lcl} \phi(e_1) & = & \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix} \\ \phi(e_2) & = & \begin{pmatrix} a & b & c \\ & e & d \\ & & a \end{pmatrix} \end{array} \right\}.$$

Since the image of ϕ consists of traceless matrices with no nonzero eigenvalues, either

$$\phi = \phi_1 \qquad \text{or} \qquad \phi(e_2) = \begin{pmatrix} 0 & b & c \\ & 0 & d \\ & & 0 \end{pmatrix}.$$

In the latter case, by replacing e_2 with $e_2 - ce_1$ we may assume $c = 0$. Then if both b and d are nonzero, we may rescale coordinates in $\mathbb{R}^3$ to obtain $\phi_1(e_1)$ (so ϕ is conjugate to ϕ_1). Otherwise, ϕ is conjugate to ϕ_2 or ϕ_3.

The embeddings ϕ_1, ϕ_2, and ϕ_3 are distinct, distinguished by the ranks and nullities of the matrices in their images.

Case 2: $\phi(\mathbb{R}^2)$ contains a matrix with a nonzero eigenvalue. Change coordinates in $\mathbb{R}^2$ to assume $\phi(e_1)$ has 1 as an eigenvalue. Then it must have one of the following Jordan forms, for some $y \in \mathbb{R}$.

$$\begin{pmatrix} 1 & y & \\ & 1 & \\ & & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & y & \\ -y & 1 & \\ & & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & & \\ & y & \\ & & -1-y \end{pmatrix}$$

Computing centralizers shows that ϕ is conjugate to ϕ_4, ϕ_5, or ϕ_6, respectively. (If $\phi(e_1)$ is diagonal with two identical diagonal entries, then some element of its centralizer isn't diagonal with two identical diagonal entries but still has one of the above forms.)

To show that none of these are conjugate to each other, observe that ϕ_4, ϕ_5, and ϕ_6 each send $(2, 1) \in \mathbb{R}^2$ to a matrix in Jordan form which is not the Jordan form of a matrix in the image of any other ϕ_i. □

The following definition is needed for Lemma 9.15's computation of cohomology; a survey can be found in [Wag10] or [AMR00, §2–4][3].

Definition 9.14 (**Lie algebra cohomology**, following [Wag10, §2])**.** Let M be a module of a Lie algebra $\mathfrak{g}$ over a field k. The *Chevalley-Eilenberg complex* is the cochain complex is the cochain complex

$$C^p(\mathfrak{g}, M) = \mathrm{Hom}_k(\Lambda^p \mathfrak{g}, M)$$

with boundary maps

$$\begin{aligned} d_p : C^p &\to C^{p+1} \\ (d_p c)(x_1, \ldots, x_{p+1}) &= \sum_{1 \le i < j \le p+1} (-1)^{i+j} c\left([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{p+1}\right) \\ &\quad + \sum_{1 \le i \le p+1} (-1)^{i+1} x_i c\left(x_1, \ldots, \hat{x_i}, \ldots, x_{p+1}\right) \end{aligned}$$

where $\hat{x_i}$ means x_i should be omitted. The cohomology of $\mathfrak{g}$ with coefficients in M is defined to be the cohomology of this complex and denoted $H^p(\mathfrak{g}, M)$.

Lemma 9.15. *If $\mathbb{R}^2$ acts on $\mathbb{R}^3$ via $\phi : \mathbb{R}^2 \to \mathfrak{sl}_3\mathbb{R}$, then $H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \mathbb{R}^3/\phi(\mathbb{R}^2)(\mathbb{R}^3)$.*

Proof. All 3-cochains are zero since $\Lambda^3\mathbb{R}^2 = 0$, so every 2-cochain is a cocycle. A 2-cochain is a map $\Lambda^2\mathbb{R}^2 \to \mathbb{R}^3$, which can be recovered from the image of a spanning element $e_1 \wedge e_2 \in \Lambda^2\mathbb{R}^2$; this identifies the 2-cocycles with $\mathbb{R}^3$.

It then suffices to identify the 2-coboundaries with $\phi(\mathbb{R}^2)(\mathbb{R}^3)$. First, for all 1-cochains c,

$$(dc)(x, y) = \phi(x)c(y) - \phi(y)c(x) \in \phi(\mathbb{R}^2)(\mathbb{R}^3).$$

Conversely, given $u \in \mathbb{R}^2$ and $w \in \mathbb{R}^3$, take $u' \in \mathbb{R}^2$ linearly independent from u and define

3. An almost identical version, generalized to super (i.e. $\mathbb{Z}/2\mathbb{Z}$-graded) Lie algebras, has been published as [AMR05].

a 1-cochain c by $c(u) = 0$ and $c(u') = w$. Then

$$(dc)(u, u') = \phi(u)c(u') = \phi(u)(w). \qquad \square$$

The proof of Prop. 9.12—that only one extension of $\mathbb{R}^2$ by $\mathbb{R}^3$ produces a maximal geometry with trivial isotropy and nilradical $\mathbb{R}^3$—is within reach, now that the above data can be used to describe the extensions in just enough detail to rule most of them out.

Proof of Prop. 9.12. Suppose $G = G/\{1\}$ is a maximal model geometry of dimension 5 and $\mathfrak{g}$ is its Lie algebra, with $\operatorname{nil}\mathfrak{g} \cong \mathbb{R}^3$. We have already established that $\mathfrak{g}$ is an extension

$$0 \to \mathbb{R}^3 \to \mathfrak{g} \to \mathbb{R}^2 \to 0$$

where $\mathbb{R}^2$ acts faithfully and tracelessly by lifting to $\mathfrak{g}$ and taking brackets (Prop. 9.8 and Lemma 9.11).

Step 1: Use $H^2(\mathbb{R}^2; \mathbb{R}^3)$ to classify extensions. For each of the actions $\phi : \mathbb{R}^2 \hookrightarrow \mathfrak{sl}_3\mathbb{R}$ (Lemma 9.13), we have by Lemma 9.15 that

$$H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \mathbb{R}^3/\phi(\mathbb{R}^2)(\mathbb{R}^3).$$

Explicitly, if $\phi_1, \ldots, \phi_6$ name the actions in Lemma 9.13 and $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$, then

$$H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \begin{cases} \mathbb{R}e_3 & \text{if } \phi = \phi_1 \text{ or } \phi_2 \\ \mathbb{R}e_2 + \mathbb{R}e_3 & \text{if } \phi = \phi_3 \\ 0 & \text{if } \phi = \phi_4, \phi_5, \text{ or } \phi_6. \end{cases}$$

Isomorphism classes of extensions of $\mathbb{R}^2$ by $\mathbb{R}^3$ are in bijection with classes $[c] \in H^2(\mathbb{R}^2; \mathbb{R}^3)$ [AMR00, Cor. 9]; so the extension with ϕ_6 splits.

Moreover, defining relations for $\mathfrak{g}$ can be recovered by using the cocycle as an $\mathbb{R}^3$-valued bracket on $\mathbb{R}^2$ [AMR00, Eqn. 5.5]. Explicitly, identify $\mathfrak{g}$ as vector space with $\mathbb{R}^3 \oplus \mathbb{R}^2$; and for $k_i \in \mathbb{R}^3$ and $q_i \in \mathbb{R}^2$, define

$$[(k_1, q_1), (k_2, q_2)] = (\phi(q_1)(k_2) - \phi(q_2)(k_1), 0).$$

Step 2: $\phi_1, \ldots, \phi_4$ produce the wrong nilradical. Let $v = (0, 1) \in \mathbb{R}^2$, and let $\mathfrak{n} = \mathbb{R}^3 + \mathbb{R}v$. Then $\mathfrak{n}$ is an ideal since $\mathfrak{g}^2 \subseteq \mathbb{R}^3 \subseteq \mathfrak{n}$.

Since v acts by a nilpotent matrix on $\mathbb{R}^3$, using the distributive law to compute $\mathfrak{n}^4$ yields

$$\mathfrak{n}^4 = [v, [v, [v, \mathbb{R}^3]]] = 0.$$

Thus $\mathfrak{n}$ is also nilpotent; so $\phi_1, \ldots, \phi_4$ do not produce $\mathfrak{g}$ where $\operatorname{nil} \mathfrak{g} \cong \mathbb{R}^3$.

Step 3: ϕ_5 produces a non-maximal geometry. If $\phi = \phi_5$, then $G \cong (\mathbb{C} \times \mathbb{R}) \rtimes \mathbb{R}^2$, where $(x, y) \in \mathbb{R}^2$ acts as scaling by e^{x+iy} on $\mathbb{C}$ and by e^{-2x} on $\mathbb{R}$. The action of $S^1 \subset \mathbb{C}$ on $\mathbb{C}$ commutes with this, so $S^1 \subseteq \operatorname{Aut}(G)$. Then $G/1 \cong G \rtimes S^1/S^1$. $\square$

The case $\phi = \phi_6$ has $\mathbb{R}^2$ acting on $\mathbb{R}^3$ by traceless diagonal matrices; the following section will show that it produces a model geometry.

9.3 The $\mathbb{R}^3 \rtimes \mathbb{R}^2$ geometry

When point stabilizers are 0-dimensional, a geometry admitting a finite-volume quotient by isometries is a Lie group admitting a lattice (Prop. 6.5). So to show this geometry is a model geometry, it suffices to prove the following.

Proposition 9.16. *The Lie group $\mathbb{R}^3 \rtimes \mathbb{R}^2$, where $\mathbb{R}^2$ acts on $\mathbb{R}^3$ as the diagonal matrices with determinant 1 and positive eigenvalues, admits a lattice.*

Proof. By applying linear changes of coordinates in $\mathbb{R}^3$ and $\mathbb{R}^2$, it suffices to construct a group isomorphic to $\mathbb{Z}^3 \rtimes \mathbb{Z}^2$ where the action is by diagonalizable matrices with positive eigenvalues. One can use the ring of integers of a number field, acted on by its group of units.

If K is a cubic number field, then its ring of integers $\mathcal{O}_K$ is isomorphic as a group to $\mathbb{Z}^3$; and if K is totally real, Dirichlet's unit theorem (see e.g. [Neu99, Thm. 7.4]) implies that the group of units $\mathcal{O}_K^\times$ has rank $3-1=2$. So one may take $\mathcal{O}_K \rtimes 2U$ where $U \subseteq \mathcal{O}_K^\times$ is free abelian of rank 2, and $2U$ consists of the squares in U so that the action will have positive eigenvalues.

To obtain such a field, take $K = \mathbb{Q}[x]/(p(x))$ where $p \in \mathbb{Z}[x]$ is a monic irreducible cubic with three distinct real roots. □

Example 9.17. Let $p(x) = 1 - 3x + x^3$, so $K \cong \mathbb{Q}[\alpha]$ where α is any root of p. Then

$$\alpha(3-\alpha^2) = 3\alpha - \alpha^3 = 1$$
$$(1-\alpha)(2-\alpha-\alpha^2) = 2 - 3\alpha + \alpha^3 = 1,$$

so α and $1-\alpha$ are units in $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_K$.

To prove that α and $1-\alpha$ are independent in $\mathcal{O}_K^\times$, let $v_1, v_2, v_3 \in K \otimes_{\mathbb{Q}} \mathbb{R}$ be a basis of eigenvectors for multiplication by α on K as a $\mathbb{Q}$-vector space. Let $U \subseteq \mathcal{O}_K^\times$ be the subgroup generated by α and $1-\alpha$, and define a homomorphism

$$\begin{aligned} \phi : U &\to \mathbb{R}^3 \\ a &\mapsto \big(\log|\lambda_1|, \log|\lambda_2|, \log|\lambda_3|\big) \text{ where } av_i = \lambda_i v_i. \end{aligned}$$

Then $\phi(\alpha) \approx (0.6, -1, 0.4)$ and $\phi(1-\alpha) \approx (1, -0.4, -0.6)$ are linearly independent since they lie in non-opposite octants; so $U \cong \phi(U) \cong \mathbb{Z}^2$.

The matrices by which $\mathbb{Z}^2$ acts on $\mathbb{Z}^3$ are products of even powers of α and $1-\alpha$, expressed in the basis $(1, \alpha, \alpha^2)$. That is, the action $\mathbb{Z}^2 \to \operatorname{Aut} \mathbb{Z}^3 = \mathrm{SL}(3, \mathbb{Z})$ is given by

$$x, y \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}^{2x} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -3 \\ 0 & -1 & 1 \end{pmatrix}^{2y}.$$

9.4 Classification of semidirect sums with $\mathbb{R}$

Prop. 9.2 has reduced the discovery of candidates to a classification of semidirect products—that is, classifying semidirect sums $\mathfrak{n} \ni \mathbb{R}$ where $\mathfrak{n}$ is nilpotent of dimension 4. The program followed here is to classify such $\mathfrak{n}$, determine possible actions of $\mathbb{R}$ on each $\mathfrak{n}$, and determine which semidirect sums produce model geometries (i.e. are tangent to Lie groups admitting lattices).

Proposition 9.18 (see also [PSWZ76, Table I])**.** *Any nilpotent Lie algebra $\mathfrak{n}$ of dimension 4 is isomorphic to $\mathbb{R}^4$ or $\mathbb{R} \oplus \mathfrak{n}_3$ or $\mathfrak{n}_4$.*

Proof. The dimension of $\mathfrak{n}^2 = [\mathfrak{n}, \mathfrak{n}]$ distinguishes the three Lie algebras above, so use this dimension to determine $\mathfrak{n}$.

- $\mathfrak{n}^2 \neq \mathfrak{n}$ since $\mathfrak{n}$ is nilpotent.
- If $\dim \mathfrak{n}^2 = 3$, then pick $x \in \mathfrak{n} \smallsetminus \mathfrak{n}^2$. Then

$$\mathfrak{n}^2 = (\mathbb{R}x + \mathfrak{n}^2)^2 = [\mathbb{R}x, \mathfrak{n}^2] + [\mathfrak{n}^2, \mathfrak{n}^2] \subseteq \mathfrak{n}^3,$$

so the lower central series stabilizes at $\mathfrak{n}^2 \neq 0$, so this never occurs for nilpotent $\mathfrak{n}$.

- If $\dim \mathfrak{n}^2 = 2$, then $\dim \mathfrak{n}^3$ is either 0 or 1.
  - If $\dim \mathfrak{n}^3 = 0$, then $\mathfrak{n}^2$ is central. Pick nonzero:

$$\begin{aligned} x_3 &\notin \mathfrak{n}^2 \\ x_2 &\notin \mathfrak{n}^2 + \mathbb{R}x_3 \\ x_1 &\in \mathfrak{n}^2 \text{ such that } [x_3, x_2] = 0 \text{ or } x_1 \\ y &\in \mathfrak{n}^2 \text{ completing these to a basis.} \end{aligned}$$

    Then $\mathfrak{n}$ is either $\mathbb{R}^4$ or $\mathbb{R} \oplus \mathfrak{n}_3$. (In fact, for both of these, $\mathfrak{n}^2$ is too small.)
  - If $\dim \mathfrak{n}^3 = 1$, then choose nonzero:

$$\begin{aligned} x_1 &\in \mathfrak{n}^3 \\ x_2 &\in \mathfrak{n}^2 \setminus \mathfrak{n}^3 \\ x_4 &\in \mathfrak{n} \setminus \mathfrak{n}^2 \text{ such that } [x_4, x_2] = x_1 \\ x_3 &\in \mathfrak{n} \setminus \mathfrak{n}^2 \text{ such that } [x_4, x_3] = x_2. \end{aligned}$$

    Since $[x_3, x_2] \in \mathfrak{n}^3$, it could be any multiple of x_1. Replacing x_3 by an element of $x_3 + \mathbb{R}x_4$ to make its bracket with x_2 zero yields a basis demonstrating $\mathfrak{n} \cong \mathfrak{n}_4$.
- If $\dim \mathfrak{n}^2 = 1$, then $\mathfrak{n}^2$ is central, as the last term in the lower central series. For any linear complement V of $\mathfrak{n}^2$, the Lie bracket induces $V \wedge V \to \mathfrak{n}^2 \cong \mathbb{R}$, which is necessarily degenerate since V has odd dimension. Then we can choose x_1 spanning $\mathfrak{n}^2$, x_2 and x_3 such that $[x_3, x_2] = x_1$, and y demonstrating the degeneracy of $V \wedge V \to \mathfrak{n}^2$. This basis demonstrates $\mathfrak{n} \cong \mathbb{R} \oplus \mathfrak{n}_3$.
- If $\dim \mathfrak{n}^2 = 0$, then $\mathfrak{n} \cong \mathbb{R}^4$.

□

9.4.1 $\mathbb{R}^4$ semidirect sums

Suppose G is a model geometry and $\mathfrak{g} = \mathfrak{n} \ni_\phi \mathbb{R}$. If $\mathfrak{n} = \mathbb{R}^4$, a linear change of coordinates puts the image of 1 under $\phi : \mathbb{R} \to \mathfrak{sl}_4\mathbb{R}$ in Jordan form. Listing Jordan forms, grouped by number of blocks, yields the following. (Omitted entries are zero, and "$*$" entries are subject only to the restriction that the whole matrix is traceless.)

$$\phi_1(1) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \qquad \phi_{1'}(1) = \begin{pmatrix} & \lambda & 1 & \\ -\lambda & & & 1 \\ & & & \lambda \\ & & -\lambda & \end{pmatrix}$$

$$\phi_2(1) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & -\lambda & 1 \\ & & & -\lambda \end{pmatrix} \qquad \phi_{2'}(1) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \\ & & & -3\lambda \end{pmatrix}$$

$$\phi_{2''}(1) = \begin{pmatrix} \lambda & \mu & & \\ -\mu & \lambda & & \\ & & -\lambda & 1 \\ & & & -\lambda \end{pmatrix} \qquad \phi_{2'''}(1) = \begin{pmatrix} \lambda & \mu_1 & & \\ -\mu_1 & \lambda & & \\ & & -\lambda & \mu_2 \\ & & -\mu_2 & -\lambda \end{pmatrix}$$

$$\phi_3(1) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & * & \\ & & & * \end{pmatrix} \qquad \phi_{3'}(1) = \begin{pmatrix} \lambda & \mu & & \\ -\mu & \lambda & & \\ & & * & \\ & & & * \end{pmatrix}$$

$$\phi_4(1) = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}$$

With this, it becomes possible to classify maximal model geometries of the form $\mathbb{R}^4 \rtimes \mathbb{R}$; computing characteristic polynomials recovers the list in Thm. 5.1(ii)(b) from the following statement.

Proposition 9.19 (**Classification of $\mathbb{R}^4 \rtimes \mathbb{R}$ geometries**)**.**

(i) If $G = G/\{1\}$ is a maximal model geometry with Lie algebra $\mathfrak{g} = \mathbb{R}^4 \ni_{\phi} \mathbb{R}$, then ϕ can be taken to be one of ϕ_1, ϕ_2 ($\lambda = 1$), $\phi_{2'}$ ($\lambda = 0$), ϕ_3 ($\lambda = 0$), or ϕ_4.

(ii) All cases listed in (i) are model geometries, except that the group with Lie algebra $\mathbb{R}^4 \ni_{\phi_4} \mathbb{R}$ is a model geometry if $\exp \phi_4(t)$ has integer characteristic polynomial for some $t \neq 0$.

Note that some of the actions ϕ_i depend on parameters, and not all of the parameter values produce model geometries. Fortunately, there is an easy-to-state necessary and sufficient condition for model geometries arising in this case: recall that G/G_p is a model geometry if and only if some lattice $\Gamma \subset G$ intersects no conjugate of G_p nontrivially (Prop. 6.5(ii)); so in the case of $G_p = \{1\}$, this reduces to the question of whether a lattice exists, which is determined by the following condition.

Lemma 9.20 ([Fil83, Cor. 6.4.3])**.** *A unimodular, non-nilpotent $\mathbb{R}^n \rtimes_{\exp \phi} \mathbb{R}$ admits a lattice if and only if there is $0 \neq t \in \mathbb{R}$ such that the characteristic polynomial of $\exp \phi(t)$ has coefficients in $\mathbb{Z}$.*

Proof of Prop. 9.19. The complete list of actions of $\mathbb{R}$ on $\mathbb{R}^4$ would produce a long list of cases, so a first step will be to eliminate actions producing non-maximal geometries. Each remaining group is then examined to determine whether it admits a lattice. The cases are grouped by the number of Jordan blocks.

Preparatory step: ignore non-maximal geometries. If $\phi(1)$ has non-real eigenvalues, then it (and the 1-parameter subgroup of $\mathrm{SL}(4, \mathbb{R})$ it generates) commutes with rotations on some 2-dimensional eigenspace. These rotations form an $S^1 \subseteq \mathrm{Aut}(G)$; so $G/\{1\}$ is not maximal, since it is subsumed by $G \rtimes S^1/S^1$. This eliminates $\phi_{1'}$, $\phi_{2''}$, $\phi_{2'''}$, and $\phi_{3'}$.

Case 1: $\phi(1)$ has 1 Jordan block. Then $\phi = \phi_1$. Since

$$\exp \phi_1(6) = \begin{pmatrix} 1 & 6 & 18 & 36 \\ & 1 & 6 & 18 \\ & & 1 & 6 \\ & & & 1 \end{pmatrix}$$

has integer entries, its characteristic polynomial has integer coefficients; so the resulting $\mathbb{R}^4 \rtimes \mathbb{R}$ admits a lattice by Lemma 9.20 above. One such lattice is a subgroup isomorphic to $\mathbb{Z}^4 \rtimes_{\exp \phi_1(6)} \mathbb{Z}$, with fundamental domain $\{((\exp \phi_1(t))x, t) \mid t \in (0,6), x \in [0,1]^4\}$.

Case 2a: $\phi = \phi_2$ (2 Jordan blocks). If $\lambda = 0$ then $\phi_2(t)$ commutes with an S^1 of rotations (it coincides with the $\lambda = 0$ case for $\phi_{1'}$) and will not produce a maximal geometry.

If instead $\lambda \neq 0$, then by rescaling ϕ_2 and changing basis to put the new $\phi_2(1)$ in Jordan form, we can assume $\lambda = 1$, so this produces at most one new geometry. Exponentiating yields the 1-parameter subgroup

$$\left\{ \begin{pmatrix} e^t & te^t & & \\ & e^t & & \\ & & e^{-t} & te^{-t} \\ & & & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathrm{SL}(\mathbb{R}^4).$$

Reordering the basis elements turns this into the block matrices

$$\left\{ \begin{pmatrix} A(t) & tA(t) \\ & A(t) \end{pmatrix} : t \in \mathbb{R}, A(t) = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \right\} \subset \mathrm{SL}(\mathbb{R}^4).$$

If B diagonalizes $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then after conjugating by the block diagonal matrix with diagonal blocks B^{-1}, this 1-parameter subgroup contains an element of the form $\begin{pmatrix} C & sC \\ & C \end{pmatrix}$, with $s \in \mathbb{R}$. After rescaling the first two basis elements, we conclude as in Case 1 above that the Lie group $\mathbb{R}^4 \rtimes_{\exp\phi_2} \mathbb{R}$ admits a lattice isomorphic to $\mathbb{Z}^4 \rtimes_{A'} \mathbb{Z}$, where

$$A' = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ & & 2 & 1 \\ & & 1 & 1 \end{pmatrix}.$$

Case 2b: $\phi = \phi_{2'}$ **(2 Jordan blocks).** We will prove in this case that the resulting $G = \mathbb{R}^4 \rtimes \mathbb{R}$ is a model geometry if and only if $\lambda = 0$.

By Lemma 9.20 above, if G is a model geometry then $\exp\phi_{2'}(t)$ has characteristic polynomial $p(x) \in \mathbb{Z}[x]$ for some $t \neq 0$. Then $e^{t\lambda}$ is a triple root of p, so its minimal polynomial over $\mathbb{Q}$ divides p at least 3 times. Since $\deg p = 4$, the minimal polynomial of $e^{t\lambda}$ is linear; so $e^{t\lambda} \in \mathbb{Q}$. Since $e^{t\lambda}$ is a rational root of a polynomial whose first and last coefficients are 1, the rational root theorem implies $e^{t\lambda} = \pm 1$. Then since $t \neq 0$ and λ is real, $\lambda = 0$.

Conversely, $\lambda = 0$ then $\exp\phi_{2'}(2)$ has integer entries; so G admits the lattice $\mathbb{Z}^4 \rtimes_{\exp\phi_{2'}(2)} \mathbb{Z}$. In the notation of [Fil83, §6.4.6], this gives the geometry $G_3 \times \mathbb{R}$.

Case 3: $\phi = \phi_3$ **(3 Jordan blocks).** Again we prove that the resulting $\mathbb{R}^4 \rtimes \mathbb{R}$ is a model geometry if and only if $\lambda = 0$; moreover, this produces only one geometry.

If a model geometry results, then $\exp\phi_3(t)$ has characteristic polynomial $p(x) \in \mathbb{Z}[x]$ for some $t \neq 0$. Then $e^{t\lambda}$ is a double root of $p(x)$, so its minimal polynomial over $\mathbb{Q}$ divides $p(x)$ at least twice. If $e^{t\lambda} \notin \mathbb{Q}$, then the two eigenvalues of $\phi_3(1)$ that aren't λ are identical; so the last two coordinates admit an S^1 of rotations, making G non-maximal. Then $e^{t\lambda} \in \mathbb{Q}$; so as in Case 2b, $\lambda = 0$ by the rational root theorem.

If $\lambda = 0$, then ϕ_3 and the first basis vector of $\mathbb{R}^4$ can be rescaled so that

$$\phi_3(1) = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \alpha & \\ & & & -\alpha \end{pmatrix}$$

where $\alpha = \ln \frac{3+\sqrt{5}}{2}$. (If the diagonal entries were all zero, then the last two coordinates would admit an S^1 of rotations.) Exponentiating this yields a 1-parameter subgroup of $\mathrm{SL}(\mathbb{R}^4)$ containing a matrix A that in some basis becomes

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 2 & 1 \\ & & 1 & 1 \end{pmatrix},$$

so G admits a lattice isomorphic to $\mathbb{Z}^4 \rtimes_A \mathbb{Z}$.

Case 4: $\phi = \phi_4$ (4 Jordan blocks). The criterion claimed in (ii) for producing a model geometry is merely Lemma 9.20 above. This completes the proof of Prop. 9.19. □

Remark 9.21 (**Alternative parametrization of the $\phi = \phi_4$ case**). The 3-dimensional family of maps ϕ_4 produces a 2-dimensional family of groups $\mathbb{R}^4 \rtimes_{\exp \phi_4} \mathbb{R}$ (due to the ability to rescale the $\mathbb{R}$ factor), but not all are model geometries—by Lemma 9.20 above, $\exp \phi_4(t)$ has to have an integer characteristic polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + 1 \in \mathbb{Z}[x]$ for some nonzero t. Allowing for rescaling the $\mathbb{R}$ factor, one could instead parametrize this family of geometries by the coefficients $(a, b, c) \in \mathbb{Z}^3$; but the correspondence is not a bijection due to the following.

(i) Since $\phi_4(t)$ has 4 real eigenvalues, $p(x)$ must have 4 nonnegative real roots.

(ii) Since duplicate eigenvalues of $\phi_4(t)$ would make $\exp\phi_4(t)$ commute with an S^1 of rotations, the roots of $p(x)$ must be distinct if it comes from a maximal geometry.

(iii) If $\exp\phi_4(t)$ has integer characteristic polynomial (i.e. is an integer matrix in some basis), then so does $\exp\phi_4(nt)$ for any $n \in \mathbb{Z}$; so each geometry corresponds to a $\mathbb{Z}$-family of polynomials.

In [Fil83, §6.4.8], Filipkiewicz gave a detailed description of this kind of parametrization for the analogous family $\mathrm{Sol}^4_{m,n}$ in dimension 4.

9.4.2 $\mathfrak{n}_4$ *semidirect sums*

The classification of $\mathfrak{n}_4 \ni \mathbb{R}$ geometries has a slightly different flavor from the $\mathbb{R}^4 \ni \mathbb{R}$ case—there are fewer derivations of $\mathfrak{n}_4$ than of $\mathbb{R}^4$; but the tradeoff is that they require a bit more work to find. The result is the following.

Proposition 9.22 (Classification of $\mathfrak{n}_4 \ni \mathbb{R}$ geometries). *Suppose $G = G/\{1\}$ is a model geometry whose Lie algebra $\mathfrak{g}$ is of the form $\mathfrak{n}_4 \ni \mathbb{R}$ and not expressible as $\mathbb{R}^4 \ni \mathbb{R}$. Then $\mathfrak{g}$ has basis $\{x_1, x_2, x_3, x_4, x_5\}$ with*

$$[x_4, x_2] = x_1 \qquad [x_4, x_3] = x_2 \qquad [x_5, x_3] = x_1 \qquad [x_5, x_4] = 0 \text{ or } x_3,$$

and all other brackets not determined by skew-symmetry are zero. Both Lie algebras thus described are the Lie algebras of model geometries.

Most of the proof lies in describing a derivation D that makes $\mathfrak{n}_4 \ni_D \mathbb{R}$ the Lie algebra of a model geometry. Predictably, this case begins with the computation of the traceless outer derivation algebra $\mathrm{sout}(\mathfrak{n}_4)$ (Lemma 9.23). This will be followed by an extra condition on D for model geometries (Lemma 9.24) and the proof of Prop. 9.22.

Lemma 9.23. *Every element of* $\operatorname{sout}(\mathfrak{n}_4)$ *is represented by a matrix of the form*

$$D_{4,a,b,c} = \begin{pmatrix} 2a & 0 & b & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -4a & c \\ 0 & 0 & 0 & 3a \end{pmatrix},$$

with respect to the basis in Definition 9.4.

Proof. Suppose $D \in \operatorname{der} \mathfrak{n}_4$ is traceless. $\operatorname{ad} \mathfrak{n}_4$ consists of the matrices

$$\begin{pmatrix} 0 & c & 0 & a \\ 0 & 0 & c & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Any derivation D is upper triangular since it preserves the following filtration by characteristic ideals.

$$\begin{aligned} \operatorname{span}(x_1) &= Z(\mathfrak{n}) \\ \operatorname{span}(x_1, x_2) &= \mathfrak{n}_4^2 \\ \operatorname{span}(x_1, x_2, x_3) &= \{x \in \mathfrak{n}_4 \mid \dim[x, \mathfrak{n}_4] < 2\} \end{aligned}$$

Hence, up to an inner derivation, we may write D as

$$\begin{aligned} Dx_1 &= a_1 x_1 \\ Dx_2 &= b_2 x_2 \\ Dx_3 &= c_1 x_1 + c_2 x_2 + c_3 x_3 \\ Dx_4 &= d_3 x_3 + d_4 x_4. \end{aligned}$$

Using the Leibniz rule,

$$\begin{aligned} a_1 x_1 = Dx_1 &= [Dx_4, x_2] + [x_4, Dx_2] = (d_4 + b_2) x_1 \\ b_2 x_2 = Dx_2 &= [Dx_4, x_3] + [x_4, Dx_3] = (d_4 + c_3) x_2 + c_2 x_1, \end{aligned}$$

so $c_2 = 0$ and $c_3 = b_2 - d_4 = a_1 - 2d_4$. Finally, if D is traceless,

$$0 = a_1 + b_2 + c_3 + d_4 = 3c_3 + 4d_4. \qquad \square$$

Lemma 9.24. *Suppose $G = G/\{1\}$ is a model geometry whose Lie algebra $\mathfrak{g}$ is of the form $\mathfrak{n} \rtimes \mathbb{R}$ where $\mathfrak{n}$ is nilpotent. Then $\mathbb{R}$ acts tracelessly on $Z(\mathfrak{n})$.*

Proof. Since $Z(\mathfrak{n})$ is a characteristic ideal, it is stable under the action of $\mathbb{R}$. Suppose $\mathbb{R}$ acts with nonzero trace on $Z(\mathfrak{n})$.

Then $\mathbb{R}$ acts non-nilpotently, so $\mathfrak{n} = \operatorname{nil} \mathfrak{g}$. Write $G = N \rtimes \mathbb{R}$ where N is the simply-connected Lie group with Lie algebra $\mathfrak{n}$.

Since G is a model geometry, it admits a lattice Γ. The nilradical of a solvable Lie group inherits lattices [Mos71, Lemma 3.9]—that is, $N \cap \Gamma$ is a lattice in N, and $\Gamma/(N \cap \Gamma)$ is a lattice in $\mathbb{R}$. The latter implies that some element of Γ acts by conjugation with determinant > 1 on $Z(N)$.

Since $Z(N)$ is a term in the upper central series of N, the intersection $Z(N) \cap \Gamma$ is a lattice in $Z(N)$ [Rag72, Prop. 2.17]. This is impossible since no lattice of $\mathbb{R}^k$ is stable under a linear map with determinant > 1. □

Given the above, the classification of model geometries with tangent algebra of the form $\mathfrak{n}_4 \rtimes \mathbb{R}$ is within reach and will recover the list in Thm. 5.1(ii)(c).

Proof of Prop. 9.22. The action of $\mathbb{R}$ on $\mathfrak{n}_4$ is represented by a derivation of the form described above in Lemma 9.23. The following steps recover the Lie algebra defined in the statement of Prop. 9.22.

1. Since G needs to be a model geometry, $a = 0$ by the tracelessness condition from Lemma 9.24.

2. If $b = 0$, then x_1, x_2, x_3, and x_5 span an abelian ideal, so G is also expressible as $\mathbb{R}^4 \rtimes \mathbb{R}$. Hence $b \neq 0$. Rescale x_5 by a factor of b^{-1}—i.e. replace x_5 with $b^{-1}x_5$—to make $b = 1$.

3. If $c \neq 0$, then similarly rescale $\operatorname{span}(x_1, x_2, x_3)$ by a factor of c to make $c = 1$.

In the end only one parameter can vary—c may be 0 or 1. Either way, $\mathfrak{g}$ is nilpotent and expressed in a basis with integral structure constants; so the corresponding simply-connected group G admits a lattice [Rag72, Thm. 2.12]. Therefore G is a model geometry. □

9.4.3 $\mathbb{R} \oplus \mathfrak{n}_3$ *semidirect sums*

Despite only producing two geometries, this case is the messiest of the three—$\mathbb{R} \oplus \mathfrak{n}_3$ has enough outer derivations to make this a long problem, yet not so many that a systematic approach is as easy as listing Jordan blocks in $\mathfrak{sl}_4\mathbb{R}$. The main result is the following.

Proposition 9.25 (Classification of $(\mathbb{R} \times \mathrm{Heis}_3) \rtimes \mathbb{R}$ geometries). *Suppose $G = G/\{1\}$ is a maximal model geometry with Lie algebra $\mathfrak{g}$ of the form $(\mathbb{R} \oplus \mathfrak{n}_3) \rtimes_D \mathbb{R}$. If $\mathfrak{g}$ contains* *no ideal isomorphic to* $\mathbb{R}^4$ *or* $\mathfrak{n}_4$*, then* $\mathfrak{g}$ *is isomorphic to a Lie algebra constructed by letting* $1 \in \mathbb{R}$ *act on* $\mathbb{R} \oplus \mathfrak{n}_3 = \operatorname{span}(y, x_1, x_2, x_3)$ *by*

$$x_2 \mapsto x_2 \qquad\qquad x_3 \mapsto -x_3 \qquad\qquad y \mapsto 0 \text{ or } x_1.$$

For both Lie algebras thus named, the corresponding simply-connected group G *is a model geometry.*

The strategy will broadly be the same as before: list elements of $\operatorname{sout}(\mathbb{R} \oplus \mathfrak{n}_3)$ by which $1 \in \mathbb{R}$ can act; use coordinate changes to cluster the resulting Lie algebras by isomorphism type; and show that each admits a lattice. To save some work, we begin with a lemma that eliminates some of $\operatorname{sout}(\mathbb{R} \oplus \mathfrak{n}_3)$ from consideration.

Lemma 9.26. *Let* $\mathfrak{n} = \mathbb{R} \oplus \mathfrak{n}_3$*. If* $G = G/\{1\}$ *is a model geometry whose Lie algebra* $\mathfrak{g}$ *is of the form* $\mathfrak{n} \ni \mathbb{R}$*, then* $\mathbb{R}$ *acts tracelessly on* $\mathfrak{n}^2 = Z(\mathfrak{n}_3)$*.*

Proof. Suppose $\mathbb{R}$ acts with nonzero trace on the characteristic ideal $Z(\mathfrak{n}_3)$. Let Γ be a lattice in $G = (\mathbb{R} \times \mathrm{Heis}_3) \rtimes \mathbb{R}$. Following the argument in Lemma 9.24, some element of Γ acts with determinant < 1 on $Z(\mathrm{Heis}_3)$, and $\Gamma \cap (\mathbb{R} \times \mathrm{Heis}_3)$ maps to a lattice in $(\mathbb{R} \times \mathrm{Heis}_3)/Z(\mathbb{R} \times \mathrm{Heis}_3) \cong \mathbb{R}^2$.

Then over some two linearly independent elements of $\mathbb{R}^2$ lie two elements of Γ; their commutator is a nontrivial element of $Z(\mathrm{Heis}_3) \cong \mathbb{R}$. This is impossible since no discrete subgroup of $\mathbb{R}$ is stable under an automorphism with determinant < 1. □

In combination with the requirement that $\mathbb{R}$ acts tracelessly on $Z(\mathfrak{n})$ (Lemma 9.24), this brings the dimension of the relevant subspace of $\operatorname{sout}(\mathbb{R} \oplus \mathfrak{n}_3)$ down to 6.

Lemma 9.27. *If* $\mathfrak{n} = \mathbb{R} \oplus \mathfrak{n}_3$ *has ordered basis* (y, x_1, x_2, x_3) *(with* $[x_3, x_2] = x_1$ *as in Definition 9.4), then every element of* $\operatorname{sout}(\mathfrak{n})$ *which also acts tracelessly on* $\mathfrak{n}^2 = \operatorname{span}(x_1)$

and $Z(\mathfrak{n}) = \operatorname{span}(x_1, y)$ *is represented by a matrix of the form*

$$\begin{pmatrix} 0 & 0 & e & f \\ d & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{pmatrix}.$$

Proof. The tracelessness assumptions, along with the fact that derivations preserve characteristic ideals, imply that $Dx_1 = 0$, that $Dy \in \operatorname{span}(x_1)$, and that the last two diagonal entries add to 0.

By subtracting multiples of the inner derivations $\operatorname{ad} x_2$ and $\operatorname{ad} x_3$, we can assume that Dx_2 and Dx_3 have zero x_1 coordinate. □

The proof of Prop. 9.25 which now follows is similar in spirit to that for $\mathfrak{n}_4$ (Prop. 9.22)—it shows that most of this space of derivations either produces no maximal model geometries or produces geometries accounted for by previous cases.

Proof of Prop. 9.25. Give $\mathfrak{g}$ the ordered basis (y, x_1, x_2, x_3, z), where the first four elements are as above in Lemma 9.27 and z spans the last $\mathbb{R}$ factor. Let $D \in \operatorname{der}(\mathbb{R} \oplus \mathfrak{n}_3)$ be the derivation by which z acts; and let D' be the lower right 2×2 block of D.

Preparatory step: Simplify D with coordinate changes. By changing basis in $\operatorname{span}(x_2, x_3)$, we can assume D' is in Jordan form. By rescaling z, we may assume this block contains only 1, 0, and -1 as entries.

For $r \neq 0$, let $\mu_r : \mathbb{R} \oplus \mathfrak{n}_3 \to \mathbb{R} \oplus \mathfrak{n}_3$ be the automorphism given by

$$x_1 \mapsto r^2 x_1 \qquad x_2 \mapsto r x_2 \qquad x_3 \mapsto r x_3 \qquad y \mapsto r y.$$

In the notation of Lemma 9.27, conjugating D by μ_r replaces d by rd while leaving the other entries unchanged; so we may also assume that d will always be 0 or 1.

Case 1: If $D' = 0$ then no new maximal geometries arise. In this case, D' commutes with the automorphisms of $\mathbb{R} \oplus \mathfrak{n}_3$ acting as rotations on span (x_2, x_3). In the notation of Lemma 9.27, if $e = f = 0$, then these automorphisms also commute with D, so they extend to automorphisms of $\mathfrak{g}$. This would make G nonmaximal, since $G \cong G \rtimes S^1/S^1$.

Thus e and f are not both zero; so we can conjugate by a rotation to make $f = 0$ and a rescaling of y to make $e = 1$. Under this modified basis, D has the matrix

$$\begin{pmatrix} 0 & & 1 & \\ d & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

where d is 0 or 1. In either case, $\mathfrak{g}$ has an ideal isomorphic to $\mathbb{R}^4$ or $\mathfrak{n}_4$:

- If $d = 0$, then span $(y, x_1, x_3, z) \subset \mathfrak{g}$ is a 4-dimensional abelian ideal.
- If $d = 1$, then span $(x_1, y, x_2, z) \subset \mathfrak{g}$ is an ideal isomorphic to $\mathfrak{n}_4$, with the basis ordered as in Definition 9.4—that is, the nonzero brackets are

$$[z, y] = x_1 \qquad\qquad [z, x_2] = y.$$

Case 2: If $D' \neq 0$ is skew-symmetric then G is non-maximal. As in Case 1, D' commuting with a compact group of automorphisms leads to the conclusion that D has matrix

$$\begin{pmatrix} 0 & & 1 & \\ d & 0 & & \\ & & & 1 \\ & & -1 & \end{pmatrix}.$$

Define

$$u_1 = x_1 \qquad u_2 = x_2 \qquad u_3 = x_3 - y \qquad v = y \qquad w = z - dx_2.$$

Then $\operatorname{span}(v, u_1, u_2, u_3) \cong \mathbb{R} \oplus \mathfrak{n}_3$, and w acts in this ordered basis by the matrix

$$\begin{pmatrix} 0 & & & \\ d & 0 & & \\ & & & 1 \\ & & -1 & \end{pmatrix},$$

which commutes with a compact group of automorphisms, implying as in Case 1 that G is non-maximal.

Case 3: If D' is a single Jordan block then no new maximal geometries arise. In the notation of Lemma 9.27, either $e = 0$, or we can replace x_3 by $x_3 - \frac{f}{e}x_2$ to make $f = 0$. If one of e and f remains nonzero, then y can be rescaled to make it equal 1. This produces three matrices for D:

$$D_{1,d,0} = \begin{pmatrix} 0 & & & \\ d & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \qquad D_{1,d,1} = \begin{pmatrix} 0 & & 1 & \\ d & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \qquad D_{1,d,2} = \begin{pmatrix} 0 & & & 1 \\ d & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

In all but one case, an ideal isomorphic to $\mathfrak{n}_4$ can be named by an ordered basis (v_0, v_1, v_2, v_3) where the nonzero brackets are $[v_3, v_2] = v_1$ and $[v_3, v_1] = v_0$.

$$D_{1,d,0} : x_1, x_2, -z, x_3$$

$$D_{1,1,1} : x_1, y, x_2, z$$

$$D_{1,d,2} : x_1, x_2 + y, dx_3 - z, x_3$$

If $D = D_{1,0,1}$, then $\mathfrak{g}$ has basis $\{y, x_1, x_2, x_3, z\}$ with brackets

$$[z, x_2] = y \qquad [x_3, x_2] = x_1 \qquad [z, x_3] = x_2.$$

In this case $\mathfrak{g}$ admits a compact group of automorphisms ϕ_θ defined by

$$z \mapsto z\cos\theta + x_3 \sin\theta \qquad y \mapsto y\cos\theta + x_1\sin\theta \qquad x_2 \mapsto x_2$$

$$x_3 \mapsto -z\sin\theta + x_3\cos\theta \qquad x_1 \mapsto -y\sin\theta + x_1\cos\theta,$$

so $G/\{1\}$ is non-maximal when $D = D_{1,0,1}$.

Case 4: If $D' \neq 0$ is diagonal then $\mathfrak{g}$ is determined up to isomorphism. In this case, D is the matrix

$$\begin{pmatrix} 0 & & e & f \\ d & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Let

$$u_1 = x_1 \qquad u_2 = x_2 + ey \qquad u_3 = x_3 - fy \qquad v = y \qquad w = z - dex_3 - dfx_2.$$

Then u_1 is central, v commutes with all u_i, and

$$[u_3, u_2] = u_1$$
$$[w, u_2] = u_2$$
$$[w, u_3] = -u_3$$
$$[w, v] = du_1.$$

This shows that e and f do not affect the isomorphism type of $\mathfrak{g}$; taking $e = f = 0$ recovers the the definition in the statement of Prop. 9.25.

Final step: Verify these are model geometries A finite-volume quotient of G by a subgroup of its isometry group is a quotient by a lattice (Prop. 6.5(ii)); so it suffices to find a lattice in G.

Suppose $s \in \mathbb{R}$ is nonzero. Putting coordinates (y, x_1, x_2, x_3) on $\mathbb{R} \times \mathrm{Heis}_3$, the semidirect product $(\mathbb{R} \times \mathrm{Heis}_3) \rtimes_{e^{tA}} \mathbb{R}$ where

$$A = \begin{pmatrix} 0 & & & \\ s^{-1}d & 0 & & \\ & & -1 & 2 \\ & & 2 & 1 \end{pmatrix}$$

is, by the preparatory step, isomorphic to G. In particular, if $s = \dfrac{\ln \frac{3+\sqrt{5}}{2}}{\ln \sqrt{5}}$, then

$$e^{sA} = \begin{pmatrix} 1 & & & \\ d & 1 & & \\ & & 1 & 1 \\ & & 1 & 2 \end{pmatrix}.$$

This is an integer matrix, so it preserves the lattice Γ in $\mathbb{R} \times \mathrm{Heis}_3$ consisting of the integer points. Then $\Gamma \rtimes_{e^{sA}} \mathbb{Z}$ is a lattice in G, with closed fundamental domain the standard unit cube of coordinates $[0,1]^5$. $\square$

9.5 Maximality and distinctness

This section proves that the geometries $G = G/\{1\}$ obtained above are both maximal and distinct. Distinctness (Prop. 9.31) mostly uses dimensions of characteristic subalgebras to distinguish the corresponding Lie algebras from each other, and is summarized in Figure 5.3. Maximality (Prop. 9.30) uses the following theorem by Gordon and Wilson.[4]

Theorem 9.28 (part of [GW88, Thm. 4.3])**.** *Suppose G is a connected unimodular solvable Lie group with Lie algebra $\mathfrak{g}$. If the elements of* $\operatorname{ad}\mathfrak{g}$ *have only real eigenvalues (i.e. $\mathfrak{g}$ has all*

4. This generalizes [Wil82, Thm. 2(3)], which is the same result for nilpotent Lie groups.

real roots), then in any invariant metric, there is some $C \subseteq \operatorname{Aut} G$ *such that* $(\operatorname{Isom} G)^0 = G \rtimes C$.

Then $G \cong G \rtimes C/C$, and C becomes the stabilizer of the identity in G; so for a maximal geometry, C is a maximal compact subgroup of $\operatorname{Aut} G$. The following computation helps to find the maximal compact subgroups.

Lemma 9.29. *Suppose* $H \subseteq \mathrm{GL}(n, \mathbb{R})$ *is a Lie group of consisting of block upper-triangular matrices, where* $k_1, \ldots, k_m$ *are the block sizes. Then its maximal compact subgroup is conjugate in* $\mathrm{GL}(n, \mathbb{R})$ *to a subgroup of* $\mathrm{SO}(k_1) \times \cdots \times \mathrm{SO}(k_m)$.

Proof. Let $G(k_1, \ldots, k_m)$ be the group of block upper-triangular matrices where $k_1, \ldots, k_m$ are the block sizes. Since the maximal compact subgroup of H is a compact subgroup of $G(k_1, \ldots, k_m)$, it suffices to compute the maximal compact subgroup of $G(k_1, \ldots, k_m)$.

The determinant of a block upper-triangular matrix is the product of the determinants of the blocks. The straight-line homotopy that sends matrix entries outside the blocks to zero also preserves the determinant, thereby specifying a deformation retract of $G(k_1, \ldots, k_m)$ onto the corresponding group of block diagonal matrices, which deformation retracts onto $\mathrm{SO}(k_1) \times \cdots \times \mathrm{SO}(k_m)$. □

Proposition 9.30. *Let* G *be the simply-connected Lie group with one of the following solvable Lie algebras* $\mathfrak{g}$. *Then* $G/\{1\}$ *is a maximal geometry.*

- *(nilpotent)* $\mathbb{R}^4 \rtimes \mathbb{R}$ *where* $1 \in \mathbb{R}$ *acts with Jordan blocks with characteristic polynomials* (x^4) *or* (x^3, x).
- *(nilpotent)* $\mathfrak{n}_4 \rtimes \mathbb{R}$ *where* $1 \in \mathbb{R}$ *acts by* $x_3 \mapsto x_1$ *and* $x_4 \mapsto 0$ *or* x_3 *(as named in Prop. 9.22).*
- $\mathbb{R}^3 \rtimes \mathbb{R}^2$ *where* $\mathbb{R}^2$ *acts by traceless diagonal matrices.*

- $\mathbb{R}^4 \rtimes \mathbb{R}$ *where* $1 \in \mathbb{R}$ *acts diagonalizably with distinct eigenvalues or with Jordan blocks with characteristic polynomials* $(x^2, x-1, x+1)$ *or* $((x-1)^2, (x+1)^2)$.

- $(\mathbb{R} \oplus \mathfrak{n}_3) \rtimes \mathbb{R}$ *(as named in Prop. 9.25) where* $1 \in \mathbb{R}$ *acts by*

$$x_2 \mapsto x_2 \qquad\qquad x_3 \mapsto -x_3 \qquad\qquad y \mapsto 0 \text{ or } x_1.$$

Proof. Using the theorem of Gordon and Wilson (Thm. 9.28) requires showing that $\mathfrak{g}$ is unimodular—which is already known since G must admit a lattice to be a geometry—and that the eigenvalues of its adjoint representation are all real.

Case 1: G is nilpotent. For a nilpotent $\mathfrak{g}$, the adjoint representation's eigenvalues are all zero; so showing that the maximal compact connected subgroup of $\operatorname{Aut} G$ is trivial will show that G is maximal. By Lemma 9.29 above, it suffices to show that each Lie algebra has a complete flag of characteristic ideals. They are as follows.

- If $\mathfrak{g} = \mathbb{R}^4 \rtimes \mathbb{R}[x^4]$:

$$\begin{aligned}
\operatorname{span}(x_1) &= \mathfrak{g}^4 \\
\operatorname{span}(x_2, x_1) &= \mathfrak{g}^3 \\
\operatorname{span}(x_3, x_2, x_1) &= \mathfrak{g}^2 \\
\operatorname{span}(x_4, x_3, x_2, x_1) &= \{x \in \mathfrak{g} \mid \dim[x, \mathfrak{g}] \leq 1\}
\end{aligned}$$

- If $\mathfrak{g} = \mathbb{R}^4 \ni \mathbb{R}[x^3, x]$:

$$\begin{aligned}
\operatorname{span}(x_1) &= \mathfrak{g}^3 \\
\operatorname{span}(x_2, x_1) &= \mathfrak{g}^2 \\
\operatorname{span}(x_4, x_2, x_1) &= \mathfrak{g}^2 + Z(\mathfrak{g}) \\
\operatorname{span}(x_3, x_4, x_2, x_1) &= \{x \in \mathfrak{g} \mid \dim[x, \mathfrak{g}] \leq 1\}
\end{aligned}$$

- If $\mathfrak{g} = \mathfrak{n}_4 \ni \mathbb{R}$ and $[x_5, x_4] = 0$:

$$\begin{aligned}
\operatorname{span}(x_1) &= \mathfrak{g}^3 \\
\operatorname{span}(x_2, x_1) &= \mathfrak{g}^2 \\
\operatorname{span}(x_5, x_2, x_1) &= \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subseteq \operatorname{span}(x_1)\} \\
\operatorname{span}(x_3, x_5, x_2, x_1) &= \{x \in \mathfrak{g} \mid [x, \operatorname{span}(x_2, x_1)] = [x, \operatorname{span}(x_5, x_2, x_1)]\}
\end{aligned}$$

- If $\mathfrak{g} = \mathfrak{n}_4 \ni \mathbb{R}$ and $[x_5, x_4] = x_3$:

$$\begin{aligned}
\operatorname{span}(x_1) &= \mathfrak{g}^4 \\
\operatorname{span}(x_2, x_1) &= \mathfrak{g}^3 \\
\operatorname{span}(x_3, x_2, x_1) &= \mathfrak{g}^2 \\
\operatorname{span}(x_5, x_3, x_2, x_1) &= \{x \in \mathfrak{g} \mid \dim[x, \mathfrak{g}] \leq 2\}
\end{aligned}$$ □

Case 2: G is not nilpotent. The descriptions of these Lie algebras are explicit enough that the eigenvalues can be verified to be real by inspection. So again it suffices to show that in each case, $\operatorname{Aut} G = \operatorname{Aut} \mathfrak{g}$ contains no nontrivial connected compact subgroups. By Lemma 9.29, it suffices to show that $(\operatorname{Aut} \mathfrak{g})^0$ is upper-triangular in some basis.

In each case, an automorphism of $\mathfrak{g}$ preserves the nilradical $\mathfrak{n}$ and the decomposition of

$\mathfrak{n}$ into generalized eigenspaces of $\mathfrak{g}/\mathfrak{n}$, up to any reordering and scaling. Each generalized eigenspace has a natural flag—the filtration by the rank of generalized eigenvectors—which is also preserved. Except in two cases, this is enough data to make $(\operatorname{Aut}\mathfrak{g})^0$ upper-triangular. The two cases and their additional data are as follows.

- In the $(\mathbb{R}\oplus\mathfrak{n}_3)\rtimes\mathbb{R}$ geometries, the flag $0 \subset \operatorname{span}(x_1) \subset \operatorname{span}(y, x_1)$ in the 0-eigenspace is preserved since $\operatorname{span}(x_1) = [\mathfrak{n}, \mathfrak{n}]$.

- In $\mathbb{R}^3\rtimes\mathbb{R}^2$, the subset of $\mathbb{R}^2 \cong \mathfrak{g}/\mathfrak{n}$ consisting of points that act with a zero eigenvalue is preserved. This is a set of three concurrent lines.

Proposition 9.31 (Claim of correctness of Figure 5.3). *All of the maximal geometries named above are distinct, with the exception that some of the geometries $\mathbb{R}^4\rtimes\mathbb{R}$ with 4 distinct real eigenvalues may coincide with each other.*

Proof. Isomorphic geometries have isomorphic transformation groups; so it suffices to show that the corresponding Lie algebras are mutually non-isomorphic.

Referring to the calculation in Prop. 9.30, the nilpotent algebras are distinguished by whether they admit 4-dimensional abelian ideals and the number of nonzero terms in their lower central series.

The non-nilpotent algebras can be subdivided according to the isomorphism type of their nilradicals.

- Only $\mathbb{R}^3\rtimes\mathbb{R}^2$ has nilradical $\mathbb{R}^3$.

- The two $(\mathbb{R}\oplus\mathfrak{n}_3)\rtimes\mathbb{R}$ geometries are distinguished from each other by the dimensions of their centers—which are $\operatorname{span}(x_1)$ if $[z, y] = x_1$, and $\operatorname{span}(x_1, y)$ if $[z, y] = 0$).

- The non-nilpotent algebras $\mathfrak{g} = \mathbb{R}^4\rtimes\mathbb{R}$ can be distinguished from each other by the Jordan blocks by which $\mathbb{R} \cong \mathfrak{g}/\mathbb{R}^4$ acts on the nilradical $\mathbb{R}^4$—up to a scale factor, to account for the ability to rescale $\mathbb{R}$. This distinguishes all but the case when one action of $\mathbb{R}$ on $\mathbb{R}^4$ has 4 distinct real eigenvalues that are a constant multiple of those of another action. $\square$

PART III

THE FIBERED GEOMETRIES

CHAPTER 10

OVERVIEW

Thurston's geometries are a family of eight homogeneous spaces that form the building blocks of 3-manifolds in Thurston's Geometrization Conjecture. Building on Thurston's classification [Thu97, Thm. 3.8.4] in dimension 3 and Filipkiewicz's classification [Fil83] in dimension 4, this paper is part of a series carrying out the classification in dimension 5.

In the sense of Thurston, a *geometry* is a simply-connected Riemannian homogeneous space $M = G/G_p$, with the additional conditions that M has a finite-volume quotient—"model"—and G is as large as possible—"maximality" (details are in Defn. 11.1). Part I outlines the division of the problem into cases, following the strategy of Thurston and Filipkiewicz, using the action of point stabilizers G_p on tangent spaces T_pM (the "linear isotropy representation"). Part II performs the classification for the case when this representation is trivial or irreducible, by leveraging other classification results.

The present paper finishes the classification by working out the case when T_pM is nontrivial and reducible. The decomposition of T_pM is used to construct a G-invariant fiber bundle structure on M—hence the name "fibering geometries"—and the classification of these bundles provides a way to access a classification of geometries. To cope with the increased richness in fiber bundle structures compared to what is possible in lower dimensions, the tools required include conformal geometry, Galois theory, and Lie algebra cohomology in addition to everything used in previous classifications. In particular, extension problems feature much more noticeably than for the analogous cases in lower dimensions. The main result is the following.

Theorem 10.1 (**Classification of 5-dimensional maximal model geometries with nontrivial, reducible isotropy**). *Let $M = G/G_p$ be a 5-dimensional maximal model geometry, and let V be an irreducible subrepresentation of $G_p \curvearrowright T_pM$ of maximal dimension.*

(i) If $\dim V = 4$ *(Chapter 13), then* M *is one of the spaces*

$$S^4 \times \mathbb{E} \qquad \mathbb{H}^4 \times \mathbb{E} \qquad \mathbb{C}P^2 \times \mathbb{E} \qquad \mathbb{C}\mathbb{H}^2 \times \mathbb{E} \qquad \widetilde{\mathrm{U}(2,1)/\mathrm{U}(2)} \qquad \mathrm{Heis}_5.$$

(ii) If $\dim V = 3$ *(Chapter 14), then* M *is a product of 2-dimensional and 3-dimensional constant-curvature geometries.*

(iii) If $\dim V = 2$ *(Chapter 15), then* M *is either a product of lower-dimensional geometries or one of the following.*

(a) The unit tangent bundles,

$$T^1\mathbb{H}^3 = \mathrm{PSL}(2,\mathbb{C})/\mathrm{PSO}(2) \qquad T^1\mathbb{E}^{1,2} = \mathbb{R}^3 \rtimes \mathrm{SO}(1,2)^\circ/\mathrm{SO}(2);$$

(b) The associated bundles (see Defn. 15.27 and Table 15.29),

$$\begin{aligned}
&\mathrm{Heis}_3 \times_{\mathbb{R}} S^3 && \widetilde{\mathrm{SL}_2} \times_\alpha S^3, \quad 0 < \alpha < \infty \\
&\mathrm{Heis}_3 \times_{\mathbb{R}} \widetilde{\mathrm{SL}_2} && \widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2}, \quad 0 < \alpha \le 1 \\
& && L(a;1) \times_{S^1} L(b;1), \quad 0 < a \le b \text{ coprime in } \mathbb{Z};
\end{aligned}$$

(c) The line bundles over $\mathbb{F}^4$*,*

$$\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2} \cong (\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}) \rtimes \mathrm{SO}(2)/\mathrm{SO}(2)$$
$$\mathbb{F}^5_a = \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SL}_2}/\{atz, \gamma(t)\}_{t\in\mathbb{R}}, \quad a = 0 \text{ or } 1;$$

(d) The indecomposable non-nilpotent solvable Lie groups $\underset{\text{polynomials}}{\mathbb{R}^4 \rtimes \mathbb{R}}$*, specified by the list of characteristic polynomials of the Jordan blocks of a matrix* A *where* $t \in \mathbb{R}$

acts on $\mathbb{R}^4$ *by* e^{tA}*,*

$$A_{5,9}^{-1,-1} = \underset{(x-1)^2,\, x+1,\, x+1}{\mathbb{R}^4 \rtimes \mathbb{R}}$$

$$A_{5,7}^{1,-1,-1} = \underset{x-1,\, x-1,\, x+1,\, x+1}{\mathbb{R}^4 \rtimes \mathbb{R}}$$

$$A_{5,7}^{1,-1-a,-1+a} = \underset{x-1,\, x-1,\, x-a+1,\, x+a+1}{\mathbb{R}^4 \rtimes \mathbb{R}} \quad \textit{where } a > 0,\ a \neq 1,\ a \neq 2,$$

$$\textit{and } \det(\lambda - e^{tA}) \in \mathbb{Z}[\lambda] \textit{ for some } t > 0;$$

(e) and the indecomposable nilpotent Lie groups,

$$A_{5,1} = \underset{x^2, x^2}{\mathbb{R}^4 \rtimes \mathbb{R}} \qquad A_{5,3} = \underset{x_3 \to x_2 \to y}{(\mathbb{R} \times \mathrm{Heis}_3) \rtimes \mathbb{R}}.$$

Moreover, all of the explicitly named spaces above are indeed maximal model geometries; and each product geometry is a model geometry, and maximal if at most one factor is Euclidean.

The solvable Lie groups M are given with names from [PSWZ76, Table II] for the classification of 5-dimensional solvable real Lie algebras by Mubarakzyanov in [Mub63]. Their isometry groups are $M \rtimes K$, where $K \cong G_p$ is a maximal compact subgroup of $\operatorname{Aut} M$. These point stabilizer subgroups G_p are listed in Table 10.2; more explicit descriptions for the solvable Lie groups can be found in Prop. 15.13 Step 4 and Prop. 15.7(iii) Step 6. The lower dimensional geometries have their usual isometry groups; e.g. $\mathbb{C}P^2$ and $\mathbb{CH}^2$ are as specified in [Fil83, Thm. 3.1.1]. A full list of point stabilizers in dimension 4 can be found in [Wal86, Table 1].

In the spirit of [Mos50, Cor. p. 624], [Gor77], [Ish55], and [Ott09, Thm. 1.0.3], one can also give a classification up to diffeomorphism (Table 10.3). Most of the diffeomorphism types are readily guessed and can be verified by a theorem of Mostow [Mos62a, Thm. A]. The exception is the family of associated bundles $L(a;1) \times_{S^1} L(b;1)$; Ottenburger names

Table 10.2: Non-product fibering geometries by isotropy group G_p (see also Fig. 11.4)

Isotropy	Geometries
$\mathrm{U}(2)$	Heis_5 and $\widetilde{\mathrm{U}(2,1)/\mathrm{U}(2)}$
$\mathrm{SO}(2) \times \mathrm{SO}(2)$	$\underset{x-1,x-1,x+1,x+1}{\mathbb{R}^4 \rtimes \mathbb{R}}$ and the associated bundles (Thm. 10.1(iii)(b))
$\mathrm{SO}(2)$	The remaining solvable groups from Thm. 10.1(iii)(d)
$S^1_{1/2}$	All line bundles over $\mathbb{F}^4$ (Thm. 10.1(iii)(c))
S^1_1	The two unit tangent bundles (Thm. 10.1(iii)(a)) and the nilpotent Lie groups from Thm. 10.1(iii)(e)

them N^{ab1} in [Ott09, §3.1] and shows that they are all diffeomorphic to $S^3 \times S^2$ in [Ott09, Cor. 3.3.2].

Table 10.3: Non-contractible non-product fibering geometries by diffeomorphism type

Type	Geometries
$S^2 \times \mathbb{R}^3$	$T^1\mathbb{H}^3$
$S^3 \times \mathbb{R}^2$	$\mathrm{Heis}_3 \times_{\mathbb{R}} S^3$ and $S^3 \times_\alpha \widetilde{\mathrm{SL}_2}$
$S^3 \times S^2$	$L(a;1) \times_{S^1} L(b;1)$

Roadmap. Chapter 11 lists basic definitions and any external results that need to be used frequently. Then Chapter 12 uses the decomposition of the linear isotropy representation to establish the existence of invariant fiber bundle structures on geometries (Prop. 12.3), introducing related notations (such as names for invariant distributions) along the way. The remaining sections each deal with base spaces of a single dimension:

- Chapter 13 handles the case where the fiber bundle has a 4-dimensional base with irreducible isotropy. The geometries are classified by curvature, using a strategy closely following Thurston's in [Thu97, Thm. 3.8.4(b)] for 3-dimensional geometries over 2-dimensional bases.

- Chapter 14 handles 3-dimensional isotropy irreducible bases. This case produces one non-product geometry that is shown not to be a model geometry by Galois theory (Section 14.2).

- Chapter 15 handles 2-dimensional bases. This case produces the "associated bundle geometries", a source of examples such as an uncountable family of geometries (Prop. 15.35) and a geometry whose maximal realization is not unique (Rmk. 15.40).

CHAPTER 11

BACKGROUND

11.1 Geometries, products, and isotropy

Recall the definition of a geometry, following Thurston and Filipkiewicz in [Thu97, Defn. 3.8.1] and [Fil83, §1.1]. This is given in terms of homogeneous spaces, since the upcoming classification will rely heavily on them; (Prop. 6.5) in Part II outlines the equivalence to earlier definitions.

Definition 11.1 (Geometries).

(i) A *geometry* is a connected, simply-connected homogeneous space $M = G/G_p$ where G is a connected Lie group acting faithfully with compact point stabilizers G_p.

(ii) M is a *model geometry* if there is some lattice $\Gamma \subset G$ that acts freely on M. Then the manifold $\Gamma\backslash G/G_p$ is said to be *modeled on* M.

(iii) M is *maximal* if it is not G-equivariantly diffeomorphic to any other geometry G'/G'_p with $G \subsetneq G'$. Any such G'/G'_p is said to *subsume* G/G_p.

Many properties of this definition—including the existence of invariant Riemannian metrics and the correspondence between quotients $\Gamma\backslash G/G_p$ and complete, finite-volume manifolds locally isometric to M—are taken for granted here but stated more explicitly in Part II (§2).

Discussion of product geometries was omitted from Part II—but among the fibering geometries are many (28 and one countable family). Some shortcuts are possible with their classification, such as the following.

Proposition 11.2 (Products are models). *If $M = G/G_p$ and $M'G'/G'_q$ are model geometries, then their product $M \times M' = (G \times G')/(G_p \times G'_q)$ is a model geometry.*

Proof. If $\Gamma\backslash G/G_p$ is modeled on M and $\Gamma'\backslash G'/G'_q$ is modeled on M', then $(\Gamma \times \Gamma')\backslash(G \times G')/(G_p \times G'_q)$ is modeled on $M \times M'$. □

Maximality is in general a more difficult question; we prove in Prop. 12.12 that the product of two maximal geometries is maximal, but under the assumption that at most one factor admits nonzero invariant vector fields and at most one factor is itself a product with a Euclidean factor. This happens to be enough for our usage in dimension 5 (Prop. 15.37).

There is, however, a shortcut to prove maximality for geometries realized by solvable Lie groups, given by the following rephrasing of a theorem of Gordon and Wilson.

Lemma 11.3 (**Maximality for solvable Lie groups**)**.** *Let M be a simply-connected, unimodular, solvable Lie group whose adjoint representation acts with only real eigenvalues. Then the maximal geometry subsuming $M/\{1\}$ is $M \rtimes K/K$ where K is a maximal compact subgroup of* $\operatorname{Aut} M$*.*

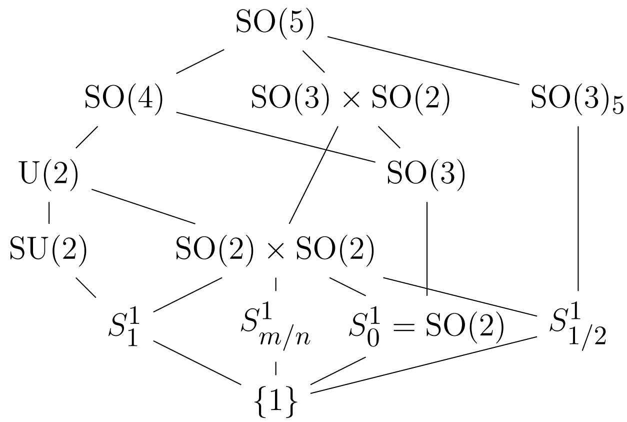
Proof. Under the provided assumptions, in any invariant metric on M, there is some $K \subseteq \operatorname{Aut} M$ such that the identity component of the isometry group of M is $\operatorname{Isom}_0 M = M \rtimes K$ [GW88, Thm. 4.3]. Since K is the point stabilizer of the identity, it is compact.

Since the transformation group of a maximal geometry is realizable as the isometry group in some invariant metric [Fil83, Prop. 1.1.2], the maximal geometry subsuming $M/\{1\}$ is of the form $M \rtimes K/K$, with K not in any larger compact group of automorphisms. □

When M is nilpotent, the adjoint eigenvalues are always 1; this case was proven earlier by Wilson in [Wil82, Thm. 2(3)].

The existence of non-product geometries forces a classification to confront fiber bundle structures. Chapter 12 details how the occurrence of these structures is controlled by the action of the point stabilizer G_p on the tangent space T_pM. Since G_p is compact, it preserves an inner product, which allows expressing such a representation by a subgroup of SO(5). Figure 11.4 recalls the classification of such subgroups from Part II in (Prop. 7.1).

Figure 11.4: Closed connected subgroups of SO(5), with inclusions. $\mathrm{SO}(3)_5$ denotes SO(3) acting on its 5-dimensional irreducible representation; and $S^1_{m/n}$ acts as on the direct sum $V_m \oplus V_n \oplus \mathbb{R}$ where S^1 acts irreducibly on V_m with kernel of order m.



11.2 Lie algebra extensions

If M is a G-invariant fiber bundle over a space B, then the isometry group G is an extension of the transformation group of B. Passing to Lie algebras permits use of the well-known classification of Lie algebra extensions by second cohomology, which this subsection recalls. For more details, a survey of Lie algebra cohomology in low degrees can be found in [Wag10] or [AMR00, §2–4][1].

Definition 11.5 (**Lie algebra cohomology**, following [Wag10, §2])**.** Let M be a module of a Lie algebra $\mathfrak{g}$ over a field k. The *Chevalley-Eilenberg complex* is the cochain complex is the cochain complex

$$C^p(\mathfrak{g}, M) = \mathrm{Hom}_k(\Lambda^p \mathfrak{g}, M)$$

1. An almost identical version, generalized to super (i.e. $\mathbb{Z}/2\mathbb{Z}$-graded) Lie algebras, has been published as [AMR05].

with boundary maps

$$d_p : C^p \to C^{p+1}$$

$$(d_p c)(x_1, \ldots, x_{p+1}) = \sum_{1 \le i < j \le p+1} (-1)^{i+j} c\left([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{p+1}\right) + \sum_{1 \le i \le p+1} (-1)^{i+1} x_i c\left(x_1, \ldots, \hat{x_i}, \ldots, x_{p+1}\right)$$

where $\hat{x_i}$ means x_i should be omitted. The cohomology of $\mathfrak{g}$ with coefficients in M is defined to be the cohomology of this complex and denoted $H^p(\mathfrak{g}, M)$.

Theorem 11.6 (**Classification of extensions by second cohomology** [AMR00, Thm. 8]). *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let*

$$\overline{\alpha} : \mathfrak{g} \to \operatorname{out} \mathfrak{h} = \operatorname{der}(\mathfrak{h}) / \operatorname{ad}(\mathfrak{h})$$

be a Lie algebra homomorphism. Then the following are equivalent.

(i) For one (equivalently: any) linear lift $\alpha : \mathfrak{g} \to \operatorname{der} \mathfrak{h}$ of $\overline{\alpha}$ choose $\rho : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}$ satisfying

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \operatorname{ad}_{\rho([X,Y])} .$$

Then the cohomology class of $d\rho$ in $H^3(\mathfrak{g}, Z(\mathfrak{h}))$ vanishes, where the action of $\mathfrak{g}$ on $Z(\mathfrak{h})$ is induced by α and d is defined by the formula in 11.5 (ignoring that $\mathfrak{h}$ may not be a $\mathfrak{g}$-module).

(ii) There exists an extension $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ inducing the homomorphism $\overline{\alpha}$.

If either occurs, then the extensions satisfying (ii) are parametrized by $[\rho] \in H^2(\mathfrak{g}; Z(\mathfrak{h}))$.

11.3 Notations

The naming of Lie groups and Lie algebras requires a number of notations; the following conventions will be used throughout, with scattered reminders.

- $T_{\mathbf{1}}G$ denotes the tangent algebra of a Lie group G—the tangent space at the identity, identified with *right*-invariant vector fields on G so that the resulting flows correspond with the action of 1-parameter subgroups by multiplication on the left.

- G° denotes the identity component of a topological group G, and $\operatorname{Isom}_0 M$ is short for the identity component of the isometry group of M.

- Fraktur letters usually denote Lie algebras, e.g. an occurrence of $\mathfrak{g}$ near G probably means $\mathfrak{g} = T_{\mathbf{1}}G$; and $\mathfrak{isom}\, M = T_{\mathbf{1}} \operatorname{Isom} M$.

CHAPTER 12

FIBER BUNDLE STRUCTURES ON GEOMETRIES

If $M = G/G_p$ admits a G-invariant fiber bundle structure $F \to M \to B$, then F and B admit transitive group actions, which affords some hope of reduction to lower-dimensional problems. The aim of this section is to prove that when $\dim M = 5$, reducibility of the isotropy representation $G_p \curvearrowright T_pM$ both implies the existence of such a fibering and constrains some of its properties (Prop. 12.3). This begins with a definition of the structure being sought.

Definition 12.1 (Fiberings). A geometry $M = G/G_p$ *fibers over* a smooth manifold B if B is diffeomoprhic to $M/\mathcal{F}$ for some G-invariant foliation $\mathcal{F}$ with closed leaves. (Equivalently, $B \cong G/H$ for some closed subgroup $H \subseteq G$ containing G_p.) The fibering is described as *isometric* if B admits a G-invariant Riemannian metric, *conformal* if B admits a G-invariant conformal structure, and *essentially conformal* (or *essential* for short) if it is conformal but not isometric.

Remark 12.2. Closedness of H ensures that G/H has a natural smooth structure and smooth action by G [Hel78, Thm. II.4.2] and, by the existence of local cross sections for $B \to G$ [Hel78, Lemma II.4.1], that M is a smooth fiber bundle $F \to M \to B$ where $F \cong H/G_p$ is a leaf of $\mathcal{F}$.

This section's main result, guaranteeing the existence of useful fiberings, is the following Proposition. Its proof will be given in the last subsection, after the first two subsections introduce the G-invariant distributions that will be used to work infinitesimally with fiberings.

Proposition 12.3 (The fibering description). *Let $M = G/G_p$ be a 5-dimensional geometry, and let d be the dimension of the largest irreducible subrepresentation of $G_p \curvearrowright T_pM$.*

(i) If $d = 4$ and M is a model geometry then M fibers isometrically over a 4-dimensional simply-connected Riemannian symmetric space.

(ii) If $d = 3$ *then* M *fibers conformally over* S^3, $\mathbb{E}^3$, *or* $\mathbb{H}^3$.

(iii) If $d = 2$ *and* M *is a model geometry then* M *fibers conformally over* S^2, $\mathbb{E}^2$, *or* $\mathbb{H}^2$.

Remark 12.4 (**Generalizations to higher dimensions**). The first two cases generalize to n dimensions without major modifications of the proof or conclusions. Case (ii) is when the restriction of T_pM to some normal subgroup of G_p decomposes with exactly one nontrivial irreducible subrepresentation, and case (i) is the sub-case when this subrepresentation has codimension 1. In both of these generalizations, the base space is an isotropy irreducible space; and if the fibering is essentially conformal, the base is Euclidean or a sphere.

Other fiberings can be produced from the "natural bundle" if M is compact [GOV93, §II.5.3.2], or by using the Levi decomposition (see e.g. [GOV94, §1.4]) as part of something like [Mos05, Thm. C].

Example 12.5 (**Reducible isotropy does not in general imply fibering**). If G/G_p has an invariant fiber bundle structure with positive-dimensional fiber and base, then the point stabilizer of the base is an intermediate group $G_p \subsetneq H \subsetneq G$. Building on Dynkin's work classifying maximal proper connected subgroups, Dickinson and Kerr classified compact Riemannian homogeneous spaces with two isotropy summands in [DK08, §6] and found examples G/G_p where G_p is maximal. So these spaces have reducible isotropy but no nontrivial fiberings.

The lowest-dimensional counterexample (that is compact and has two isotropy summands) is $\mathrm{Sp}(3)/\mathrm{Sp}(1)$, in dimension 18 (Example V.10), where the embedding $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(3)$ is given by the irreducible representation of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ on $\mathbb{C}^6$.

12.1 Invariant distributions and foliations

This subsection introduces the fixed distributions (Defn. 12.6) that will usually become the vertical distributions—i.e. tangent distributions to the fibers in a fiber bundle. Showing that such a distribution can be integrated to a foliation by submanifolds will require some tools such as the *integrability tensor* (Lemma 12.8).

The action of G on TM induces the correspondence

$$\left\{ \begin{array}{c} G_p\text{-invariant subspaces} \\ \text{(subrepresentations) of } T_pM \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} G\text{-invariant} \\ \text{distributions on } M = G/G_p \end{array} \right\},$$

which provides a convenient way to refer to G-invariant distributions, including the following.

Definition 12.6 (**Fixed distributions of normal subgroups**)**.** Given a geometry $M = G/G_p$ and a normal subgroup $H \trianglelefteq G_p$, let TM^H be the G-invariant distribution defined at p by T_pM^H (the subspace on which H acts trivially). Let TM^G denote the case $H = G_p$.

The most visibly useful property of TM^H is that it can be integrated to produce the fibers of a fiber bundle. That is (see also Rmk. 12.2 above),

Proposition 12.7. *TM^H is integrable to a G-invariant foliation $\mathcal{F}^H$ with closed leaves.*

The proof uses a construction known in the context of Riemannian submersions as the "integrability tensor" (see e.g. [O'N66, Lemma 2] and the discussion before [Pet06, Thm. 3.5.5]). Its definition, its tensoriality, and its compatibility with the action of point stabilizers are covered by the following rephrasing of [Fil83, Lemma 3.2.2].

Lemma 12.8 (**Integrability tensor as a representation homomorphism**)**.** *Let D be a distribution on a manifold M. Then at each point p, the Lie bracket of vector fields induces a linear map $\mu : \Lambda^2 D_p \to T_pM/D_p$. If D is G-invariant on a homogeneous space $M = G/G_p$, then $\Lambda^2 D_p \to T_pM/D_p$ is a G_p-representation homomorphism.*

Proof. Suppose $\{X_1, \ldots, X_k\}$ is a local basis of D, and Y_1 and Y_2 are vector fields in D with $\operatorname{span}(Y_1(p), Y_2(p)) \subseteq \operatorname{span}(X_1(p), X_2(p))$. Pick functions $\{a_i\}$ and $\{b_i\}$ so that

$$Y_1 = \sum_{i=1}^{m} a_i X_i \qquad\qquad Y_2 = \sum_{i=1}^{m} b_i X_i$$

in a neighborhood of p. Using the Leibniz rule for Lie brackets,

$$[Y_1, Y_2] = \sum_{i,j} a_i b_j [X_i, X_j] + a_i X_i(b_j) X_j - b_j X_j(a_i) X_i.$$

The last two terms are pointwise linear combinations of $X_1, \ldots, X_m$, so they add to some vector field Z in D. At p, only a_1, a_2, b_1, and b_2 can be nonzero, so

$$[Y_1, Y_2](p) = (a_1 b_2 - a_2 b_1)[X_1, X_2](p) + Z(p),$$

which ensures that μ is well-defined. The G_p-equivariant version then follows by recalling that diffeomorphisms respect Lie brackets. □

Proof of Prop. 12.7. The integrability tensor $\mu : \Lambda^2 T_p M^H \to T_p M/(T_p M)^H$ is zero, since the domain is a trivial representation of H, and the codomain is a representation with no trivial summands. So by the Frobenius condition, TM^H is integrable.

A G-invariant foliation on a geometry $M = G/G_p$ whose tangent distribution contains TM^G has closed leaves [Fil83, Prop. 2.1.1, 2.1.2]. Since $H \subseteq G_p$ implies $T_p M^H \supseteq T_p M^{G_p}$, the foliation integrating TM^H has closed leaves. □

12.2 Complementary distributions and conformal actions

If the distribution TM^H is interpreted as vertical—i.e. tangent to the fibers of a fibering $M \to B = M/\mathcal{F}^H$—then a complementary horizontal distribution should offer some understanding of the target B. This subsection introduces such a distribution and its applications to maximality of products (Prop. 12.12) and conformal fiberings (Lemma 12.13).

Definition 12.9 (Complementary distributions). Given a geometry $M = G/G_p$ and a normal subgroup $H \trianglelefteq G_p$, let $(TM^H)^\perp$ denote the distribution defined at p by the complementary H-representation to $(T_p M)^H$ in $T_p M$.

Conveniently, this always agrees with the other distribution deserving the same name—the orthogonal complement—which makes TM^H and $(TM^H)^\perp$ useful together for studying invariant metrics on M. Explicitly,

Lemma 12.10. *$(TM^H)^\perp$ is the orthogonal complement to TM^H in every G-invariant metric on M.*

Proof. A G_p-invariant metric on T_pM induces a representation isomorphism $T_pM \cong_H T_p^*M$. Since $(TM^H)^\perp_p$ contains no trivial subrepresentation by definition, its image in the trivial representation $(T_p^*M)^H$ is zero; so the pairing of $(TM^H)^\perp_p$ with $(T_pM)^H$ is zero. □

Remark 12.11. This property makes any isometric fibering $M \to M/\mathcal{F}^H$ a *Riemannian submersion* (see e.g. the discussion before [Pet06, Example 1.1.5]) for some invariant metric on M for each invariant metric on $M/\mathcal{F}^H$.

Our main purposes in defining $(TM^H)^\perp$ are to understand properties of fiberings given properties of the isotropy representation (Lemma 12.13 below, used to establish the fibering description in Prop. 12.3) and to prove the following statement about maximality of products.

Proposition 12.12 (Products are often maximal). *Given maximal geometries $M = G/G_p$ and $M' = G'/G'_q$, their product*

$$M \times M' = (G \times G')/(G_p \times G'_q)$$

is maximal provided that both of the following hold.

(i) At most one of M and M' is itself a product with a Euclidean factor.

(ii) At least one of M and M' admits no nonzero invariant vector field.

Proof. Hano proved (see e.g. [KN63, Thm. VI.3.5]) that the de Rham decomposition of a complete, simply-connected Riemannian manifold (see e.g. [KN63, Thm. IV.6.2]) also decomposes the isometry group. So if M and M' are complete, connected, simply-connected

Riemannian manifolds, one of which is not isometric to any product $M'' \times \mathbb{E}^k$, then

$$(\mathrm{Isom}(M \times M'))^0 = (\mathrm{Isom}\, M)^0 \times (\mathrm{Isom}\, M')^0.$$

Since the transformation group of a maximal geometry is realizable as the isometry group in some metric [Fil83, Prop. 1.1.2], it suffices to know that all invariant metrics on $M \times M'$ are realizable as product metrics. Condition (ii) implies this as follows.

To say that M admits no invariant vector field means that TM^G is zero. This implies $\left(T_{(p,q)}(M \times M')\right)^{G_p} = T_q M'$, whose orthogonal complement in every invariant metric is the complementary G_p-representation $T_p M$ (Lemma 12.10). $\square$

For non-product fiberings, the main use of $(TM^H)^\perp$ is in understanding metrics on $M/\mathcal{F}^H$, using the following slight generalization of part of [Fil83, Thm. 4.1.1].

Lemma 12.13 (**Irreducible + trivial isotropy produces conformal fibering**). *Let $M = G/G_p$ be a geometry and H a normal subgroup of G_p whose action on $(TM^H)^\perp_p$ is irreducible. Then there is a metric on $M/\mathcal{F}^H$ with respect to which G acts conformally.*

A key ingredient in similar results such as [Fil83, Thm. 4.1.1] or the start of the proof of [Thu97, Thm. 3.8.4(b)] is the observation that all points of the same fiber F have the same subgroup of G for stabilizers. Then one can speak of representation isomorphisms, which the following standard fact turns into conformal maps.

Lemma 12.14 (**Existence** [BD85, Thm. II.1.7] **and uniqueness** [KN63, App. 5 Thm. 1] **of invariant inner products**). *Every finite-dimensional irreducible representation V of a compact Hausdorff group K over $\mathbb{R}$ has a unique K-invariant inner product (i.e. positive-definite symmetric bilinear form) up to scaling.*

With this control over possible metrics, Filipkiewicz then constructs the appropriate metric using a partition-of-unity argument. Here is the proof, with details of the steps outlined above.

Proof of the conformal fibering Lemma (12.13). This proof repeats most of the result it extends, [Fil83, Thm. 4.1.1]; the new material is mostly the first step, which works around the unavailability of the previously mentioned "key ingredient": points of the same fiber may not have the same stabilizer in G.

Step 1: Find a group G_F to replace G_p. Given p (thus G_p) and $H \trianglelefteq G_p$, let F be the leaf of $\mathcal{F}^H$ containing p, and let G_F denote the subgroup of G that acts as the identity on a leaf F of $\mathcal{F}^H$. Then $H \subseteq G_F \subseteq G_p$, so G_F is a compact group acting irreducibly on $(TM^H)^\perp_q$ for every $q \in F$.

Step 2: Isomorphisms of G_F-irreps become conformal linear maps. Fix a G-invariant metric μ on M. Its restriction $\mu|_{(TM^H)^\perp}$ to the (invariant) distribution $(TM^H)^\perp$ is G-invariant, hence G_F-invariant for every leaf F of $M/\mathcal{F}^H$.

The projection $\pi : M \to M/\mathcal{F}^H$ induces G_F-representation isomorphisms

$$(TM^H)^\perp_q \to T_{\pi(q)}(M/\mathcal{F}^H),$$

each producing from $\mu|_{(TM^H)^\perp}$ a G_F-invariant inner product on a tangent space to $M/\mathcal{F}^H$. Since invariant inner products on irreducible representations are unique up to scaling (Lemma 12.14 above), all such pushed-forward inner products on the same tangent space are scalar multiples of each other.

Step 3: Give sufficient conditions for a conformal structure to be G-invariant. If a metric on $M/\mathcal{F}^H$ is pointwise a linear combination of the pushforwards described in Step 2, then G preserves its conformal class since G preserves μ, $\mathcal{F}^H$, and $(TM^H)^\perp$. What remains is to construct such a metric—i.e. to show that these pushforwards can be chosen in a smoothly varying way—using the partition-of-unity method from [Fil83, Thm. 4.1.1].

Step 4: Construct the metric on $M/\mathcal{F}^H$. Let $k = \dim(TM^H)^\perp$, and choose a collection of k-discs $V_i \subset M$ that

1. are transverse to TM^H,
2. each map diffeomorphically to $M/\mathcal{F}^H$, and
3. cover $M/\mathcal{F}^H$ with their images.

The cover of $M/\mathcal{F}^H$ can be made locally finite since $M/\mathcal{F}^H$ is a manifold; so there is a partition of unity $\{\phi_i\}$ subordinate to $\{\pi(V_i)\}$.

Let $\rho : TM \to (TM^H)^\perp$ be the projection with kernel TM^H. On each V_i, let μ_i be the metric defined pointwise as the pullback of $\mu|_{(TM^H)^\perp}$ by ρ. Then the sum

$$\bar{\mu} = \sum_i \phi_i \pi_* \mu_i$$

defines a metric on $M/\mathcal{F}^H$ with the property described in Step 3. □

12.3 Proof of the fibering description (Proposition 12.3)

Two more facts will be needed for the proof of Prop. 12.3. First, the base is simply-connected in every case of Prop. 12.3: by definition, a geometry M is simply-connected and a leaf F of $\mathcal{F}$ is connected. So the homotopy exact sequence for $F \to M \to M/\mathcal{F}$ implies $M/\mathcal{F}$ is simply-connected. Hence the proofs to follow will skip proving simply-connectedness.

The second fact is a lemma extracted from Thurston's 3-dimensional classification, used to reason about invariant vector fields in cases (i) and (iii).

Lemma 12.15 (see e.g. [Thu97, Proof of 3.8.4(b)]). *A G-invariant vector field X on a model geometry $M = G/G_p$ is divergence-free.*

Proof. Let ϕ_t be the G-equivariant flow on M integrating X. If N is any finite-volume manifold modeled on M, then X and ϕ_t descend to some $\overline{X}$ and $\overline{\phi_t}$ on N, along with any G-invariant metric. Since $\operatorname{vol} N$ is finite, $\overline{\phi_t}$ is globally volume-preserving; so $\int_N \operatorname{div} \overline{X} \, d \operatorname{vol} = 0$. Since G acts transitively on M, the value of $\operatorname{div} X = \operatorname{div} \overline{X}$ is constant—thus 0. □

The proof of the fibering description (Prop. 12.3) can now proceed. The three cases exhibit somewhat different behavior, so they are handled separately. In particular, case (iii) (fiberings over 2-dimensional bases) is a bit more work to prove than the others, due to isotropy representations $G_p \curvearrowright T_pM$ in which there is not a canonical choice of a 2-dimensional irreducible summand to form the horizontal distribution.

12.3.1 Case (i): over dimension 4

Proof of 12.3(i). Let $M = G/G_p$ be a model geometry for which $G_p \curvearrowright T_pM$ has irreducible subrepresentations of dimensions 1 and 4. This proof mostly follows the argument in [Thu97, Thm. 3.8.4(b)].

Case (i), Step 1: M fibers over a 4-dimensional space. The distribution TM^G is 1-dimensional and integrates to a foliation with closed leaves (Prop. 12.7), so M fibers over a 4-dimensional space $M/\mathcal{F}^G$.

Case (i), Step 2: M fibers isometrically. A nonzero vector in $(T_pM)^G$ pushes forward by the action of G to a G-invariant vector field X on M, with corresponding G-equivariant flow ϕ_t. Then ϕ_t commutes with the action of G, so all points in the same ϕ_t-orbit have the same stabilizer in G. Therefore $d_p\phi_t : T_pM \to T_{\phi_t(p)}M$ is an isomorphism of G_p-representations.

Since M is a model geometry, ϕ_t is volume-preserving (Lemma 12.15). Combined with the fact that ϕ_t preserves the metric (the fiberwise inner product) on TM^G since it preserves its own velocity field X, this implies ϕ_t preserves volume on the orthogonal complement $(TM^G)^\perp$. Since $(TM^G)^\perp$ is irreducible, ϕ_t preserves the metric on it (Lemma 12.14). So the metric on $(TM^G)^\perp$ descends to a G-invariant metric on $M/\mathcal{F}^G$.

Case (i), Step 3: $M/\mathcal{F}^G$ is Riemannian symmetric. By the classification of isotropy groups (Fig. 11.4), G_p is SO(4), U(2), or SU(2). In their standard representations on $\mathbb{R}^4$, all of these contain the scalar -1, which reverses tangent vectors—and thus geodesics—at the image of p in $M/\mathcal{F}^G$. □

12.3.2 *Case (ii): over dimension 3*

Proof of 12.3(ii). Suppose $M = G/G_p$ is a geometry where $G_p \curvearrowright T_pM$ contains an irreducible 3-dimensional subrepresentation. By the classification of isotropy groups (Fig. 11.4), G_p contains SO(3) as a characteristic subgroup. Therefore M fibers conformally over $B = M/\mathcal{F}^{\mathrm{SO}(3)}$ (Lemma 12.13).

Then G acts transitively on B, with SO(3) in the point stabilizers. If the fibering $M \to B$ is isometric, then B is one of the 3-dimensional constant-curvature spaces. Otherwise,

1. a manifold B with a transitive[1] essential conformal automorphism group $\operatorname{Conf} B$ is conformally flat [Oba73, Lemma 1]; and

2. if B is conformally flat and the identity component of $\operatorname{Conf} B$ acts essentially, then B is conformally equivalent to a sphere or Euclidean space [Laf88, Thm. D.1].

So if the fibering $M \to B$ is essential, then B is S^3 or $\mathbb{E}^3$. □

1. A stronger version of Obata's theorem (i.e. without assuming transitivity) could instead be used here, but its proof involves some analytic subtleties—see [Fer96] for an overview. The transitive action of G provides a way to sidestep these issues.

12.3.3 Case (iii): over dimension 2

Proof of 12.3(iii). Suppose $M = G/G_p$ is a model geometry where the irreducible subrepresentations of $G_p \curvearrowright T_pM$ have dimensions 1 and 2.

If G_p is SO(2) or SO(2) × SO(2), then a strategy like that of case (ii) applies: since G_p contains SO(2) as a normal subgroup, M fibers conformally over $B = M/\mathcal{F}^{\mathrm{SO}(2)}$ (Lemma 12.13). Since B is simply-connected by the homotopy exact sequence, the Uniformization Theorem (see e.g. [Ahl10, Thm. 10-3]) implies it is conformally equivalent to S^2, $\mathbb{E}^2$, or $\mathbb{H}^2$.

The remaining case is when $G_p = S^1_{m/n}$ (a 1-parameter subgroup of SO(2) × SO(2)). By the same application of the Uniformization Theorem, it suffices to find a G-invariant foliation $\mathcal{F}$ of codimension 2 such that M fibers conformally over $M/\mathcal{F}$. The strategy consists of the following two steps. First, a foliation $\mathcal{F}$ with closed leaves and codimension 2 is found by examining Lie brackets of vector fields tangent to subrepresentations of $G_p \curvearrowright T_pM$. Then if M does not fiber conformally over $M/\mathcal{F}$, this information is used to find an alternative fibering of M.

Case (iii), Step 1: Finding a foliation with closed leaves. $G_p = S^1_{m/n}$ means that

$$T_pM = (T_pM)^G \oplus ((T_pM)^G)^\perp = (T_pM)^G \oplus V_m \oplus V_n,$$

where $(T_pM)^G$ is a 1-dimensional trivial representation of S^1, and V_m is an irreducible representation on which S^1 acts with kernel of size m.

If $m \neq n$, then $V_m \ncong V_n$. The integrability tensor (Lemma 12.8) is a G_p-representation homomorphism

$$\mathbb{R} \oplus V_m \cong \Lambda^2((T_pM)^G \oplus V_m) \to T_pM/((T_pM)^G \oplus V_m) \cong V_n$$

that agrees with Lie brackets of vector fields. This is zero since $\mathbb{R}$, V_m, and V_n are nonisomorphic irreducible representations; so Lie brackets preserve tangency to the 3-plane $(T_pM)^G \oplus V_m$. Hence the corresponding G-invariant distribution is integrable. The resulting foliation has closed leaves since its tangent distribution contains TM^G, ([Fil83, Prop. 2.1.2]; also used earlier in Prop. 12.7).

If $m = n$ then V_m and V_n are both isomorphic to the standard representation V_1 of $SO(2)$. This $SO(2)$ action induces the structure of a complex vector space on $V_m \oplus V_n = ((T_pM)^G)^\perp$.

Since TM^G is 1-dimensional, there is a canonical (up to rescaling) G-invariant vector field v along TM^G, with zero divergence (Lemma 12.15). Fixing $p \in M$, choose $w \in T_{\mathbf{1}}G \hookrightarrow \operatorname{Vect} M$ such that $w(p) = v(p)$ (possible since $G \to M$ is a submersion). Since G acts by isometries, w satisfies the following.

- w is also divergence-free.
- $\exp tw$ preserves v, so $[v, w] = 0$.
- $\exp tw$ preserves the $(\exp tv)$-orbit of p, so it preserves G_p. Conjugation induces a map

$$t \in \mathbb{R} \to \operatorname{Aut} G_p \cong \{\pm 1\}.$$

  Since $\mathbb{R}$ is connected, this is the trivial homomorphism; so $\exp tw$ commutes with G_p.

Then $v - w$ is divergence-free, G_p-invariant, and zero at p; so $\exp t(v - w)$ is a G_p-equivariant operator on T_pM—i.e. a $\mathbb{C}$-linear operator on $((T_pM)^G)^\perp$. It has an eigenvector, whose span V_p (over $\mathbb{C}$) is G_p-invariant. Let V be the G-invariant distribution whose 2-plane at p is V_p.

Since V_p is an eigenspace of $\exp t(v - w)$ (thus of $v - w$) and V is G-invariant,

$$[v, V](p) \subseteq [v - w, V](p) + [w, V](p) \subseteq V_p + V_p = V_p.$$

The map $\Lambda^2 V \to T_pM/V$ from Lemma 12.8 lands in a trivial representation since $\Lambda^2 V$ is 1-dimensional; so $[V, V]_p \subseteq (T_pM)^G$. Therefore the G-invariant distribution of 3-planes

$TM^G \oplus V$ is integrable. As above, it contains TM^G so the resulting foliation has closed leaves.

Case (iii), Step 2: M fibers conformally over some space. Let $\mathcal{F}$ be the foliation tangent to the distribution produced in Step 1, and suppose M does not fiber conformally over $B = M/\mathcal{F}$—i.e. there is no metric on B with respect to which G acts by conformal automorphisms.

The subgroup $G_b \subset G$ fixing $b \in B$ acts on the tangent space $T_b B$ by some

$$G_b \to \mathrm{GL}(T_b B)) \cong \mathrm{GL}(2, \mathbb{R}).$$

Since $G_p \subseteq G_b$ and G preserves no conformal structure on B, there are two non-coincident copies of SO(2) acting on $T_b B$. Together they generate all of $\mathrm{SL}(2, \mathbb{R})$ (compare this to rotation groups around two distinct points of $\mathbb{H}^2$ generating all of $\mathrm{Isom}_0 \mathbb{H}^2$), so the semisimple part of G contains a subgroup covering $\mathrm{PSL}(2, \mathbb{R})$. Since G is 6-dimensional, the classification of simple Lie groups [Hel78, Ch. X, §6 (p. 516)], implies that G either surjects onto $\mathrm{PSL}(2, \mathbb{R})$ or covers $\mathrm{PSL}(2, \mathbb{C})$.

In both $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$, the maximal torus of the maximal compact subgroup is 1-dimensional; so all copies of S^1 are conjugate, and $G_p \cong S^1$ lands in some copy. Thus M fibers isometrically over (or is) $\mathrm{PSL}(2, \mathbb{R}) / \mathrm{PSO}(2) \cong \mathbb{H}^2$ or $\mathrm{PSL}(2, \mathbb{C}) / \mathrm{PSO}(2)$. The latter fibers conformally over S^2, as

$$S^2 \cong \mathrm{Conf}^+ S^2 / \mathrm{Conf}^+ \mathbb{E}^2 \cong \mathrm{PSL}(2, \mathbb{C}) / \mathrm{Conf}^+ \mathbb{E}^2.$$

Fibers are closed since the maps used are continuous, and $SO(2)$ is in the point stabilizers on the base since G surjects onto $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{C})$. □

Example 12.16. The necessity of Step 2 is demonstrated by the fibering

$$\mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}^2.$$

Since $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{R}^2$ with $\mathrm{SL}(2,\mathbb{R})$ point stabilizers, it cannot preserve any conformal structure. If Step 1 had produced this fibering, then Step 2 would find the conformal fibering

$$\mathbb{R}^2 \rtimes \mathrm{SO}(2) \to \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}) \to \mathbb{H}^2.$$

CHAPTER 13

GEOMETRIES FIBERING OVER 4D GEOMETRIES

This section proves part (i) of Theorem 10.1—that is,

Proposition 13.1 (Thm. 10.1(i))**.** *The* 5*-dimensional maximal model geometries* $M = G/G_p$ *for which the isotropy representation* $G_p \curvearrowright T_pM$ *contains an irreducible* 4*-dimensional summand are the product geometries*

$$S^4 \times \mathbb{E} \qquad \mathbb{H}^4 \times \mathbb{E} \qquad \mathbb{C}P^2 \times \mathbb{E} \qquad \mathbb{CH}^2 \times \mathbb{E}$$

and the homogeneous spaces

$$\widetilde{\mathrm{U}(2,1)/\,\mathrm{U}(2)} \qquad \mathrm{Heis}_5 = \mathrm{Heis}_5 \rtimes \mathrm{U}(2)/\,\mathrm{U}(2).$$

Our approach to proving Prop. 13.1 closely resembles the classification of 3-dimensional geometries with an irreducible 2-dimensional isotropy summand in [Thu97, Thm. 3.8.4(b)]. Recall that $B = M/\mathcal{F}^G$ is a 4-dimensional Riemannian symmetric space over which M fibers isometrically (Prop. 12.3). Curvature determines M once B is known. That is:

Proposition 13.2 (**Base and curvature determine the geometry**)**.** *A* 5*-dimensional maximal model geometry* $M = G/G_p$ *whose isotropy representation contains an irreducible* 4*-dimensional summand is determined by the following two pieces of information:*

1. *the geometry* $B = M/\mathcal{F}^G$*; and*

2. *whether* $(TM^G)^\perp$*, as a connection on the fiber bundle* $M \to B$*, has nonzero curvature.*

Moreover, if $G_p = \mathrm{SO}(4)$*, then* $(TM^G)^\perp$ *has zero curvature.*

This key fact, proven in section 13.1, reduces the classification problem to listing pairs (B, x) and checking whether each arises from a maximal model geometry. Section 13.2 carries this out, finding and then verifying the candidates listed in Table 13.3.

Table 13.3: Candidate geometries with irreducible 4-dimensional isotropy summand

Base	Flat (product)	Curved
S^4	$S^4 \times \mathbb{E}$	
$\mathbb{E}^4$	non-maximal $\mathbb{E}^5$	
$\mathbb{H}^4$	$\mathbb{H}^4 \times \mathbb{E}$	
$\mathbb{C}P^2$	$\mathbb{C}P^2 \times \mathbb{E}$	non-maximal S^5
$\mathbb{C}^2$	non-maximal $\mathbb{E}^5$	Heis_5
$\mathbb{CH}^2$	$\mathbb{CH}^2 \times \mathbb{E}$	$\widetilde{\mathrm{U}(2,1)/\,\mathrm{U}(2)}$

13.1 Reconstructing geometries from base and curvature information

This subsection proves Prop. 13.2 in two steps: one recovers a connection from its curvature using some general theory, and one normalizes any nonzero curvature to a single value. The latter interprets $(TM^G)_p^\perp$ as the quaternions $\mathbb{H}$ and of $\mathrm{SU}(2)$ as the unit quaternions in order to write down and work with the isotropy representation, as follows. Since the action

$$\widetilde{\mathrm{SO}(4)} \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \curvearrowright \mathbb{H}$$
$$(p, q)(x) = pxq^{-1}$$

descends to the standard representation of $\mathrm{SO}(4)$, all three of the isotropy representations with a 4-dimensional irreducible summand—$\mathrm{SU}(2) \subset \mathrm{U}(2) \subset \mathrm{SO}(4)$ (Fig. 11.4)—can be written using quaternion multiplication and subgroups of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Proof of Prop. 13.2. Let $M = G/G_p$ be a 5-dimensional maximal model geometry whose isotropy representation contains an irreducible 4-dimensional summand, and let $B = M/\mathcal{F}^G$. Then $M \to B$ is a G-invariant principal S^1- or $\mathbb{R}$-bundle, with vertical distribution TM^G and a natural connection (horizontal distribution) $(TM^G)^\perp$. The curvature is a G-invariant 2-form with values in the Lie algebra of S^1 or $\mathbb{R}$—that is, an element of $(\Omega^2 B \otimes \mathbb{R})^G \cong (\Lambda^2 (TM^G)^\perp_p)^{G_p}$.

Step 1: Nonzero curvature can be normalized to a single value. By counting weight vectors, $\Lambda^2(TM^G)^\perp_p \cong_{\mathrm{SU}(2)} 3\mathbb{R} \oplus \mathbb{R}^3$ (i.e. SU(2) acts trivially on $3\mathbb{R}$ and as $SO(3)$ on $\mathbb{R}^3$). One checks by inspection that the $3\mathbb{R}$ is spanned by

$$1 \wedge i - j \wedge k \qquad 1 \wedge j - k \wedge i \qquad 1 \wedge k - i \wedge j.$$

Under the action of conjugation by unit quaternions, $(TM^G)^\perp_p$ decomposes instead as $\mathbb{R} \oplus \mathbb{R}^3$; so its second exterior power decomposes as $2\mathbb{R}^3$. In fact one of these copies of $\mathbb{R}^3$ is the $3\mathbb{R}$ above, since conjugation by $1+i$ takes $1 \wedge j - k \wedge i$ to $1 \wedge k - i \wedge j$.

Then if $G_p \cong SU(2)$, applying some inner automorphism of $SU(2)$ makes the curvature a scalar multiple of $1 \wedge i - j \wedge k$; and the fibers can be rescaled to make the curvature either 0 or $1 \wedge i - j \wedge k$.

If instead $G_p \cong U(2)$, take the scalar factor to act as multiplication by $e^{i\theta}$ on the right. This acts on the span of $1 \wedge j - k \wedge i$ and $1 \wedge k - i \wedge j$ by rotation, so $1 \wedge i - j \wedge k$ lies in the only invariant direction.

Finally, if $G_p \cong SO(4)$, then the action of $SU(2)$ by conjugation of quaternions factors through G_p, so $\Lambda^2 \mathbb{R}^4$ contains no invariant directions—so the curvature is 0.

Step 2: Base and curvature determine a geometry. Let π be the projection $M \to B$. To x in a neighborhood U of p, assign the coordinates $(t, \pi(x))$ where the shortest path from $\pi(x)$ to $\pi(p)$ lifts to a horizontal path from x to $\phi_t(p)$. (For these coordinates to be well-defined, it suffices to have the radius of $\pi(U)$ at most the injectivity radius of B.)

If M and N have matching curvature and base, then choose $p \in M$ and $q \in N$ lying over the same point $b_0 \in B$; and define a map f from $U \ni p$ to $V \ni q$ by $(t, b) \mapsto (t, b)$. Since homogeneous spaces are analytic [KN63, Prop. I.4.2], and the isometry type of a complete, connected, simply-connected, analytic Riemannian manifold is determined by its local isometry type [KN63, Cor. VI.6.4], two geometries are isometric if they contain isometric open sets. Since a maximal geometry is determined by any invariant Riemannian metric [Fil83, Prop. 1.1.2], it suffices to check that f is an isometry.

Since f descends to the identity on B, and the metric on M is the direct sum of its restrictions to TM^G and $(TM^G)^\perp$, it suffices to check that f takes the horizontal distribution $(TM^G)^\perp$ on M to the horizontal distribution on N. (We'll say "f is horizontal".)

Since f takes horizontal lifts of geodesics through p to horizontal lifts of geodesics through q, it's horizontal at p. At points other than p, since M and N have matching curvature and base, it suffices to check that there is only one 4-plane distribution with the prescribed 4-plane at p and the prescribed curvature.

Let $\tilde{S}$ be a circular sector in $T_{b_0}B$ with its vertex at the origin, and let $S = \exp \tilde{S}$. The displacement along the fiber $\mathcal{F}_p$ incurred by traveling around a horizontal lift of ∂S is the integral of the curvature over S. So if s denotes the distance along the circular arc in ∂S, then computing $\frac{dt}{ds}$ in enough directions recovers the slope of the horizontal distribution relative to the coordinates (t, b).

□

13.2 Classifying and verifying geometries

Having established that the base and curvature determine a 5-dimensional maximal model geometry with irreducible 4-dimensional isotropy (Prop. 13.2), this subsection carries out the classification (i.e. the proof of Prop. 13.1) according to the plan outlined at the start of the section.

Proof of Thm. 10.1(i)/Prop. 13.1. Let $M = G/G_p$ be a 5-dimensional maximal model geometry whose isotropy representation contains an irreducible 4-dimensional summand. The base $M/\mathcal{F}^G$ and the curvature of $(TM^G)^\perp$ determine M (Prop. 13.2); so M occurs in Table 13.3, provided that the list of base spaces is exhaustive (Step 1) and that every entry has the claimed base and curvature (Step 2). Determining which candidates are maximal and model (Step 3) finishes the proof.

Step 1: Classify base spaces for Table 13.3. $M/\mathcal{F}^G$ is a Riemannian symmetric space (Prop. 12.3). The isotropy in M descends to irreducible isotropy in $M/\mathcal{F}^G$, so $M/\mathcal{F}^G$ is either irreducible or Euclidean. Consulting the classification of irreducible Riemannian symmetric spaces in [Hel78, X.6, p.515–518] yields the following non-Euclidean base spaces.

$$\mathbb{H}^4 \qquad S^4 \qquad \mathbb{C}P^2 \cong \mathrm{SU}(3)/\,\mathrm{S}(\mathrm{U}(2)\times \mathrm{U}(1)) \qquad \mathbb{CH}^2 \cong \mathrm{SU}(2,1)/\,\mathrm{S}(\mathrm{U}(2)\times \mathrm{U}(1))$$

If $M/\mathcal{F}^G$ is Euclidean, it may not be a maximal geometry—its isotropy may be some proper but irreducibly-acting subgroup of SO(4). Hence Table 13.3 also lists $\mathbb{C}^2$ as a base, to stand for Euclidean space with SU(2) or U(2) isotropy.

Step 2: Verify the candidates in Table 13.3. This step proves that every spot in Table 13.3 is occupied by a space with the correct base and curvature (or empty if no such space exists).

For the product spaces, there is nothing to prove. For the bases with SO(4) isotropy, no non-product geometries occur (Prop. 13.2). For the remaining bases, geometries with nonzero curvature of $(TM^G)^\perp$ are realized by the following.

- Over $\mathbb{C}P^2$ is its tautological circle bundle S^5, since $\mathbb{C}P^2$ is the quotient of $S^5 \subset \mathbb{C}^3$ by the action of norm-1 scalars. Its curvature is nonzero since a flat S^1-bundle over the simply-connected $\mathbb{C}P^2$ is the product bundle—which, unlike S^5, has nontrivial π_1.

- Over $\mathbb{C}^2$ is Heis_5, the 5-dimensional Heisenberg group, which can be written as the set $\mathbb{C}^2 \times \mathbb{R}$ with the product

$$(v_1, t_1)(v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \mathrm{Im}\,\langle v_1, v_2 \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian product. Dropping the $\mathbb{R}$ coordinate is a fibering over $\mathbb{C}^2$, for which an invariant connection with nonzero curvature is given by the kernel of the invariant contact form

$$\alpha_{(v,t)} = dt - \mathrm{Im}\,\langle v, dv \rangle \,.$$

- Over $\mathbb{C}\mathbb{H}^2 \cong U(2,1)/(U(2) \times U(1))$ is the line bundle $\widetilde{U(2,1)}/U(2)$. That its curvature is nonzero is Prop. 13.4 below.

Step 3: Determine maximal model geometries. Except when multiple Euclidean factors are involved, products of maximal model geometries are maximal model geometries (Prop. 12.12 and surrounding discussion); so it suffices to consider the three non-product geometries.

- S^5 as a circle bundle over $\mathbb{C}P^2$ is not maximal—it is a homogeneous space $\mathrm{SU}(3)/\mathrm{SU}(2)$ or $\mathrm{U}(3)/\mathrm{U}(2)$, both of which are subsumed by the geometry $\mathrm{SO}(6)/\mathrm{SO}(5)$.

- Heis_5 is a nilpotent Lie group. Its maximal realization is $\mathrm{Heis}_5 \rtimes K/K$ where $K \subset \mathrm{Aut}\,\mathrm{Heis}_5$ is maximal compact (Lemma 11.3). Since $\mathrm{Aut}\,\mathrm{Heis}_5$ must preserve $Z(\mathrm{Heis}_5)$, it is block triangular in some basis with blocks of size 4 and 1; so its maximal compact subgroup is conjugate to a subgroup of $\mathrm{SO}(4) \times \mathrm{SO}(1)$ (Lemma 9.29). Moreover, $\mathrm{Aut}\,\mathrm{Heis}_5$ has to preserve the antisymmetric pairing on $\mathrm{Heis}_5/Z(\mathrm{Heis}_5)$ used to define

Heis_5 in Step 2. Then

$$K \subseteq \mathrm{SO}(4) \cap \mathrm{Sp}(4, \mathbb{R}) = \mathrm{U}(2)$$

since SO(4) preserves the real part of a Hermitian form and $\mathrm{Sp}(4, \mathbb{R})$ preserves the imaginary part. Conversely, $K \supseteq \mathrm{U}(2)$ since U(2) preserves the Hermitian product used to define Heis_5. So $\mathrm{Heis}_5 \rtimes \mathrm{U}(2)/\mathrm{U}(2)$ is maximal.

- $\widetilde{\mathrm{U}(2,1)/\mathrm{U}(2)}$ is a model geometry since $\mathbb{CH}^2$ is—the deck group Γ of any compact $\Gamma\backslash\mathbb{CH}^2$ acts by elements of $\mathrm{SU}(2,1) \subset \mathrm{U}(2,1)$, so the circle bundle $\mathrm{U}(2,1)/\mathrm{U}(2)$ over $\mathbb{CH}^2$ descends to a circle bundle N over $\Gamma\backslash\mathbb{CH}^2$. To prove maximality it suffices to distinguish $\widetilde{\mathrm{U}(2,1)/\mathrm{U}(2)}$ from geometries with larger isotropy group; consulting Figure 11.4, these are just SO(5) and SO(4). Geometries $M = G/G_p$ with irreducible 4-dimensional isotropy are distinguished from each other by $M/\mathcal{F}^G$ and the curvature of $(TM^G)^\perp$ (Prop. 13.2), so only the constant-curvature spaces need to be checked.
  - $M = \widetilde{\mathrm{U}(2,1)/\mathrm{U}(2)}$ is not S^5 since M is contractible, being a line bundle over the contractible space $\mathbb{CH}^2$.
  - M is not $\mathbb{E}^5$ since $\mathrm{SU}(2,1)$ is semisimple and noncompact, whereas every semisimple subgroup of $\mathrm{Isom}_0\,\mathbb{E}^5$ is compact as its semisimple part is SO(5).
  - M is not $\mathbb{H}^5$: a circle bundle N over a compact $\mathbb{CH}^2$ manifold has quotients of arbitrarily small volume (by the scalar action of $e^{2\pi i/m}$ for large m); whereas for fixed $n \geq 4$, there is a minimum volume for hyperbolic n-manifolds by a theorem of Wang [BP92, Thm. E.3.2].

Hence the maximal model geometries in Table 13.3 are the products other than $\mathbb{E}^5$ and the two non-product geometries Heis_5 and $\widetilde{\mathrm{U}(2,1)/\mathrm{U}(2)}$. □

Proposition 13.4. *For the fibering of $M = \mathrm{U}(1,n)/\mathrm{U}(n)$ over $\mathbb{CH}^n$, the curvature of the connection $(TM^{\mathrm{U}(n)})^\perp$ is nonzero.*

Proof. Let $\{e_0, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^{n+1}$. Embed M in $\mathbb{C}^{n+1}$ as the preimage of 1 under the $(1, n)$ Hermitian form

$$\sum_{i=0}^{n} z_i e_i \mapsto |z_0|^2 - \sum_{i=1}^{n} |z_i|^2,$$

so that M fibers as a circle bundle over $\mathbb{CH}^2$ where the fibers are the intersections of M with complex lines.

Embed $U(n)$ in $U(1, n)$ as the copy of $U(n)$ fixing $e_0 \in \mathbb{C}^{n+1}$, and let V be the span of $\{e_1, \ldots, e_n\}$. Then $(TM^G)^{\perp}_{e_0} = V$; and we can find $(TM^G)^{\perp}$ elsewhere by translating by elements of $U(1, n)$—explicitly, for $p \in M \subset \mathbb{C}^{n+1}$,

$$(TM^G)^{\perp}_p = p^{\perp}.$$

If $v \in V$ has unit length, then there is a 1-parameter subgroup of $U(1, n)$ defined by

$$e_0 \mapsto (\cosh t)e_0 + (\sinh t)v$$
$$v \mapsto (\sinh t)e_0 + (\cosh t)v$$
$$w \mapsto w \text{ if } w \in V \text{ and } w \perp v.$$

Fix a small $t > 0$ and define a path $\gamma(\theta)$ in M by

$$\gamma(\theta) = (\cosh t)e^{i\theta(\tanh t)^2} e_0 + (\sinh t)e^{i\theta} e_1.$$

Then

$$\gamma'(\theta) = i(\tanh t)\left((\sinh t)e^{i\theta(\tanh t)^2} e_0 + (\cosh t)e^{i\theta} e_1\right)$$

so γ is horizontal. At $\theta = \frac{\pi}{1-(\tanh t)^2}$, both $\gamma(\theta)$ and $\gamma(0)$ are in the same fiber, but $\gamma(0) \neq \gamma(\theta)$. So horizontal lifts of closed paths in $\mathbb{CH}^n$ are not necessarily closed. $\square$

CHAPTER 14

GEOMETRIES FIBERING OVER 3D GEOMETRIES

This section proves part (ii) of Theorem 10.1—that is, the 5-dimensional maximal model geometries $M = G/G_p$ for which the isotropy representation $G_p \curvearrowright T_pM$ contains an irreducible 3-dimensional summand are products. Major ingredients of the proof include some reasoning about fiber bundles (set up in Chapter 12), Lie group extension problems, and some facts about transformation groups of spaces of constant curvature (postponed to Section 14.3).

While the classification seeks only model geometries, it does encounter one geometry that satisfies the weaker condition of having a unimodular isometry group. This geometry is the homogeneous space

$$\mathbb{R} \rtimes \operatorname{Conf}^+ \mathbb{E}^3 / \operatorname{SO}(3),$$

where the action on $\mathbb{R}$ is chosen to make $\mathbb{R} \rtimes \operatorname{Conf}^+ \mathbb{E}^3$ unimodular. Its failure to be a model geometry is proven using Galois theory; see Section 14.2 for details.

The proof splits into four parts with the following preparation. Since M is a model geometry, G must admit a lattice (Defn. 11.1), which requires G to be unimodular [Fil83, Prop. 1.1.3]. Under the assumption that $G_p \curvearrowright T_pM$ has an irreducible 3-dimensional summand, G_p contains a characteristic copy of SO(3) (Fig. 11.4). So the fibering $M \to B = M/\mathcal{F}^{\operatorname{SO}(3)}$ is conformal, with $B = S^3$, $\mathbb{E}^3$, or $\mathbb{H}^3$ (Prop. 12.3(ii)). Hence to prove Theorem 10.1(ii) it will suffice to show the following.

Proposition 14.1. *Let $M = G/G_p$ be a 5-dimensional maximal geometry where G is unimodular and $G_p = \operatorname{SO}(3)$ or $\operatorname{SO}(3) \times \operatorname{SO}(2)$; and let $B = M/\mathcal{F}^{\operatorname{SO}(3)}$.*

(i) If $\pi_B : M \to B$ is an isometric fibering, then M is a product of 2-dimensional geometries and 3-dimensional constant-curvature geometries.

(ii) The products in (i) that are maximal model geometries are those for which both factors have constant curvature and at least one factor is not Euclidean.

(iii) If π_B is essential, then $M = \mathbb{R} \rtimes \operatorname{Conf}^+ \mathbb{E}^3 / \operatorname{SO}(3)$, where $A \in \operatorname{Conf}^+ \mathbb{E}^3$ acts on $\mathbb{R}$ as dilation by $(\det A)^{-1}$.

(iv) $\mathbb{R} \rtimes \operatorname{Conf}^+ \mathbb{E}^3 / \operatorname{SO}(3)$ is not a model geometry.

The rest of this section is devoted to proving Prop. 14.1.

14.1 Geometries fibering isometrically are products

This section classifies the isometrically fibering geometries, in two steps as delineated by the first two parts of Prop. 14.1. The first is to show that all isometrically fibering geometries are products, and the second is to determine which products are maximal model geometries. Standard facts about groups acting on constant-curvature spaces will be stated where used, with references either to the literature or to proofs deferred to Section 14.3.

Proof of Prop. 14.1(i). Let $M = G/(\operatorname{SO}(3) \times \operatorname{SO}(2))$ or $M = G/\operatorname{SO}(3)$ be a 5-dimensional maximal geometry with G unimodular, and suppose $\pi_B : M \to B = M/\mathcal{F}^{\operatorname{SO}(3)}$ is isometric.

Step 1: $M \to B$ is a product bundle $F \times B$. The G_p-invariant integrability tensor (Lemma 12.8)

$$\Lambda^2 \left((TM^{\operatorname{SO}(3)})_p^{\perp} \right) \to T_p M^{\operatorname{SO}(3)}$$

is zero since the left side is the standard representation of SO(3) while the right side is a trivial representation; so $(TM^{\operatorname{SO}(3)})^{\perp}$ is a flat connection on $M \to B$. Since B is simply-connected and flat bundles are classified by the monodromy representation $\pi_1(B) \to \operatorname{Diff} F$, the fiber bundle $M \to B$ admits an isomorphism to a product bundle $F \times B$ taking $(TM^{\operatorname{SO}(3)})^{\perp}$ to the 3-planes tangent to copies of B. This produces a second projection $\pi_F : M \to F$.[1]

Step 2 (some general theory): It suffices to show that F has a G-invariant metric. If G-invariant metrics on F exist, they correspond one-to-one with invariant inner products on $TM^{\mathrm{SO}(3)}$, since both are determined by an inner product on a single 2-plane. The same holds for $(TM^{\mathrm{SO}(3)})^\perp$ and B. Then since $TM^{\mathrm{SO}(3)}$ and $(TM^{\mathrm{SO}(3)})^\perp$ are orthogonal in any invariant metric (Lemma 12.10), every invariant metric on M is a direct sum of invariant inner products on $TM^{\mathrm{SO}(3)}$ and $(TM^{\mathrm{SO}(3)})^\perp$. So if both π_B and π_F are both isometric fiberings, then any invariant metric on M is isometric to some invariant metric on $F \times B$.

Step 3 (an extension problem): Describe point stabilizers G_f of $G \curvearrowright F$. Let $G_f \subseteq G$ be the subgroup fixing a point $f \in F$. Identifying B with $\pi_F^{-1}(f)$, the image of G_f in $\operatorname{Diff} B$ is all of $\operatorname{Isom}_0 B$, since it contains $\mathrm{SO}(3)$ in the point stabilizers and G acts transitively. If $G_p \cong \mathrm{SO}(3)$, then counting dimensions shows that $G_f \cong \operatorname{Isom}_0 B$.

Otherwise, the kernel of $G_f \to \operatorname{Isom}_0 B$ is the $\mathrm{SO}(2)$ factor in G_p; so passing to Lie algebras, $T_{\mathbf{1}} G_f$ is an extension

$$0 \to \mathfrak{so}_2\mathbb{R} \to T_{\mathbf{1}} G_f \to \mathfrak{isom}\, B \to 0.$$

A 1-dimensional representation of $\mathfrak{isom}\, B$ must be trivial for $B = \mathbb{E}^3$, S^3, or $\mathbb{H}^3$ (Corollary 14.3(iii)), so $\mathfrak{isom}\, B$ acts trivially on $\mathfrak{so}_2\mathbb{R} \cong \mathbb{R}$. Then since $H^2(T_{\mathbf{1}} \operatorname{Isom}_0 B; \mathbb{R}) = 0$ (Lemma 14.5), the classification of Lie algebra extensions by second cohomology (Thm. 11.6) implies G_f is covered by the product $\mathbb{R} \times \widetilde{\operatorname{Isom}_0} B$.

1. This alone does not make M a product geometry, as demonstrated in dimension 3 by the (non-model) geometry $\operatorname{Conf}^+ \mathbb{E}^2 / \mathrm{SO}(2)$; so there still remains something nontrivial to prove.

Step 4 (a study of transformation groups): G_f preserves a metric on F. The action of $G_f \subseteq G$ on F defines a homomorphism

$$\phi : \widetilde{G_f} \to \operatorname{Aut} T_f F \cong \mathrm{GL}(2, \mathbb{R}).$$

Since $\widetilde{\operatorname{Isom}_0} B$ has no quotients of dimension 1 or 2 (Corollary 14.3(i)), it acts with determinant 1 on $\mathbb{R}^2$—and thus as a subgroup of $\mathrm{SL}(2, \mathbb{R})$ of dimension 0 or 3. In fact $\widetilde{\operatorname{Isom}_0} B$ does not admit $\mathrm{SL}(2, \mathbb{R})$ as a quotient, since:

1. proper quotients of $\widetilde{\operatorname{Isom}_0} \mathbb{E}^3$ factor through $\mathrm{SU}(2)$ (Lemma 14.2);
2. $\widetilde{\operatorname{Isom}_0} S^3 \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$; and
3. $\operatorname{Isom}_0 \mathbb{H}^3 = \mathrm{SO}(3, 1)$ is simple.

Therefore $\phi\left(\widetilde{\operatorname{Isom}_0} B\right) = \{1\}$. If $G_p \cong \mathrm{SO}(3)$, then this means ϕ has trivial image. If $G_p \cong \mathrm{SO}(3) \times \mathrm{SO}(2)$, then this implies ϕ factors through the $\mathrm{SO}(2) \subset G_f$ covered by the $\mathbb{R}$ factor in $\widetilde{G_f} \cong \mathbb{R} \times \widetilde{\operatorname{Isom}_0} B$. Either way, the action of G_f on $T_f F$ factors through a compact group, so it preserves an inner product on $T_f F$. Then since G acts transitively, it preserves a metric on F, which finishes the proof by Step 2. □

Completing the classification of geometries fibering isometrically over 3-dimensional constant curvature geometries requires determining which of the products $F \times B$ from (i) are actually maximal model geometries. So far, B is known to be $\mathbb{E}^3$, S^3, or $\mathbb{H}^3$, while F is just a geometry—a manifold with a smooth transitive group action with compact point stabilizers. The first step will be to determine all possibilities for F; then general statements such as the maximality of most products (Prop. 12.12) will finish the proof.

Proof of Prop. 14.1(ii) (determining maximal model geometries). All 2-dimensional geometries with $\mathrm{SO}(2)$ isotropy have constant curvature; and those with trivial isotropy are the two simply-connected real Lie groups in dimension 2, namely $\mathbb{R}^2$ and $\mathrm{Aff}^+ \mathbb{R}$. Since $\mathbb{R}^2$ is a non-maximal $\mathbb{E}^2$ and $\mathrm{Aff}^+ \mathbb{R}$ is a non-maximal $\mathbb{H}^2$,[2] the only maximal $F \times B$ from part (i) are those where both factors are maximal model constant-curvature geometries. Since $\mathbb{E}^2 \times \mathbb{E}^3$ is a non-maximal $\mathbb{E}^5$, only those products with at most one Euclidean factor remain.

Conversely, suppose $F \times B$ is a product of maximal model constant-curvature geometries of dimensions 2 and 3, at most one of which is Euclidean. Then $F \times B$ is a model geometry since products are models (Prop. 11.2); and F, having $\mathrm{SO}(2)$ isotropy, has no invariant vector fields, so $F \times B$ is maximal (Prop. 12.12). □

14.2 Case study: The geometry fibering essentially conformally

Let $M = G/G_p$ be a 5-dimensional maximal geometry with G unimodular, and suppose the fibering $\pi_B : M \to B = M/\mathcal{F}^{\mathrm{SO}(3)}$ is essentially conformal. This section proves Prop. 14.1(iii)–(iv): that $M = \mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3 / \mathrm{SO}(3)$ and that M is not a model geometry.

Proof of Prop. 14.1(iii). We will determine M by solving the following extension problem to find G.

$$0 \to \mathbb{R} \to T_{\mathbf{1}} G \to T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3 \to 0$$

Such an extension is determined by a homomorphism $\phi : T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3 \to \mathrm{Der}\, \mathbb{R}$ (realized by lifting to $\mathfrak{g}'$ and taking brackets, where $\mathrm{Der}\, \mathbb{R}$ denotes the algebra of derivations of $\mathbb{R}$) and a class in $H^2(T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3; \mathbb{R})$.

Step 1: G is an abelian extension of $\mathrm{Conf}^+ \mathbb{E}^3$. In an essential fibering, B is $\mathbb{E}^3$ or S^3 (since $\mathrm{Conf}^+ \mathbb{H}^3 = \mathrm{Isom}_0 \mathbb{H}^3$ [BP92, Thm. A.4.1]). Since the only connected, transitive, essential subgroup of $\mathrm{Conf}^+ B$ containing $\mathrm{SO}(3)$ in the point stabilizers is the entire group (Lemma 14.4), the image of G in $\mathrm{Diff}\, B$ is either $\mathrm{Conf}^+ \mathbb{E}^3 \cong \mathbb{R}^3 \rtimes (SO(3) \times \mathbb{R})$ (7-

2. Take $(x, y) \in \mathbb{R} \rtimes \mathbb{R}$ to the upper-half plane by $(x, y) \mapsto (x, e^y)$.

dimensional) or $\mathrm{Conf}^+ S^3 \cong \mathrm{Isom}_0 \mathbb{H}^4$ (10-dimensional). Since G_p is SO(3) or SO(3)×SO(2), the dimension of G is 8 or 9; so $B = \mathbb{E}^3$, and G is an extension of $\mathrm{Conf}^+ \mathbb{E}^3$ by a unimodular group H of dimension 1 or 2.

Step 2: G_p is SO(3). As a G_p-representation, $T_{\mathbf{1}}G \cong T_{\mathbf{1}}G_p \oplus T_pM$. Since $T_{\mathbf{1}}H \subset T_{\mathbf{1}}G$ is an ideal, it is also a subrepresentation. Then letting V denote the standard representation of SO(3) and $\mathbb{R}$ the trivial representation,

$$T_{\mathbf{1}}G \cong_{\mathrm{SO}(3)} T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3 \oplus T_{\mathbf{1}}H \cong_{\mathrm{SO}(3)} 2V \oplus \mathbb{R} \oplus T_{\mathbf{1}}H.$$

The $\mathbb{R}$ is tangent to a group of dilations in $\mathrm{Conf}^+ \mathbb{E}^3$; and the two copies of V are $\mathfrak{so}_3\mathbb{R} \subset T_{\mathbf{1}}G_p$ and the 3-dimensional subspace of T_pM on which it acts.

If $G_p = \mathrm{SO}(3) \times \mathrm{SO}(2)$, then $\mathbb{R} + T_{\mathbf{1}}H$ must consist of a 2-dimensional subspace of T_pM and the $\mathfrak{so}_2\mathbb{R}$ acting on it. Then since $T_{\mathbf{1}}H$ is an abelian ideal of $T_{\mathbf{1}}G$, $\mathbb{R} + T_{\mathbf{1}}H$ is a subalgebra of $T_{\mathbf{1}}G$ isomorphic to $T_{\mathbf{1}} \mathrm{Isom}_0 \mathbb{E}^2$ with the $\mathbb{R}$ corresponding to $\mathfrak{so}_2\mathbb{R}$. This cannot occur since this $\mathbb{R}$ is the tangent algebra to a group of dilations in $\mathrm{Conf}^+ \mathbb{E}^3$, while $\mathfrak{so}_2\mathbb{R}$ is the tangent algebra to the compact group $\mathrm{SO}(2) \subset G_p$. Hence $G_p = \mathrm{SO}(3)$, and $\dim H = 1$; so $T_{\mathbf{1}}G$ is an extension

$$0 \to \mathbb{R} \to T_{\mathbf{1}}G \to T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3 \to 0.$$

Step 3: Unimodularity determines the action of $\mathrm{Conf}^+ \mathbb{E}^3$ on H. The above extension problem induces, via Lie brackets, an action $\phi : T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3 \to \mathrm{Der}\, \mathbb{R}$. Every ideal in $T_{\mathbf{1}} \mathrm{Conf}^+ \mathbb{E}^3 \cong \mathbb{R}^3 \ni (\mathfrak{so}_3 \oplus \mathbb{R})$ either:

- is contained in $\mathbb{R}^3$, in which case it's 0 or all of $\mathbb{R}^3$ since $\mathfrak{so}_3$ acts irreducibly; or
- contains some v projecting nontrivially to $\mathfrak{so}_3 \oplus \mathbb{R}$, in which case it still contains $\mathbb{R}^3$ since $[v, \mathbb{R}^3]$ is a nonzero subspace of $\mathbb{R}^3$.

Since $\mathfrak{so}_3$ is simple, the ideals of $T_{\mathbf{1}}\operatorname{Conf}^+\mathbb{E}^3$ are 0, $\mathbb{R}^3$, $\mathbb{R}^3\rtimes\mathfrak{so}_3$, $\mathbb{R}^3\rtimes\mathbb{R}$, and $T_{\mathbf{1}}\operatorname{Conf}^+\mathbb{E}^3$, corresponding to the quotients $T_{\mathbf{1}}\operatorname{Conf}^+\mathbb{E}^3$, $\mathfrak{so}_3\oplus\mathbb{R}$, $\mathbb{R}$, $\mathfrak{so}_3$, and 0. Of these, only $\mathbb{R}$ and 0 can occur as images in $\operatorname{Der}\mathbb{R}\cong\mathbb{R}$; so ϕ is either zero or of the form

$$\mathbb{R}^3\rtimes(\mathfrak{so}_3\mathbb{R}\oplus\mathbb{R})\ni(v,r,s)\mapsto ks$$

for some $k\in\mathbb{R}$. Since $\operatorname{ad}(0,0,1)$ is diagonal with three 1s and four 0s, unimodularity of G requires $k=-3$. Then G is an extension

$$1\to\mathbb{R}\to G\to\operatorname{Conf}^+\mathbb{E}^3\to 1$$

where $A\in\operatorname{Conf}^+\mathbb{E}^3$ acts on $\mathbb{R}$ as dilation by $(\det A)^{-1}$. ($H\cong\mathbb{R}$ since S^1 admits no dilations.)

Step 4: The extension splits—i.e. G is the semidirect product. A spanning set for $T_{\mathbf{1}}\operatorname{Conf}^+\mathbb{E}^3$ is given by translations $t_i\in\mathbb{R}^3$ $(1\le i\le 3)$, rotations $r_{ij}\in\mathfrak{so}_3$ ($[r_{ij},t_i]=t_j=[-r_{ji},t_i]$), and a scaling $s\in\mathbb{R}$. Of these, only s acts nontrivially; so given a 2-cocycle c, we can subtract a coboundary to make its restriction to $\mathbb{R}^3\rtimes\mathfrak{so}_3$ zero using Lemma 14.5.

Then when we apply the cocycle condition to $c(t_i,s)=c(r_{ij},t_j],s)$ and $c(r_{ij},s)=c([r_{ik},r_{kj}],s)$, we obtain a sum of terms of the following forms where the blanks are in $\mathbb{R}^3\rtimes\mathfrak{so}_3$.

- $sc(*,*)$, which is zero since $c|_{\mathbb{R}^3\rtimes\mathfrak{so}_3}=0$;
- $*c(*,s)$, which is zero since $\mathbb{R}^3\rtimes\mathfrak{so}_3$ acts trivially; and
- $c(*,[*,s])$, which is zero since $[s,T_{\mathbf{1}}\operatorname{Conf}^+\mathbb{E}^3]=\mathbb{R}^3$.

Thus $c=0$, so $H^2=0$.

By the classification of abelian extensions by second cohomology (Thm. 11.6), the Lie algebra extension splits. Then G is covered by $\mathbb{R} \rtimes \widetilde{\mathrm{Conf}^+ \mathbb{E}^3}$, whose center (consisting of the elements lying over the identity in $\mathrm{SO}(3) \subset \mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3$) has order 2. So G is either $\mathbb{R} \rtimes \widetilde{\mathrm{Conf}^+ \mathbb{E}^3}$ or $\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3$; these deformation retract to their maximal compact subgroups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, respectively. Since G must contain $G_p \cong \mathrm{SO}(3)$ and all maximal compact subgroups are conjugate, the geometry M is $\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3 / \mathrm{SO}(3)$. □

To show that $\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3 / \mathrm{SO}(3)$ is not a model geometry, it suffices to show that $\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3$ admits no lattice.

Proof of Prop. 14.1(iv) ($\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3 / \mathrm{SO}(3)$ is not a model geometry). Suppose there existed a lattice

$$\Gamma \subset G = \mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3 \cong (\mathbb{R} \times \mathbb{R}^3) \rtimes (\mathrm{SO}(3) \times \mathbb{R}).$$

Step 1: By general theory, produce an integer matrix A. Recall the following facts.

1. For a connected Lie group in which every compact semisimple subgroup acts nontrivially on the solvable radical, the intersection of the nilradical with a lattice is a lattice in the nilradical [Mos71, Lemma 3.9].

2. For a closed normal subgroup H of G, the group $\Gamma/(\Gamma \cap H)$ is a lattice in G/H if and only if $\Gamma \cap H$ is a lattice in H [OV00, Thm. I.1.4.7].

The nilradical N of G is $\mathbb{R} \times \mathbb{R}^3$; so $\Gamma \cap N \cong \mathbb{Z}^4$, and $\Gamma/(\Gamma \cap N)$ is a lattice in $G/N \cong \mathrm{SO}(3) \times \mathbb{R}$. Then some $g \in \Gamma$ projects nontrivially to the $\mathbb{R}$ factor (the dilation part) in $\mathrm{SO}(3) \times \mathbb{R}$. This g acts by conjugation on $\Gamma \cap N$ as an integer matrix A acting on $\mathbb{Z}^4$.

Step 2: A has a real eigenvalue $\lambda \neq \pm 1$. Since G is unimodular, $\det A = 1$. Then the action of $\mathrm{SO}(3) \times \mathbb{R}$ on $\mathbb{R} \times \mathbb{R}^3$ requires the eigenvalues to be of the form λ, $\lambda e^{i\theta}$, $\lambda e^{-i\theta}$, and λ^{-3} where $\lambda \in \mathbb{R}$. Since Step 1 selected A to act with nontrivial dilation, λ is not 1 or -1.

Step 3: Using Galois theory, conclude $\lambda = \pm 1$. Since λ is real and not ± 1 (and not 0 as A is invertible), the magnitudes of λ, λ^{-3}, and λ^9 are all distinct; so A and A^{-3} share only the eigenvalue λ^{-3}. Since each eigenvalue of an integer matrix must occur with all of its Galois conjugates, $\lambda^{-3} \in \mathbb{Q}$. Applying the rational root theorem to the characteristic polynomial of A implies $\lambda^{-3} = \pm 1$. Then $\lambda = \pm 1$, which contradicts the conclusion of Step 2. □

14.3 Groups acting on constant-curvature spaces

This subsection collects some facts used above, concerning groups acting on spaces of constant curvature. First, the classification required some results about low-dimensional representations of $\operatorname{Isom}_0 \mathbb{E}^n$. Their proof begins with the following observation, which will be reused later to classify some extensions of $\mathfrak{isom}\,\mathbb{E}^2$.

Lemma 14.2. *Every nonzero ideal of $\mathfrak{isom}\,\mathbb{E}^n \cong \mathbb{R}^n \rtimes \mathfrak{so}_n$ contains the translation subalgebra $\mathbb{R}^n$.*

Proof. An ideal $\mathfrak{n}$ of $\mathfrak{isom}\,\mathbb{E}^n$ containing no nonzero elements of $\mathbb{R}^n$ acts trivially on $\mathbb{R}^n$—since $\mathbb{R}^n$ is also an ideal,

$$[\mathfrak{n}, \mathbb{R}^n] \subseteq \mathfrak{n} \cap \mathbb{R}^n = 0.$$

Since $\mathbb{R}^n$ is a faithful representation of $\mathfrak{so}_n$, any ideal containing no nonzero elements of $\mathbb{R}^n$ is zero.

Since $\mathbb{R}^n$ is an irreducible representation of $\mathfrak{so}_n$, any ideal that does contain some nonzero $v \in \mathbb{R}^n$ also contains $[\mathfrak{so}_n, v] \cong \mathbb{R}^n$. □

Since $\mathrm{SO}(k)$ ($k \neq 4$) is simple, Lemma 14.2 above implies that for $n \neq 4$, the connected normal subgroups of $\operatorname{Isom}_0 \mathbb{E}^n$ are $\{1\}$, $\mathbb{R}^n$, and $\operatorname{Isom}_0 \mathbb{E}^n$. Since both $\operatorname{Isom}_0 S^n \cong \mathrm{SO}(n+1)$ and $\operatorname{Isom}_0 \mathbb{H}^n \cong \mathrm{SO}(n,1)$ are simple, statments about quotients and representations of the isometry groups can be made for all of $\mathbb{E}^n$, S^n, and $\mathbb{H}^n$ at once, as follows.

Corollary 14.3. *Let* $G = \widetilde{\mathrm{Isom}_0 M}$ *where* M *is* $\mathbb{E}^n$*,* S^n*, or* $\mathbb{H}^n$*.*

(i) If $n \geq 3$*, then* G *has no quotient groups of dimension* 1 *or* 2*.*

(ii) If $n > 4$*, then* G *has no nontrivial quotient groups of dimension less than* $\binom{n}{2}$*.*

(iii) All 1*-dimensional real representations of* $T_{\mathbf{1}}G$ *are trivial.*

The remainder of this section's facts are only used in the above classification of geometries fibering over $\mathbb{E}^3$, S^3, and $\mathbb{H}^3$—starting with the following classification of sufficiently large groups acting by conformal automorphisms on S^k and $\mathbb{E}^k$.

Lemma 14.4. *Let* $B = \mathbb{E}^k$ *or* S^k *for some* $k \geq 2$*. The only connected, transitive, essential subgroup* H *of* $\mathrm{Conf}^+ B$ *with a copy of* $SO(k)$ *in its point stabilizer* H_p *is* $\mathrm{Conf}^+ B$*.*

Proof. The two cases are handled separately but with broadly the same strategy: the goal is to show that H_p is the entirety of a point stabilizer of $\mathrm{Conf}^+ B$.

Case 1: $B = \mathbb{E}^k$**.** A point stabilizer of $\mathrm{Conf}^+ \mathbb{E}^k \cong \mathbb{R}^k \rtimes (\mathrm{SO}(k) \times \mathbb{R})$ [BP92, Thm. A.3.7] is $\mathrm{SO}(k) \times \mathbb{R}$. Since H acts essentially, $H_p \subset \mathrm{SO}(k) \times \mathbb{R}$ projects nontrivially to the $\mathbb{R}$ factor. The homotopy exact sequence for $H_p \to H \to B$, along with the assumption that B is simply-connected and H is connected, implies H_p is connected; so $H_p = \mathrm{SO}(k) \times \mathbb{R}$. Then H has the same point stabilizers as $\mathrm{Conf}^+ \mathbb{E}^k$, so $H = \mathrm{Conf}^+ \mathbb{E}^k$.

Case 2, preparatory claim: No transitive subgroup $H \subset \mathrm{Conf}^+ S^k$ **acts on** S^k **with point stabilizers** $H_p \cong \mathrm{SO}(k) \times \mathbb{R}$**.** A point stabilizer of $\mathrm{Conf}^+ S^k \curvearrowright S^k$ is $\mathrm{Conf}^+ \mathbb{E}^k$ [BP92, Cor. A.3.8]. Up to conjugacy, H_p is the standard $\mathrm{SO}(k) \times \mathbb{R} \subset \mathrm{Conf}^+ \mathbb{E}^k$, since $\mathrm{SO}(k)$ is maximal compact and $\mathbb{R}$ is its centralizer. This $\mathrm{SO}(k) \times \mathbb{R}$ fixes *two* points on S^k, whose stabilizers in H coincide since point stabilizers of a transitive action are isomorphic. Then H preserves a pairing of points in S^k. In particular, if $\mathrm{SO}(k) \times \mathbb{R}$ preserves p and q then it acts transitively on $S^k \smallsetminus \{p, q\}$ while preserving this pairing. So paired points

1. lie in the same S^{k-1} in $S^k \smallsetminus \{p,q\} \cong S^{k-1} \times \mathbb{R}$ since they are exchanged by an order-2 element; and

2. are antipodal in this S^{k-1} since the $\mathrm{SO}(k-1)$ fixing one member of the pair must fix the other.

Interpret S^k as the boundary at infinity of $\mathbb{H}^{k+1}$, following [BP92, Prop. A.5.13(4)]. Any two geodesics in $\mathbb{H}^{k+1}$ joining paired points of S^k must intersect, since H acts transitively and they all intersect the geodesic joining p and q. However, only geodesics whose endpoints lie in the same $S^{k-1} \subset S^k \smallsetminus \{p,q\}$ can intersect since each S^{k-1} bounds a totally geodesic $\mathbb{H}^k \subset \mathbb{H}^{k+1}$. Therefore the pairing of points required by an action with $\mathrm{SO}(k) \times \mathbb{R}$ stabilizers cannot be defined on all of S^k.

Case 2: $B = S^k$. Since H acts essentially, it preserves no Riemannian metric; so some point stabilizer H_p preserves no inner product on the tangent space at p. So the quotient map $\pi : \mathrm{Conf}^+ \mathbb{E}^k \to \mathrm{SO}(k) \times \mathbb{R}$ is surjective when restricted to H_p. Since $H_p \ncong \mathrm{SO}(k) \times \mathbb{R}$, it cannot also be injective. Then H_p meets the translation subgroup $\ker \pi \cong \mathbb{R}^k \subset \mathrm{Conf}^+ \mathbb{E}^k$ nontrivially; so it contains all of $\mathbb{R}^k$ since $\mathrm{SO}(k) \curvearrowright \mathbb{R}^k$ is irreducible. Then $H_p = \mathrm{Conf}^+ \mathbb{E}^k$, which implies as in Case 1 that $H = \mathrm{Conf}^+ S^k$. □

Finally, the classification of geometries fibering over 3-dimensional isotropy-irreducible geometries required the following computation of second cohomology.

Lemma 14.5. *$H^2(\mathfrak{isom}\, M; \mathbb{R}) = 0$ for $M = \mathbb{E}^3$, S^3, or $\mathbb{H}^3$.*

Proof. This is a computation using a spanning set, though for S^3 and $\mathbb{H}^3$ one could instead appeal to the vanishing of H^2 for semisimple algebras [GOV94, Thm. 1.3.2]. The spanning elements of $\mathfrak{isom}\, M$ will be denoted r_{ij} and t_i ($1 \le i, j \le n = \dim M$); the linear dependency relations are $r_{ij} = -r_{ji}$, and the nonzero brackets are

$$\begin{aligned} [r_{ij}, r_{jk}] &= r_{ki} \quad && \text{if } i, j, k \text{ distinct} \\ [r_{ij}, t_i] &= t_j && \text{if } i \neq j \\ [t_i, t_j] &= K r_{ij} && \text{if } i \neq j, \end{aligned}$$

where K is 0 or ± 1 (the sectional curvature of M).

The action of $\mathfrak{isom}\, M$ on $\mathbb{R}$ is trivial (Corollary 14.3(iii)); so the cocycle condition for a 2-cocycle $c : \Lambda^2 \mathfrak{isom}\, M \to \mathbb{R}$ becomes

$$c([x_1, x_2], x_3) + c([x_2, x_3], x_1) + c([x_3, x_1], x_2) = 0,$$

and the coboundaries are of the form

$$c(x_1, x_2) = f([x_1, x_2]), \quad f \in (\mathfrak{isom}\, M)^*.$$

Suppose c is a 2-cocycle, and i, j, and k are distinct. Applying the cocycle condition to the spanning elements yields

$$\begin{aligned} c(r_{ij}, t_k) &= c([r_{jk}, r_{ki}], t_k) \\ &= c(r_{jk}, t_i) + c(r_{ki}, t_j) \\ c(r_{ij}, t_i) &= c([r_{jk}, r_{ki}], t_i) \\ &= c(r_{kj}, t_k) \\ K c(r_{jk}, r_{ki}) &= c(K r_{jk}, r_{ki}) \\ &= c([t_j, t_k], r_{ki}) \\ &= c(t_i, t_j). \end{aligned}$$

A linear combination of cyclic permutations of the first equality is $c(r_{ij}, t_k) = 3c(r_{ij}, t_k)$, so $c(r_{ij}, t_k) = 0$. The second equality implies that $c(r_{ij}, t_i)$ only depends on i. Since $n = 3$, distinctness of i, j, and k implies $c(r_{jk}, r_{ki})$ only depends on i and j, which permits defining[3]

$$\begin{aligned} f : \mathfrak{isom}\, M &\to \mathbb{R} \\ r_{ij} &\mapsto c(r_{jk}, r_{ki}) \\ t_i &\mapsto c(r_{ji}, t_j). \end{aligned}$$

This definition and the third equality imply $c(x, y) = f([x, y])$ for all x and y in the spanning set, and therefore on all of $\mathfrak{isom}\, M$. So every cocycle c is a coboundary, i.e. $H^2(\mathfrak{isom}\, M; \mathbb{R}) = 0$. □

3. Independence of $c(r_{jk}, r_{ki})$ from k can be proven for $n > 3$ by computing $c([r_{k\ell}, r_{\ell j}], r_{ki})$ and using the $n = 3$, $K = 1$ case to show that $c(r_{ij}, r_{k\ell}) = 0$ for distinct i, j, k, ℓ. This extends Lemma 14.5 to dimensions other than 3.

CHAPTER 15

GEOMETRIES FIBERING OVER 2D GEOMETRIES

This section carries out the (unfortunately long) task of proving part (iii) of Theorem 10.1. That is, it classifies the 5-dimensional maximal model geometries $M = G/G_p$ in case (iii) of the fibering description (Prop. 12.3)—those for which the irreducible subrepresentations of $G_p \curvearrowright T_pM$ have dimensions 1 and 2. The first step is to set up an extension problem that can be solved to find G.

Lemma 15.1 (**The isometry group as an extension**). *If $M = G/G_p$ is a 5-dimensional model geometry for which $G_p \curvearrowright T_pM$ decomposes into 1-dimensional and 2-dimensional summands, then G is an extension*

$$1 \to H \to G \to Q \to 1$$

where:

(i) Q *is* $\mathrm{Conf}^+ S^2$, $\mathrm{Conf}^+ \mathbb{E}^2$, $\mathrm{Isom}_0 S^2$, $\mathrm{Isom}_0 \mathbb{E}^2$, *or* $\mathrm{Isom}_0 \mathbb{H}^2$;

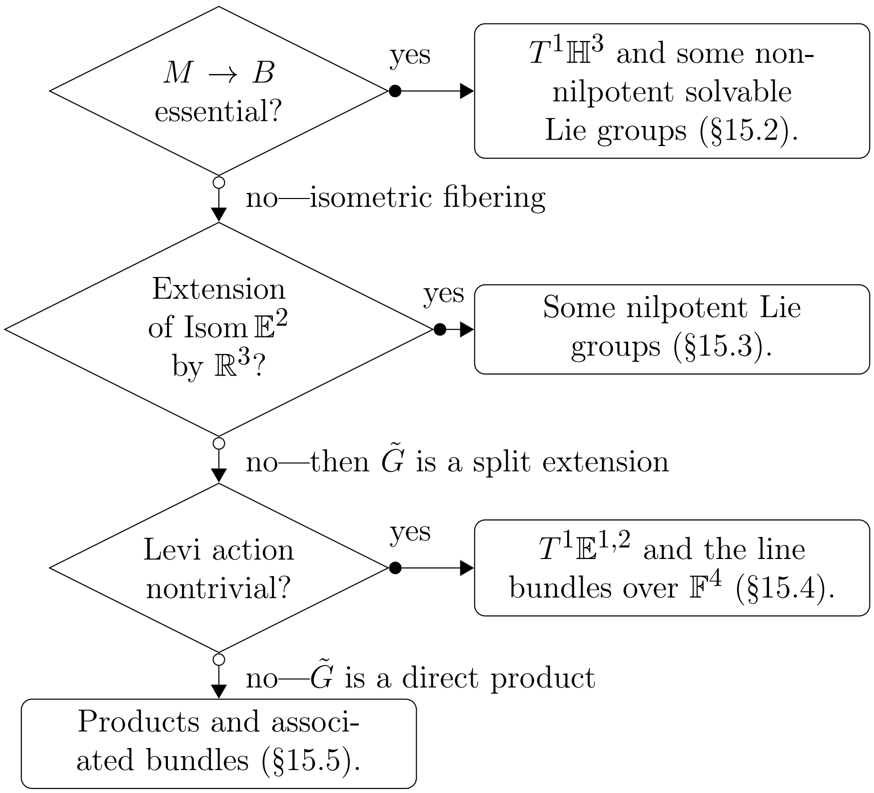
(ii) if $G_p \cong S^1$, *then the identity component of* H *is covered by* $\mathbb{R}$, $\mathbb{R}^2$, S^3, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Sol^3, $\widetilde{\mathrm{Isom}_0}\, \mathbb{E}^2$, Heis_3, *or* $\mathbb{R}^3$; *and*

(iii) if $G_p = \mathrm{SO}(2) \times \mathrm{SO}(2)$, *then the identity component of* H *is either:*

- *one of* $SO(3)$, $\mathrm{PSL}(2,\mathbb{R})$, *and* $\mathrm{Isom}_0 \mathbb{E}^2$; *or*
- *covered by* $S^3 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathbb{R}$, $\widetilde{\mathrm{Isom}_0}\, \mathbb{E}^2 \times \mathbb{R}$, *or* $\widetilde{\mathrm{Isom}_0}\, \mathrm{Heis}_3$.

To organize the solution of what could be fifty extension problems, the classification proceeds by skimming off classes of geometries until only those with $\tilde{G} = \tilde{H} \times \tilde{Q}$ remain; these either are product geometries or can be described as associated bundles (Section 15.5.1). The plan is illustrated in Figure 15.2.

Figure 15.2: Classification strategy for geometries fibering over 2-D spaces.



15.1 Setting up the extension problem

This section proves Lemma 15.1, which describes the extension problem that will be solved to determine the transformation groups G of the geometries G/G_p. Having obtained a fibering over a 2-dimensional space in Prop. 12.3, the bulk of the proof is in establishing the lists of quotients Q and kernels H. One lemma is needed, in the form of the following observation about geometries with abelian isotropy—which will also be useful later in recovering a faithfully-acting G from its universal cover $\tilde{G}$.

Lemma 15.3. *If $M = G/G_p$ be a connected homogeneous space where G is connected and G_p is compact and abelian, then the following are equivalent.*

(i) G acts faithfully on M (one of the requirements for a geometry)

(ii) G_p *acts faithfully on* T_pM

(iii) G_p *acts faithfully by conjugation on* G

Proof. There are two equivalences to verify.

(i) $\iff$ **(ii):** Since G_p is compact, M has an invariant Riemannian metric [Thu97, Prop. 3.4.11]. An isometry of a connected Riemannian manifold is determined by its value and derivative at a point [BP92, Prop. A.2.1], so the action of G_p on M is determined by the action of G_p on T_pM. Then a nontrivial $g \in G$ acts as the identity on M if and only if it lies in G_p and acts as the identity on T_pM; so G acts faithfully on M if and only if G_p acts faithfully on T_pM.

(ii) $\iff$ **(iii):** As a G_p-representation, the Lie algebra of G decomposes as

$$T_{\mathbf{1}}G = T_{\mathbf{1}}G_p \oplus T_pM.$$

Since G_p is abelian, $T_{\mathbf{1}}G_p$ is trivial; so $T_{\mathbf{1}}G$ under the adjoint action T_pM is a faithful G_p-representation if and only if $T_{\mathbf{1}}G$ under the adjoint action is too. The equivalence for the conjugation action on G follows since a homomorphism from a connected Lie group is determined by its derivative at the identity. □

Remark 15.4. Condition (iii) is equivalent to $G_p \cap Z(G) = \{1\}$. So an abelian-isotropy homogeneous space G/G_p satisfying all the conditions for a geometry except faithfulness of the G-action can be made into a geometry by passing from G to $G/(G_p \cap Z(G))$.

Proof of Lemma 15.1 (the extension problem for G). Let $M = G/G_p$ be a 5-dimensional model geometry for which $G_p \curvearrowright T_pM$ decomposes into 1-dimensional and 2-dimensional summands. Then there is a conformal fibering (Prop. 12.3(iii)) $M \to B$ where B is S^2, $\mathbb{E}^2$, or $\mathbb{H}^2$. If Q denotes the image of G in $\operatorname{Conf} B$, then G is an extension

$$1 \to H \to G \to Q \to 1.$$

Step 1: The image Q is $\operatorname{Isom}_0 B$ or $\operatorname{Conf}^+ B$. If $G_p = \mathrm{SO}(2)$, then $B = M/\mathcal{F}^{\mathrm{SO}(2)}$ so Q contains a copy of SO(2). Otherwise, the trivial subrepresentation of $G_p \curvearrowright T_pM$ is 1-dimensional, so G_p acts nontrivially on B. In either case, Q acts transitively on B and contains a copy of SO(2) fixing a point.

Then $Q = \operatorname{Isom}_0 B$ if B admits a G-invariant metric; otherwise, Q is a connected, transitive, essential subgroup of $\operatorname{Conf}^+ B$ and must therefore be all of $\operatorname{Conf}^+ B$ (Lemma 14.4). The list in the original statement (Lemma 15.1) omits $\operatorname{Conf}^+ \mathbb{H}^2$ since $\operatorname{Conf}^+ \mathbb{H}^2 = \operatorname{Isom}_0 \mathbb{H}^2$ [BP92, Thm. A.4.1].

Step 2: If $G_p \cong S^1$, then H is unimodular of dimension at most 3. If $G_p \cong S^1$, then $\dim G = 6$. Since $\dim Q \geq 3$ by the previous step, $\dim H \leq 3$. Since G is unimodular [Fil83, Prop. 1.1.3] and H is a closed normal subgroup, H is itself unimodular [Fil83, Prop. 1.1.4]. The explicit list in Lemma 15.1(ii) can be found by consulting Bianchi's classification of all 3-dimensional real Lie algebras; see e.g. [FH91, Lec. 10], [PSWZ76, Table I], or [Mac99, Table 21.3].[1]

Step 3: If $G_p = \mathrm{SO}(2) \times \mathrm{SO}(2)$, then some $\mathrm{SO}(2) \subset H$ acts faithfully by conjugation. For each Q in Step 1, the maximal compact subgroup has rank 1; so $H \cap G_p$ is a closed subgroup of dimension at least 1. Since G acts with SO(2) stabilizers on B, $H \cap G_p$ has dimension at most 1. Therefore the identity component of $H \cap G_p$ is isomorphic to SO(2).

1. Alternatively, to carry out Bianchi's classification from scratch, notice that the only two real semisimple Lie algebras of dimension at most 3 are $\mathfrak{so}_3\mathbb{R}$ and $\mathfrak{sl}_2\mathbb{R}$. The other 3-dimensional real Lie algebras are therefore solvable. A nilradical in a solvable Lie algebra is at least half the dimension [GOV94, Thm. 2.5.2, attributed to Mubarakzyanov], so these algebras are of the form $\mathbb{R}^2 \rtimes \mathbb{R}$, which can be systematically handled using Jordan forms.

Since G_p is a point stabilizer of a geometry, it acts faithfully on $T_{\mathbf{1}}G$ (Lemma 15.3). Since $H \cap G_p \subseteq H$, it acts trivially on $T_{\mathbf{1}}Q \cong_{G_p} T_{\mathbf{1}}G/T_{\mathbf{1}}H$; so its action on $T_{\mathbf{1}}H$—and therefore H—must be faithful.

Step 4: Classify possible H when $G_p = \mathrm{SO}(2) \times \mathrm{SO}(2)$. First, $\dim H \leq 4$ since $\dim Q \geq 3$ and $\dim G = \dim M + \dim(\mathrm{SO}(2) \times \mathrm{SO}(2)) = 5 + 2 = 7$. If $\dim H \leq 3$, applying the restriction from Step 3 to the list from Step 2 yields the 3-dimensional groups H in Lemma 15.1(iii).

The remainder of the groups occur if $\dim H = 4$. Then $H/(H \cap G_p)$ is a 3-dimensional homogeneous space, with $H \cap G_p = SO(2)$ point stabilizers by Step 3 and unimodular isometry group H. The proof of [Thu97, Thm. 3.8.4(b)] (classifying 3-dimensional geometries with $SO(2)$ point stabilizer), finds the spaces listed in Table 15.5. To obtain the final list in Lemma 15.1(iii), eliminiate duplicates by observing that semidirect products with inner action are isogenous with direct products. □

Table 15.5: 3-dimensional simply-connected homogeneous spaces $H/\,\mathrm{SO}(2)$ with unimodular H

$H/(H \cap G_p)$	H
S^3	$S^3 \rtimes S^1$
$\widetilde{\mathrm{PSL}(2,\mathbb{R})}$	$\widetilde{\mathrm{PSL}(2,\mathbb{R})} \rtimes \mathrm{SO}(2)$
Heis_3	$\mathrm{Heis}_3 \rtimes \mathrm{SO}(2)$
$S^2 \times \mathbb{R}$	$\mathrm{SO}(3) \times \mathbb{R}$
$\mathbb{H}^2 \times \mathbb{R}$	$\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$
$\widetilde{\mathrm{Isom}_0 \mathbb{E}^2}$	$\widetilde{\mathrm{Isom}_0 \mathbb{E}^2} \rtimes \mathrm{SO}(2)$

Remark 15.6. The list of 4-dimensional groups in Step 4 may also be obtained by computing with a convenient basis in the Lie algebra: let r generate $H \cap G_p$, let x and y span the nontrivial $(H \cap G_p)$-subrepresentation of $T_{\mathbf{1}}H$, and let z (together with r) span the trivial subrepresentation. Then one can work out the values of $[x, y]$, $[x, z]$, and $[y, z]$ that satisfy the Jacobi identity (and rescale the basis if it helps).

15.2 Geometries fibering essentially

This section handles the first side branch of Figure 15.2: proving the following classification of geometries that fiber essentially over 2-dimensional spaces.

Proposition 15.7. *Suppose $M = G/G_p$ is a 5-dimensional model geometry for which $G_p \curvearrowright T_pM$ decomposes into 1-dimensional and 2-dimensional summands. Furthermore suppose that the fibering $M \to B$ from Prop. 12.3(iii) is essential.*

(i) If $B = S^2$, then $M = T^1\mathbb{H}^3 \cong \mathrm{PSL}(2,\mathbb{C})/\mathrm{PSO}(2)$, with isotropy $G_p = S^1_1$ in the notation of Figure 11.4.

(ii) $T^1\mathbb{H}^3$ is a maximal model geometry.

(iii) If $B = \mathbb{E}^2$, then M is a solvable Lie group of the form $\mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}$. Moreover, M is maximal if and only if the multiset of characteristic polynomials of the Jordan blocks of A is one of the following.

(a) $\{x-1,\ x-1,\ x+1,\ x+1\}$

(b) $\{(x-1)^2,\ x+1,\ x+1\}$

(c) $\{x-1,\ x-1,\ x,\ x+2\}$ *(This is $\mathrm{Sol}^4_0 \times \mathbb{E}$.)*

(d) $\{x-1,\ x-1,\ x-a+1,\ x+a+1\}$; $a > 0$, $a \neq 1$, $a \neq 2$ *(This is a family of geometries.)*

(iv) The geometries in (iii).(a)–(c) are model geometries; and (iii).(d) is a model geometry if and only if e^{tA} has a characteristic polynomial in $\mathbb{Z}[x]$ for some $t > 0$.

Remark 15.8. In (iii), the proof of maximality will reveal that the isotropy consists of an SO(2) acting by automorphisms on M for every pair of identical Jordan blocks in A.

The case when $B = S^2$ is considerably easier than that of $B = \mathbb{E}^2$, which has some complexities not immediately apparent in the above statement; we treat these cases separately.

15.2.1 Over the sphere

This section contains the proof that $T^1\mathbb{H}^3$ is a maximal model geometry (ii), and that it is the only one fibering essentially conformally over the sphere (i). The extension problem from Lemma 15.1 is used only to determine that the isometry group covers $\mathrm{PSL}(2,\mathbb{C})$; the rest of the proof is built on properties of $\mathrm{PSL}(2,\mathbb{C})$.

Proof of Prop. 15.7(i). If $M = G/G_p$ fibers essentially over $B = S^2$, then G is an extension

$$1 \to H \to G \to \mathrm{Conf}^+ S^2 \to 1$$

where H is as named in Lemma 15.1. Since $\dim \mathrm{Conf}^+ S^2 = 6$, either $G_p \cong S^1$ and $\dim H = 0$ or $G_p = \mathrm{SO}(2)^2$ and $\dim H = 1$. The list of possible H for $\mathrm{SO}(2)^2$ includes no 1-dimensional entries; so $\dim H = 0$, $G_p \cong S^1$, and G covers Q.

As maximal tori in maximal compact subgroups, all copies of S^1 in $\mathrm{Conf}^+ S^2 \cong \mathrm{PSL}(2,\mathbb{C})$ are conjugate, as are all copies of S^1 in the 2-sheeted universal cover $\mathrm{SL}(2,\mathbb{C})$. Hence $M \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SO}(2) \cong \mathrm{PSL}(2,\mathbb{C})/\mathrm{PSO}(2)$. Since $\mathrm{PSL}(2,\mathbb{C})$ is centerless, the geometry M expressed with faithful transformation group (Rmk. 15.4) is indeed $\mathrm{PSL}(2,\mathbb{C})/\mathrm{PSO}(2)$.

Since $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}_0\,\mathbb{H}^3$, choosing a basepoint in $T^1\mathbb{H}^3$ identifies $\mathrm{PSL}(2,\mathbb{C})/\mathrm{PSO}(2)$ with $T^1\mathbb{H}^3$ (hence the name). The point stabilizers have slope 1 because PSO(2) rotates $\mathbb{H}^3$ and a tangent space the same way (or one can explicitly decompose $\mathfrak{sl}_2\mathbb{C}$ into S^1-representations). □

Proof of Prop. 15.7(ii). $T^1\mathbb{H}^3$ is a model geometry since it models the unit tangent bundle of any finite-volume hyperbolic 3-manifold.

For maximality, suppose G were a larger connected group of isometries for $T^1\mathbb{H}^3$ under some metric.

If $\dim G = 7$, then $G_p = \mathrm{SO}(2)^2$. From the classification of simple Lie groups [Hel78, Ch. X, §6 (p. 516)], the only connected semisimple Lie group containing $\mathrm{PSL}(2,\mathbb{C})$ of dimension at most 7 is $\mathrm{PSL}(2,\mathbb{C})$. Then using the Levi decomposition [GOV94, §1.4] and the fact that $\mathrm{PSL}(2,\mathbb{C})$ is centerless, G admits $\mathrm{PSL}(2,\mathbb{C})$ as a quotient. Since $\mathrm{PSO}(2)$ is a maximal torus in $\mathrm{PSL}(2,\mathbb{C})$, it contains the image of G_p. Then G/G_p fibers essentially over $S^2 \cong \mathrm{Conf}^+ S^2 / \mathrm{Conf}^+ \mathbb{E}^2$, which by part (i) is incompatible with $\dim G = 7$.

If $\dim G > 7$, then $\dim G_p > 2$, and previous sections already listed maximal geometries with $\dim G_p > 2$. Of those, only $S^2 \times \mathbb{E}^3$ and $S^2 \times \mathbb{H}^3$ have the same diffeomorphism type. Since $\mathrm{PSL}(2,\mathbb{C})$ admits no nontrivial image in $\mathrm{SO}(3) = \mathrm{Isom}_0 S^2$ (both are simple and the domain has larger dimension), it cannot act transitively by isometries on either of these products. □

15.2.2 Over the plane

Suppose $M = G/G_p$ fibers essentially over $\mathbb{E}^2$. The description of G as an extension (Lemma 15.1) is

$$1 \to H \to G \to \mathrm{Conf}^+ \mathbb{E}^2 \to 1$$

where, since $\mathrm{Conf}^+ \mathbb{E}^2$ is 4-dimensional, H is $\mathbb{R}^2$, $SO(3)$, $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{Isom}_0 \mathbb{E}^2$. This section classifies the resulting geometries (Prop. 15.7(iii)–(iv)), all of which will be solvable Lie groups of the form $\mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}$. The geometry named in Prop. 15.7(iii).(a) occurs when $H = \mathrm{Isom}_0 \mathbb{E}^2$; the rest will come from $H = \mathbb{R}^2$.

Passing to Lie algebras, we aim to solve the corresponding extension problem

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{q} \to 0.$$

The proof relies on the following three computations: the outer derivation algebra of $\mathfrak{h}$ (Lemma 15.9), the action $\mathfrak{q} \to \operatorname{out} \mathfrak{h}$ (Lemma 15.10), and the cohomology $H^2(\mathfrak{q}; \mathbb{R}^2)$ with the action in Lemma 15.10 (Lemma 15.11). For the Lie algebra $\mathfrak{q}$ of $\operatorname{Conf}^+ \mathbb{E}^2$, we will use the basis $\{x, y, r, s\}$ where x and y generate the translations, r generates rotations ($[r, x] = y$), and s generates scaling ($[s, x] = x$).

Lemma 15.9 (**Outer derivation algebras**). *The Lie algebras of the above groups H have the following algebras of outer derivations:*

$$\operatorname{out} \mathfrak{so}_3 = 0 \qquad \operatorname{out} \mathbb{R}^2 = \mathfrak{gl}_2\mathbb{R}$$
$$\operatorname{out} \mathfrak{sl}_2 = 0 \qquad \operatorname{out} \left(\mathfrak{isom}\, \mathbb{E}^2\right) = \mathbb{R}$$

Proof. $\mathbb{R}^2$ is abelian, and the outer derivation algebra of a semisimple Lie algebra is zero [Hum72, §5.3], so only the last one needs any computation.

Let $\{x, y, r\}$ be a basis for $\mathfrak{isom}\, \mathbb{E}^2$ where $[r, x] = y$ and x and y generate the translations. For a derivation d, the Leibniz rule $d[v, w] = [dv, w] + [v, dw]$ implies that d preserves the lower central series and the derived series—so in this case it takes translations to translations. Subtract inner derivations to ensure $dr = ar$ and $dx = bx$ for some a and b; then

$$dy = [dr, x] + [r, dx] = ay + by$$
$$bx = dx = [dy, r] + [y, dr] = (2a + b)x.$$

The second line implies $a = 0$; then d is zero on r and scales by b on the translations. □

Lemma 15.10 (**Restrictions on actions on** H)**.** *In the action of* $\mathfrak{q} = T_1 \operatorname{Conf}^+ \mathbb{E}^2$ *on* $\mathfrak{h}$*, s acts with trace -2, x and y act trivially, and r generates a compact subgroup of* $\operatorname{Out} \mathfrak{h}$*.*

In particular, if $H = \mathbb{R}^2$, then up to conjugacy in $\mathrm{GL}(2, \mathbb{R}) = \operatorname{Aut} \mathfrak{h}$*, s acts by one of the following matrices where a is a real parameter.*

$$\begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1+a & 0 \\ 0 & -1-a \end{pmatrix}$$

And if r acts nontrivially by rotations (or some conjugate thereof), then only the first of these can commute with r.

Proof. There are three claims to prove in the first sentence of the Lemma statement:

1. Since s acts with trace 2 on $\mathfrak{q}$ and $\mathfrak{g}$ is unimodular, s must act with trace -2 on $\mathfrak{h}$.

2. The restriction of $\mathfrak{q} \to \operatorname{out} \mathfrak{h}$ to $\mathfrak{isom}\, \mathbb{E}^2$ is either injective or contains the translation ideal (Lemma 14.2). Of the outer derivation algebras (Lemma 15.9), only $\mathfrak{gl}_2 \cong \mathfrak{sl}_2 \oplus \mathbb{R}$ has high enough dimension to contain an injective image of $\mathfrak{isom}\, \mathbb{E}^2$. For the kernel not to contain the translation ideal, the projection to at least one of the summands must be injective; but $\mathbb{R}$ has too low dimension, and $\mathfrak{sl}_2$ contains no subalgebras of dimension 3 other than itself. Hence $\mathfrak{isom}\, \mathbb{E}^2 \to \operatorname{out} \mathfrak{h}$ always factors through $\mathfrak{so}_2$—i.e. x and y act trivially.

3. Since the Lie algebra extension is induced by a Lie group extension, the homomorphism $\mathfrak{q} \to \operatorname{out} \mathfrak{h}$ is induced by some $\operatorname{Conf}^+ \mathbb{E}^2 = Q \to \operatorname{Out} H$, which sends the compact $\mathrm{SO}(2) \subset \operatorname{Conf}^+ \mathbb{E}^2$ generated by r to a compact subgroup of $\operatorname{Out} H \subseteq \operatorname{Out} \mathfrak{h}$.

The claims when $H = \mathbb{R}^2$ then follow from listing 2×2 Jordan forms with trace -2 and the fact that the centralizer of $\mathrm{SO}(2) \curvearrowright \mathbb{R}^2$ is generated by itself and the real scalars. □

Since $\mathbb{R}^2$ is abelian, $\operatorname{out} \mathbb{R}^2 = \operatorname{der} \mathbb{R}^2$; so the only additional data needed for $H = \mathbb{R}^2$ is second cohomology.

Lemma 15.11. *The Lie algebra cohomology* $H^2(T_{\mathbf{1}}\operatorname{Conf}^+\mathbb{E}^2;\mathbb{R}^2)$ *has the following values.*

- *If* s *acts with an eigenvalue* 2*, then* H^2 *is* 1*-dimensional, represented by cocycles* c *with* $c(x,y)$ *in the* 2*-eigenspace.*
- *If* s *acts with an eigenvalue* 0*, then* H^2 *is* 1*-dimensional, represented by cocycles* c *with* $c(r,s)$ *in the* 0*-eigenspace.*
- *Otherwise,* $H^2 = 0$*.*

Proof. If β is a 1-cochain, then the coboundary $d\beta$ has values

$$\begin{aligned}
d\beta(x,y) &= x\beta(y) - y\beta(x) - \beta([x,y]) = 0\\
d\beta(r,x) &= r\beta(x) - x\beta(r) - \beta([r,x]) = r\beta(x) - \beta(y)\\
d\beta(r,y) &= r\beta(y) - y\beta(r) - \beta([r,y]) = r\beta(y) + \beta(x)\\
d\beta(s,x) &= s\beta(x) - x\beta(s) - \beta([s,x]) = (s-1)\beta(x)\\
d\beta(s,y) &= s\beta(y) - y\beta(s) - \beta([s,y]) = (s-1)\beta(y)\\
d\beta(r,s) &= r\beta(s) - s\beta(r) - \beta([s,r]) = r\beta(s) - s\beta(r).
\end{aligned}$$

If c is a cocycle, then

$$\begin{aligned}
c(x,y) &= -c(x,[y,s])\\
&= c(y,[s,x]) + c(s,[x,y]) + xc(y,s) + yc(s,x) + sc(x,y)\\
&= c(y,x) + 0 + 0 + 0 + sc(x,y)\\
2c(x,y) &= sc(x,y).
\end{aligned}$$

Thus either $c(x,y) = 0$ or s acts with an eigenvalue 2. Similarly, applying the cocycle condition to each of the equalities

$$
\begin{aligned}
c(x,s) &= -c(s,[y,r]) \\
c(y,s) &= -c(s,[r,x]) \\
c(x,y) &= -c(x,[x,r]) \\
c(x,y) &= -c(y,[y,r]) \\
c(r,x) &= -c(r,[x,s]) \\
c(r,y) &= -c(r,[y,s])
\end{aligned}
$$

yields mostly vacuous equalities, except for the following.

$$
\begin{aligned}
c(x,s) &= -rc(y,s) + (1-s)c(r,y) \\
c(y,s) &= rc(x,s) + (s-1)c(r,x)
\end{aligned}
\tag{15.1}
$$

Using this information, we will define β to match $d\beta$ as closely as possible with c.

- If r acts as 0, then set $\beta(x) = c(r,y)$ and $\beta(y) = -c(r,x)$. Then $c - d\beta$ is zero on $r \wedge x$, $r \wedge y$, $x \wedge s$, and $y \wedge s$ (by equations 15.1).
  - If s acts with eigenvalues 0 and -2, then $c(x,y) = 0$. Set $\beta(r) = \frac{1}{2}c(r,s)$, so that the only potentially nonzero value of $c - d\beta$ is on $r \wedge s$ and lies in the 0-eigenspace of s.
  - If s acts with eigenvalues 2 and -4, setting $\beta(r) = -s^{-1}c(r,s)$ makes $c - d\beta$ zero on $r \wedge s$. The only nonzero contribution to H^2 is from $c(x,y)$ lying in the 2-eigenspace of s.
  - Otherwise, setting $\beta(r) = -s^{-1}c(r,s)$ makes $c - d\beta = 0$, so $H^2 = 0$.
- If r acts by rotation, then since s commutes with it, we reinterpret $\mathbb{R}^2$ as $\mathbb{C}$ on which

r acts by ik for some real $k \neq 0$. Defining

$$z_s = c(x,s) + ic(y,s)$$
$$z_r = c(r,x) + ic(r,y)$$
$$w_s = c(x,s) - ic(y,s)$$
$$w_r = c(r,x) - ic(r,y),$$

equations 15.1 become

$$z_s = -kz_s + i(s-1)z_r$$
$$w_s = kw_s - i(s-1)w_r.$$

Since $s \neq 1$ (by having to act with trace -2), $\beta(x)$ and $\beta(y)$ can be selected to make $d\beta$ reproduce z_s and w_s; then z_r and w_r follow dependently.

We further set $\beta(r) = 0$ and $\beta(s) = r^{-1}c(r,s)$ to make $(c - d\beta)(r,s) = 0$. Finally, $c(x,y) = 0$ since s has no eigenvalue 2 (both its eigenvalues have real part 1).

□

Proof of Prop. 15.7(iii). The proof begins by determining the possibilities for H.

Step 1: H is $\operatorname{Isom}_0 \mathbb{E}^2$ or $\mathbb{R}^2$. The outer derivation algebras for the Lie algebras of SO(3) and PSL$(2,\mathbb{R})$ are zero [GOV94, Cor. to Thm. 1.3.2]; and being unimodular, they have no inner derivations acting with trace -2. Since $T_{\mathbf{1}} \operatorname{Conf}^+ \mathbb{E}^2$ contains an element acting on $\mathfrak{h}$ with trace -2 (Lemma 15.10), this rules out SO(3) and PSL$(2,\mathbb{R})$ as candidates for H. Then G is an extension of the solvable group $\operatorname{Conf}^+ \mathbb{E}^2$ by either $\mathbb{R}^2$ or $\operatorname{Isom}_0 \mathbb{E}^2$.

Step 2: If $H = \operatorname{Isom}_0 \mathbb{E}^2$, then $M = \left[(\mathbb{R}^4 \rtimes \mathbb{R}) \rtimes \mathrm{SO}(2)^2\right] / \mathrm{SO}(2)^2$. Since $Z(\mathfrak{isom}\, \mathbb{E}^2) = 0$, the Lie algebra cohomology determining extensions is identically 0; so every homomorphism

$$T_{\mathbf{1}} \operatorname{Conf}^+ \mathbb{E}^2 \to \operatorname{out} T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2 \cong \mathbb{R}$$

can be realized by an extension, and every such extension splits on the Lie algebra level. Since r has to generate a compact subgroup and the nonzero part of $\operatorname{out} T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2$ scales the translation subalgebra, r maps to 0. Since s acts with trace -2, it maps to the scalar -1.

Then all of the resulting group extensions can be covered by the semidirect product

$$(\mathbb{C}^2 \rtimes \mathbb{R}) \rtimes (S^1 \times S^1),$$

where the first $\mathbb{R}$ acts by scaling the two copies of $\mathbb{C}$ by reciprocal factors, and the $S^1 \times S^1$ acts on $\mathbb{C}^2$ through the inclusion of S^1 as the unit circle.

(Instead of $\mathbb{R} \times \mathbb{R}$ we have $S^1 \times S^1$—the first S^1 is in $\operatorname{Isom}_0 \mathbb{E}^2 \subset G$, and the second S^1 is due to the fact that $SO(2) \subset \operatorname{Conf}^+ \mathbb{E}^2$ acts trivially on $\operatorname{Isom}_0 \mathbb{E}^2$, which allows the extension to split on the Lie group level too.)

Since this has trivial center, this group *is* G, and $G_p \cong \mathrm{SO}(2)^2$ is any maximal compact subgroup.

Step 3: Non-split extensions of $\operatorname{Conf}^+ \mathbb{E}^2$ by $\mathbb{R}^2$ produce no model geometries. The non-split extensions are with r acting trivially, and with s acting diagonally with an eigenvalue of 0 or 2. We'll show that if 0 is an eigenvalue of s, then $SO(2)$ fails to extend to a compact point stabilizer; and if 2 is an eigenvalue of s, then G fails to admit lattices.

When s acts with an eigenvalue of 0, a non-split extension is given by a nonzero value of $c(r, s)$ in the 0-eigenspace; i.e. $[r, s]$ is nonzero in $\mathbb{R}^2 \subset \mathfrak{g}$. Then conjugation by $\{\exp tr\}_{t \in \mathbb{R}} \subset G$ acts by sending $\exp us$ ($u \in \mathbb{R}$) to $tuc(r, s)$. The result is that $\{\exp tr\}_{t \in \mathbb{R}}$ cannot be compact, in G or any group covered by it. Since $[\mathbb{R}^2, s]$ lies in the -2-eigenspace and is thus independent from $c(r, s)$, this holds for any group surjecting to $SO(2) \subset \operatorname{Conf}^+ \mathbb{E}^2$. So

$SO(2) \subset \mathrm{Conf}^+ \mathbb{E}^2$ fails to lift to a compact point stabilizer in G.

When s acts with an eigenvalue of 2, a non-split extension is given by a nonzero value of $c(x, y)$ in the 2-eigenspace. So in $\mathfrak{g}$, the elements x and y generate a copy of the Heisenberg algebra. Inspecting the actions of the other basis elements shows that this is an ideal, and in fact $\tilde{G}$ can be written as

$$(\mathrm{Heis}_3 \times \mathbb{R}) \rtimes (\mathbb{R} \times \mathbb{R}),$$

where the first $\mathbb{R}$ rotates the xy-plane of the Heisenberg group and the second $\mathbb{R}$ scales diagonally with exponents 1, 1, 2, and -4.

Since $\mathbb{R} \times \mathbb{R}$ acts non-nilpotently, the nilradical of this is $\mathrm{Heis}_3 \times \mathbb{R}$. Since G is solvable, a theorem of Mostow (see [Fil83, Prop 6.4.2]) ensures that any lattice Γ in its universal cover $\tilde{G}$ intersects the nilradical in a lattice and projects to the quotient as a closed cocompact group. Then since $[\mathrm{Heis}_3, \mathrm{Heis}_3] = Z(\mathrm{Heis}_3)$ and $Z(\mathrm{Heis}_3)$ is a copy of $\mathbb{R}$, cocompactness of Γ ensures $[\Gamma, \Gamma] \cap Z(\mathrm{Heis}_3)$ is nontrivial. Since Γ has cocompact image in $\mathbb{R} \times \mathbb{R}$, some $g \in \Gamma$ acts by nontrivial scaling on $\mathrm{Heis}_3 \times \mathbb{R}$; in particular, this action on $Z(\mathrm{Heis}_3)$ implies $[\Gamma, \Gamma] \cap Z(\mathrm{Heis}_3)$ is not discrete; so Γ fails to be a lattice.

Step 4: Semidirect products where s acts by a complex scalar produce non-maximal geometries. These are the extensions where s acts as $\begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix}$ for some real a. We can express the resulting $\tilde{G}$ as the semidirect product

$$(\mathbb{C}^2 \rtimes \mathbb{R}) \rtimes \mathbb{R}$$

where the second $\mathbb{R}$ acts via the unit circle on one or both coordinates of $\mathbb{C}^2$, and each element of the first $\mathbb{R}$ acts on the first coordinate by some complex scalar z and on the second coordinate by $|z|^{-1}$. Similarly to the reasoning in Step 3, to have a compact group we can use as the point stabilizer, the second $\mathbb{R}$ must intersect the center nontrivially; so we have G covered by

$$(\mathbb{C}^2 \rtimes \mathbb{R}) \rtimes S^1$$

where the S^1 action on each $\mathbb{C}$ may have some degree other than 1. Since this has trivial center, this actually is G (and G_p is its maximal compact subgroup S^1). Whatever geometry it produces is subsumed by the homogeneous space

$$[(\mathbb{C}^2 \rtimes \mathbb{R}) \rtimes T^2]/T^2$$

where T^2 acts via the unit circle on each coordinate of $\mathbb{C}^2$, and $t \in \mathbb{R}$ acts on $\mathbb{C}^2$ by the diagonal matrix with eigenvalues e^t and e^{-t}.

Step 5: Identify the geometries that remain. Similarly to Step 4, the remaining groups G are all described as the semidirect product

$$((\mathbb{R}^4) \rtimes \mathbb{R}) \rtimes S^1$$

(again, to obtain G from the universal cover, we replaced a second $\mathbb{R}$ factor by S^1 and noted that the result has trivial center), where $t \in \mathbb{R}$ acts by one of the matrices below (omitted entries are zero) and S^1 acts as $\mathrm{SO}(2)$ on the last two coordinates.

$$\begin{pmatrix} e^t & e^t & & \\ & e^t & & \\ & & e^{-t} & \\ & & & e^{-t} \end{pmatrix} \qquad \begin{pmatrix} e^{(1+a)t} & & & \\ & e^{(1-a)t} & & \\ & & e^{-t} & \\ & & & e^{-t} \end{pmatrix}$$

Step 4 eliminates $a = 0$; and since $S^1 \subset G$ is maximal compact, $M \cong \mathbb{R}^4 \rtimes \mathbb{R}$ where $t \in \mathbb{R}$ acts by one of the above matrices. The case $a = 1$ is $\mathrm{Sol}_0^4 \times \mathbb{E}$; and the case $a = 2$ is a non-maximal form of $\mathbb{R} \rtimes \mathrm{Conf}^+ \mathbb{E}^3/\,\mathrm{SO}(3)$ (Section 14.2).

Step 6: Maximality. The maximal geometry realizing a solvable Lie group M with real roots is of the form $M \rtimes K/K$, where $K \subseteq \operatorname{Aut} M$ is maximal compact (Lemma 11.3).

So for these $M = \mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}$, it suffices to determine the maximal compact subgroup of $\operatorname{Aut} M \cong \operatorname{Aut} T_{\mathbf{1}} M$. An automorphism of the Lie algebra $T_{\mathbf{1}} M$ must preserve its nilradical $\mathbb{R}^4$, each of the generalized eigenspaces by which $\mathbb{R} \cong T_{\mathbf{1}} M/\mathbb{R}^4$ acts on $\mathbb{R}^4$, and the filtration by rank on each generalized eigenspace. So in coordinates the matrices in $\operatorname{Aut} T_{\mathbf{1}} M$ are block upper-triangular, and the maximal compact subgroup $K \subset \operatorname{Aut} T_{\mathbf{1}} M$ is conjugate into a group whose only nonzero entries are in the diagonal blocks (Part II, Lemma 9.29). Since K matches the dimension of the isotropy groups computed above in the classification, we conclude that the maximal geometries are

$$\begin{aligned} &\underset{(x-1)^2,\, x+1,\, x+1}{\mathbb{R}^4 \rtimes \mathbb{R}} \rtimes \mathrm{SO}(2)/\,\mathrm{SO}(2) \\ &\underset{x-1,\, x-1,\, x+1,\, x+1}{\mathbb{R}^4 \rtimes \mathbb{R}} \rtimes \mathrm{SO}(2)^2/\,\mathrm{SO}(2)^2 \\ &\underset{x-1,\, x-1,\, x-a+1,\, x+a+1}{\mathbb{R}^4 \rtimes \mathbb{R}} \rtimes \mathrm{SO}(2)/\,\mathrm{SO}(2). \end{aligned} \qquad \square$$

Proof of Prop. 15.7(iv) (model geometries). Let $M = G/G_p \cong \mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}$ be a geometry named in Prop. 15.7. The proof splits into cases according to characteristic polynomials of Jordan blocks of A.

Case (a): $x-1$, $x-1$, $x+1$, $x+1$. This geometry models the solvmanifold $\mathbb{C}^2 \rtimes \mathbb{R}/(\Lambda \rtimes \mathbb{Z})$ where $1 \in \mathbb{Z}$ acts by a matrix conjugate to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\Lambda \cong \mathbb{Z}^4$ is preserved by this action.

Case (b): $(x-1)^2$, $x+1$, $x+1$. When A is the matrix below, [Fil83, Corollary 6.4.3] ensures the existence of a lattice, if we choose t to be the logarithm of an invertible quadratic integer, e.g. $\ln(3+2\sqrt{2})$.

$$\begin{pmatrix} e^t & e^t & & \\ & e^t & & \\ & & e^{-t} & \\ & & & e^{-t} \end{pmatrix}$$

Cases (c, d): $x-1-a$, $x-1+a$, $x+1$, $x+1$. We need a slightly stronger version of the same result; backtracking to Mostow's theorem [Fil83, Prop 6.4.2], any lattice in $\tilde{G}$ intersects the nilradical $\mathbb{R}^4$ in a lattice and projects to $\mathbb{R}\times\mathbb{R}$ as a closed cocompact subgroup (hence a lattice). So if G admits a lattice Γ, then it lifts to a lattice in $\tilde{G}$ containing the elements of $\mathbb{R}=\tilde{S^1}$ lying over the identity in S^1; furthermore $\Gamma\cap(\mathbb{R}^2\times\mathbb{C})$ is a lattice in $\mathbb{R}^2\times\mathbb{C}$ preserved by two independent elements of $\mathbb{R}\times\mathbb{R}$. One of these can be chosen to lie over the identity in S^1 so the other just needs to be independent from the S^1 direction.

Hence G of the second form admits a lattice if and only if, for some real θ and $t\neq 0$, the matrix with diagonal entries $e^{(1+a)t}$, $e^{(1-a)t}$, $e^{-t+i\theta}$, and $e^{-t-i\theta}$ has a characteristic polynomial with integer coefficients—i.e. these four numbers are roots of an integer polynomial $p\in\mathbb{Z}[x]$. Then the extension L of $\mathbb{Q}$ containing these roots is Galois. Let $\lambda=e^t$ and $z=e^{i\theta}$, so that these roots can be written as λ^{1+a}, λ^{1-a}, $\lambda^{-1}z$, and $\lambda^{-1}\overline{z}$.

If z is real, then λ appears twice but the other roots appear only once, so λ has no Galois conjugates—i.e. $\lambda\in\mathbb{Q}$. By Gauss's lemma, $x-\lambda\in\mathbb{Z}[x]$ so $\lambda\in\mathbb{Z}$. then $\lambda=1$, which contradicts $t\neq 0$. Hence z is not real.

If $\lambda^{1-a}\in\mathbb{Q}$, then by the rational root theorem, $\lambda^{1-a}=1$; so $a=1$ since $\lambda\neq 1$. Then the resulting geometry is $\mathrm{Sol}_0^4\times\mathbb{E}$ (in Filipkiewicz's notation, $G_5\times\mathbb{E}$).

If λ^{1-a} has degree 2 over $\mathbb{Q}$, then its Galois conjugate must be the other real root of p, so p factors—again in $\mathbb{Z}[x]$ due to Gauss's lemma—as

$$\left(x^2-(\lambda^{1+a}+\lambda^{1-a})x+\lambda^2\right)\left(x^2-(2\lambda^{-1}\cos\theta)x+\lambda^{-2}\right),$$

which again implies that $\lambda = 1$. If λ^{1-a} has degree 3 over $\mathbb{Q}$, then λ^{1+a} has degree 1 over $\mathbb{Q}$, again yielding $\mathrm{Sol}_0^4 \times \mathbb{E}$.

The last case is when λ^{1-a} has degree 4 over $\mathbb{Q}$. Then

$$p(x) = x^4 + ax^3 + bx^2 + cx + 1$$

is irreducible with exactly two real roots $\phi_1 = \lambda^{1-a}$ and $\phi_2 = \lambda^{1+a}$, both positive. From the values of ϕ_1 and ϕ_2, we can recover a by

$$a = \left| \frac{\ln \phi_1 - \ln \phi_2}{\ln(\phi_1 \phi_2)} \right|.$$

The condition on p having exactly two real roots is that its discriminant is negative.

One can use Sturm's theorem or computer assistance to derive the condition that the two real roots are positive when $a + c < 0$. □

15.3 Geometries requiring non-split extensions

Having handled the essentially-fibering geometries in the previous section, the remaining geometries $M = G/G_p$ fiber isometrically. So in the description of G as an extension $1 \to H \to G \to Q \to 1$ (Lemma 15.1), $Q = \mathrm{Isom}_0\, B$ where $B = S^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$—thus reducing the number of extension problems from fifty to thirty.

This section's contribution is the second branch in Figure 15.2—that is, it proves the following two claims.

Lemma 15.12. *If $M = G/G_p$ is a maximal model geometry for which*

(i) $G_p \curvearrowright T_pM$ decomposes into 1-dimensional and 2-dimensional summands, and

(ii) $\tilde{G}$ is not a split extension of $\widetilde{\mathrm{Isom}_0}\, B$ ($B = S^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$),

then G is an extension

$$1 \to \mathbb{R}^3 \to G \to \operatorname{Isom}_0 \mathbb{E}^2 \to 1.$$

Proposition 15.13. *The 5-dimensional maximal model geometries $M = G/G_p$ where $T_{\mathbf{1}}G$ is an extension*

$$0 \to \mathbb{R}^3 \to T_{\mathbf{1}}G \to \mathfrak{isom}\,\mathbb{E}^2 \to 0$$

are the nilpotent Lie groups

(i) $M = \mathbb{R}^4 \rtimes_A \mathbb{R}$ *with* $A = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$

(ii) M *has Lie algebra with basis* $\{x, y, u, v, w\}$ *and*

$$[x, w] = u \qquad [y, w] = v \qquad [x, y] = w.$$

On both of these, the point stabilizer is an S^1's worth of Lie group automorphisms at the identity, acting as the diagonal circle in $\mathrm{SO}(2) \times \mathrm{SO}(2)$*; and the full isometry group is* $M \rtimes S^1$.

The proofs will set up the tools for solving all thirty extension problems—i.e. for classifying extensions of Q by H for all combinations of the three remaining values of Q and the ten values of H (See Lemma 15.1). The strategy is as follows. Passing to Lie algebras, a (non-split) extension

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{q} \to 0$$

is given by the homomorphism $\mathfrak{q} \to \operatorname{out} \mathfrak{h}$ and a (nonzero) class in $H^2(\mathfrak{q}; Z(\mathfrak{h}))$ (Thm. 11.6). Since each Q is unimodular and a model geometry's transformation group is unimodular [Fil83, Prop. 1.1.3], the action on $\mathfrak{h}$ must be by traceless derivations (Part II, Lemma 9.9).

Hence a starting point would be to compute $Z(\mathfrak{h})$ and the traceless outer derivation algebra sout($\mathfrak{h}$) for each $\mathfrak{h}$ (Table 15.14).

This data and the actions of $\mathfrak{q}$ on $\mathfrak{h}$ are computed in Section 15.3.1. The reduction to extensions of $\mathfrak{isom}\,\mathbb{E}^2$ by $\mathbb{R}^3$ (Lemma 15.12) is proven in Section 15.3.2). Finally, Section 15.3.3) classifies the geometries resulting from this extension (Prop. 15.13).

Table 15.14: Centers and traceless outer derivation algebras. $T_{\mathbf{1}}\text{Heis}_3$ is written with basis $\{x, y, z\}$ where $[x, y] = z$. See Prop. 15.15 for details.

$\mathfrak{h}$	$Z(\mathfrak{h})$	sout($\mathfrak{h}$)
$\mathfrak{so}_3 \oplus \mathbb{R}$	$0 \oplus \mathbb{R}$	0
$\mathfrak{sl}_2 \oplus \mathbb{R}$	$0 \oplus \mathbb{R}$	0
$T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2 \oplus \mathbb{R}$	$0 \oplus \mathbb{R}$	$\mathbb{R}\ni\mathbb{R}$
$T_{\mathbf{1}} \operatorname{Isom}_0 \text{Heis}_3$	$\mathbb{R}z$	$\mathbb{R}$
$\mathfrak{so}_3$	0	0
$\mathfrak{sl}_2$	0	0
$T_{\mathbf{1}} \text{Sol}^3$	0	0
$T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2$	0	0
$T_{\mathbf{1}}\text{Heis}_3$	$\mathbb{R}z$	$\mathfrak{sl}_2$
$\mathbb{R}^3$	$\mathbb{R}^3$	$\mathfrak{sl}_3$

15.3.1 Data for the extension problem

This section establishes the list of traceless outer derivation algebras in Table 15.14 and classifies the homomorphisms $\mathfrak{q} \to \operatorname{sout} \mathfrak{h}$.

Lemma 15.15. *The centers and traceless outer derivation algebras for the Lie algebras* $\mathfrak{h}$ *are as given in Table 15.14.*

Proof. Explicit calculation of the centers is omitted; they are verifiable in any convenient basis.

Calculation of the derivation algebras uses the following shortcut: A derivation d satisfies a Leibniz rule with respect to the Lie bracket; so if d preserves some subset A of $\mathfrak{h}$, then it also preserves the derived series, the lower central series, and the centralizer of A.

Lemma 15.9 provides the derivation algebras for $\mathfrak{so}_3$, $\mathfrak{sl}_2$, and $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2$. The entry for $\mathfrak{so}_3 \oplus \mathbb{R}$ follows since $\mathfrak{so}_3$ is the first term of the derived series and $\mathbb{R}$ is the center; $\mathfrak{sl}_2 \oplus \mathbb{R}$ is handled likewise. Since $\mathbb{R}^3$ is abelian, $\operatorname{sout} \mathbb{R}^3 = \mathfrak{sl}(\mathbb{R}^3)$. Only four cases then remain.

Case 1: $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2 \times \mathbb{R}$**.** The first term of the derived series is the translation subalgebra, and the center is $\mathbb{R}$. The same calculation as in Lemma 15.9 ensures that an outer derivation can be represented by d which scales the translation subalgebra uniformly and has $dr \in \mathbb{R}$. Then if d acts as the scalar b on the translation subalgebra, tracelessness requires it to act as $-2b$ on $\mathbb{R}$. Writing this algebra as $\mathbb{R} \ni \mathbb{R}$ is now just writing the matrices by which it acts on the span of $\mathbb{R}$ and r.

Case 2: $T_{\mathbf{1}} \operatorname{Sol}^3$**.** For the basis $\{x, y, z\}$ of $T_{\mathbf{1}} \operatorname{Sol}^3$ where $[z, x] = x$ and $[z, y] = -y$, the first term of the derived series is $\mathbb{R}x + \mathbb{R}y$. Subtracting inner derivations, we may assume $dz = az$ for some a. Then

$$dx = d[z, x] = [dz, x] + [z, dx] = ax + [z, dx];$$

so $(d - a)x = [z, dx]$, which implies $dx \in \mathbb{R}x$. Then $[z, dx] = dx$ so the above implies $a = 0$, and we can subtract a multiple of $\operatorname{ad} z$ to ensure $dx = 0$. Similarly, $dy \in \mathbb{R}y$, and tracelessness then implies $dy = 0$.

Case 3: $T_{\mathbf{1}}\mathrm{Heis}_3$**.** (This calculation is outlined quickly in [Fil83, §6.3].) With the basis $\{x, y, z\}$ in which $[x, y] = z$, inner derivations account for the $\mathbb{R}x + \mathbb{R}y \to \mathbb{R}z$ component; so a derivation d can be taken to preserve $\mathbb{R}x + \mathbb{R}y$. Since $\mathbb{R}z$ is central, d preserves $\mathbb{R}z$. Then

$$(\operatorname{tr} d)z = [dx, y] + [x, dy] + dz = d[x, y] + dz = 2dz,$$

so $dz = 0$ and d acts with trace zero on $\mathbb{R}x + \mathbb{R}y$.

Case 4: $T_{\mathbf{1}} \operatorname{Isom}_0 \operatorname{Heis}_3$**.** Since $T_{\mathbf{1}} \operatorname{Isom}_0 \operatorname{Heis}_3$ admits $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2$ as a quotient, any outer derivation of the former induces an outer derivation of the latter. Hence (consulting Lemma 15.9) any outer derivation of the former can be represented by a derivation d such that

$$\begin{aligned} dx &= bx + c_1 z \\ dy &= by + c_2 z \\ dr &= c_3 z \\ dz &= c_4 z \end{aligned}$$

for some real b and c_i. Tracelessness implies $c_4 = -2b$, and

$$-2bz = dz = [dx, y] + [x, dy] = 2b[x, y] = 2bz$$

implies $b = 0$. Then

$$dx = d[r, -y] = [dr, -y] + [r, -dy] = 0$$

and similarly $dy = 0$; so all that remains is c_3. □

Lemma 15.16. *If* $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{q} \to 0$ *arises from a* 5*-dimensional geometry, then*

$\mathfrak{q} \to \operatorname{sout}\mathfrak{h}$ *is nonzero only in the following cases, all of which lift to maps* $\mathfrak{q} \to \operatorname{Der}\mathfrak{h}$*.*

$$\mathfrak{sl}_2 \xrightarrow{\sim} \operatorname{sout} T_{\mathbf{1}}\mathrm{Heis}_3$$

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3 \cong \operatorname{sout}\mathbb{R}^3$$

$$\mathfrak{sl}_2 \cong \mathfrak{so}_{2,1} \hookrightarrow \mathfrak{sl}_3 \cong \operatorname{sout}\mathbb{R}^3$$

$$\mathfrak{so}_3 \hookrightarrow \mathfrak{sl}_3 \cong \operatorname{sout}\mathbb{R}^3$$

$$T_{\mathbf{1}}\operatorname{Isom}_0\mathbb{E}^2 \cong \mathbb{R}^2 \ni \mathfrak{so}_2 \hookrightarrow \mathfrak{sl}_3 \cong \operatorname{sout}\mathbb{R}^3 \text{ and its negative transpose}$$

$$T_{\mathbf{1}}\operatorname{Isom}_0\mathbb{E}^2 \to \mathfrak{so}_2 \overset{t\neq 0}{\hookrightarrow} \mathfrak{sl}_2 \cong \operatorname{sout} T_{\mathbf{1}}\mathrm{Heis}_3$$

$$T_{\mathbf{1}}\operatorname{Isom}_0\mathbb{E}^2 \to \mathfrak{so}_2 \overset{t>0}{\hookrightarrow} \mathfrak{sl}_3 \cong \operatorname{sout}\mathbb{R}^3$$

Proof. If $\mathfrak{q}$ is semisimple then only $\operatorname{sout} T_{\mathbf{1}}\mathrm{Heis}_3 \cong \mathfrak{sl}_2\mathbb{R}$ and $\operatorname{sout}\mathbb{R}^3 = \mathfrak{sl}_3\mathbb{R}$ have high enough dimension to admit nonzero images of $\mathfrak{q}$. By counting representations, the only such are the first four maps listed above.

Otherwise, $\mathfrak{q} = T_{\mathbf{1}}\operatorname{Isom}_0\mathbb{E}^2$. The proof proceeds similarly to Lemma 15.10, making note of the following facts.

1. Any homomorphism from $T_{\mathbf{1}}\operatorname{Isom}_0\mathbb{E}^2$ either is faithful or factors through the rotation part $\mathfrak{so}_2$ (Lemma 14.2).

2. Since $\mathfrak{q} \to \operatorname{sout}\mathfrak{h}$ comes from the conjugation action $\operatorname{Isom}_0\mathbb{E}^2 \to \operatorname{Out} H$, the generator $r \in \mathfrak{q}$ of $\mathrm{SO}(2)$ generates a compact subgroup of $\operatorname{Out} H$.

Lifting to $\operatorname{Der}\mathfrak{h}$ will follow from observing that $\operatorname{out}\mathbb{R}^3 = \operatorname{Der}\mathbb{R}^3$, and $\operatorname{out} T_{\mathbf{1}}\mathrm{Heis}_3$ embeds in $\operatorname{Der} T_{\mathbf{1}}\mathrm{Heis}_3$ as the subalgebra preserving $\mathbb{R}x + \mathbb{R}y$.

Case 1: ρ is faithful. Only $\operatorname{sout} T_{\mathbf{1}}\mathrm{Heis}_3 \cong \mathfrak{sl}_2$ (dimension 3) and $\operatorname{sout}\mathbb{R}^3 \cong \mathfrak{sl}_3$ (dimension 8) have high enough dimension to admit a faithful image of $T_{\mathbf{1}}\operatorname{Isom}_0\mathbb{E}^2$. Since

$\mathfrak{sl}_2 \not\cong T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2$ (only the latter is solvable), a faithful image must land in $\mathfrak{sl}_3$.

Let $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2$ have basis $\{x, y, r\}$ with nonzero brackets $[r, x] = y$ and $[r, y] = -x$. An embedding sends x and y to commuting, similar matrices. Consulting the list of 2-dimensional abelian subalgebras of $\mathfrak{sl}_3\mathbb{R}$ (Part II, Lemma 9.13) yields exactly two embeddings $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^2 \hookrightarrow \mathfrak{sl}_3$ up to conjugacy (omitted entries are zero):

$$ax + by + cr \mapsto \begin{pmatrix} & -c & a \\ c & & b \\ & & 0 \end{pmatrix} \text{ or } \begin{pmatrix} & -c & \\ c & & \\ -a & -b & 0 \end{pmatrix}.$$

Case 2: ρ factors through $\mathfrak{so}_2$. From Table 15.14, the volume-preserving parts of $\operatorname{Out}(\widetilde{\operatorname{Isom}}_0 \mathbb{E}^2 \times \mathbb{R})$ and $\operatorname{Out}(\widetilde{\operatorname{Isom}}_0 \operatorname{Heis}_3)$ are isomorphic to $\mathbb{R} \rtimes \mathbb{R}$ and $\mathbb{R}$ respectively, neither of which has nontrivial compact subgroups. So for a map factoring through $\mathfrak{so}_2$ by which r generates a compact subgroup of $\operatorname{Out} H$, still only $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ admit nonzero images. In both of these, all images of $\mathfrak{so}_2$ are conjugate, so the only flexibility is in how $\mathfrak{so}_2$ maps to this image. □

15.3.2 Reduction to one extension problem

Using the data computed in the previous section, this section proves Lemma 15.12—the claim that it suffices to consider extensions of $\operatorname{Isom}_0 \mathbb{E}^2$ by $\mathbb{R}^3$. The idea is to show that the maximal model geometries produced from all other non-split extensions can be produced from split extensions that will be handled in later sections.

Proof of Lemma 15.12. Under the standing assumptions, G is a extension of $Q = \operatorname{Isom}_0 B$ where $B = S^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$; and passing to Lie algebras yields a non-split extension

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{q} \to 0.$$

Step 1: $\mathfrak{q}$ is $T_1 \operatorname{Isom}_0 \mathbb{E}^2$. The other possibilities for $\mathfrak{q}$ are semisimple. The first and second cohomology of a semisimple Lie algebra vanish in any coefficient system [GOV94, Thm. 1.3.2]. So if $\mathfrak{q}$ were semisimple, the classification of these extensions by $H^2(\mathfrak{q}; Z(\mathfrak{h}))$ (Thm. 11.6) would imply that $\mathfrak{g}$ is a split extension of $\mathfrak{q}$.

Step 2: If $\mathfrak{h} \neq \mathbb{R}^3$, then $\mathfrak{q}$ acts trivially on a 1-dimensional $Z(\mathfrak{h})$. Since $H^2(\mathfrak{q}; Z(\mathfrak{h}))$ must be nonzero for a non-split extension to exist, $Z(\mathfrak{h})$ is nonzero. Consulting the list of derivation algebras (Table 15.14) and the nontrivial actions (Lemma 15.16), the dimension of $Z(\mathfrak{h})$ is 1, and the action of $\mathfrak{q}$ on it is trivial. We now aim to show that the resulting geometries are either non-maximal or realized with direct-product transformation groups $\tilde{G}$, which are handled later in Section 15.5.

Step 3: $H^2(\mathfrak{q}; Z(\mathfrak{h})) \cong \mathbb{R}$. With trivial action, the cocycle condition reduces to

$$\sum_{\text{cyc}} c(x_1, [x_2, x_3]) = 0.$$

This expression is trilinear, so to check that it is zero, it suffices to check basis elements. If two of the basis elements are the same, then antisymmetry of c and the Lie bracket imply the value is zero. Since $\mathfrak{q}$ is 3-dimensional, only one combination then needs to be checked. Letting $\mathfrak{q}$ have basis $\{x, y, r\}$ where $[r, x] = y$ and $[r, y] = -x$,

$$c(x, [y, r]) + c(y, [r, x]) + c(r, [x, y]) = c(x, x) + c(y, y) + c(r, 0) = 0 + 0 + 0 = 0;$$

so *every* linear map $\Lambda^2 \mathfrak{q} \to Z(\mathfrak{h})$ is a cocycle.

With trivial action, coboundaries are maps $\Lambda^2 \mathfrak{q} \to Z(\mathfrak{h})$ which factor through the Lie bracket; so coboundaries account for any nonzero values on (and only on) $r \wedge x$ and $r \wedge y$. Therefore $H^2(\mathfrak{q}; Z(\mathfrak{h})) \cong \mathbb{R}$, generated by a cocycle that is nonzero on $x \wedge y$ and zero on $r \wedge x$ and $r \wedge y$.

Step 4: Describe the resulting groups G. Since $\mathfrak{q} \to \operatorname{sout}\mathfrak{h}$ lifts to $\operatorname{Der}\mathfrak{h}$ (Lemma 15.16), the Lie algebra structure of $\mathfrak{g}$ can be recovered from by interpreting the cocycle as a $Z(\mathfrak{h})$-valued bracket on $\mathfrak{q}$, as follows. On the vector space $\mathfrak{h} \oplus \mathfrak{q}$, define, following [AMR00, Eqn. 5.5],

$$[h_1 + q_1, h_2 + q_2] = q_1(h_2) - q_2(h_1) + c(q_1, q_2) + [q_1, q_2]_{\mathfrak{q}}.$$

If $\mathfrak{h}$ is one of the direct sums $\mathfrak{h}' \oplus \mathbb{R}$, then $Z(\mathfrak{h})$ is this $\mathbb{R}$ summand. Using the above formula,

$$\mathfrak{g} \cong \mathfrak{h}' \oplus (\mathfrak{n}_3 \rtimes \mathfrak{so}_2) \cong \mathfrak{h}' \oplus T_{\mathbf{1}} \operatorname{Isom}_0 \operatorname{Heis}_3.$$

Then $\tilde{G}$ is one of the direct products below, handled later in Section 15.5.

$$S^3 \times \widetilde{\operatorname{Isom}_0 \operatorname{Heis}_3} \qquad \widetilde{\operatorname{PSL}(2,\mathbb{R})} \times \widetilde{\operatorname{Isom}_0 \operatorname{Heis}_3} \qquad \widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \widetilde{\operatorname{Isom}_0 \operatorname{Heis}_3}$$

Otherwise, $\mathfrak{h}$ is $T_{\mathbf{1}}\operatorname{Heis}_3$ or $T_{\mathbf{1}} \operatorname{Isom} \operatorname{Heis}_3$. Then rescaling the x, y plane gives an isomorphism of the resulting simply-connected group $\tilde{G}$ with either $\operatorname{Heis}_5 \rtimes \mathbb{R}$ or $\operatorname{Heis}_5 \rtimes \mathbb{R}^2$, with the action by rotations. Point stabilizers $\tilde{G}_p$ act semisimply since they act by some sort of rotations, while the nilradical Heis_5 acts nilpotently; so Heis_5 meets the point stabilizer trivially and therefore acts freely on any resulting geometry $\tilde{G}/\tilde{G}_p$. Then $\tilde{G}/\tilde{G}_p \cong \operatorname{Heis}_5$, which is non-maximal since $\operatorname{Heis}_5 \rtimes \operatorname{U}(2)/\operatorname{U}(2)$ subsumes it. □

15.3.3 The geometries

Having established that geometries requiring non-split extensions in the extension problem (Lemma 15.1) are accounted for by extensions of $\mathfrak{isom}\,\mathbb{E}^2$ by $\mathbb{R}^3$ (Lemma 15.12), we now classify the resulting geometries.

Proof of Prop. 15.13 (Classification when $T_{\mathbf{1}}G$ is an extension of $\mathfrak{isom}\,\mathbb{E}^2$ by $\mathbb{R}^3$). The key observation is the first step, that M is a nilpotent Lie group; after that the problem reduces to computing some cohomology (Step 2) and some automorphism groups (Step 3).

Step 1: M is the simply-connected nilpotent Lie group whose Lie algebra is the induced extension of $\mathbb{R}^2$ by $\mathbb{R}^3$. The inclusion of $\mathbb{R}^2$ as the translation subalgebra of $\mathfrak{isom}\,\mathbb{E}^2$ induces an inclusion of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & T_{\mathbf{1}}G & \longrightarrow & \mathfrak{isom}\,\mathbb{E}^2 & \longrightarrow & 0 \\ & & \wr\Vert & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathfrak{m} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & 0. \end{array}$$

Since $\mathbb{R}^2$ and $\mathbb{R}^3$ are abelian and $\mathbb{R}^2$ acts nilpotently (see the list of actions in Lemma 15.16), $\mathfrak{m}$ is nilpotent. Moreover $\mathbb{R}^2$ is an ideal in $\mathfrak{isom}\,\mathbb{E}^2$, so $\mathfrak{m}$ is an ideal in $T_{\mathbf{1}}G$, with codimension 1. Since $T_{\mathbf{1}}G$ is not nilpotent, having the non-nilpotent $\mathfrak{isom}\,\mathbb{E}^2$ as a quotient, $\mathfrak{m}$ is the nilradical of $T_{\mathbf{1}}G$. Then $T_{\mathbf{1}}G$ is the extension

$$0 \to \mathfrak{m} \to T_{\mathbf{1}}G \to \mathfrak{so}_2 \to 0,$$

which splits since any linear map from $\mathfrak{so}_2 \cong \mathbb{R}$ is a Lie algebra homomorphism.

Since G_p is compact, it acts semisimply in the adjoint representation on $T_{\mathbf{1}}G$; while the nilradical of $T_{\mathbf{1}}G$ acts nilpotently. Then $T_{\mathbf{1}}G_p \cap \mathfrak{m} = 0$—so $T_{\mathbf{1}}G_p$ lies over $\mathfrak{so}_2$ in the above extension, and $\mathfrak{m}$ is tangent to a group acting transitively on $M = G/G_p$. Since $\pi_1(M) = 0$ and $\dim M = \dim \mathfrak{m}$, this action is free, which identifies M with the simply-connected group whose Lie algebra is $\mathfrak{m}$ once a basepoint is chosen.

Step 2: List candidate nilpotent Lie algebras $\mathfrak{m}$. This step proceeds using the classification of extensions by second cohomology (Thm. 11.6). Since $\mathbb{R}^2$ is 2-dimensional, it has no 3-cocycles; so every $\Lambda^2\mathbb{R}^2 \to \mathbb{R}^3$ is a cocycle. Let $\{x, y\}$ be a basis for $\mathbb{R}^2$; then coboundaries take the form $x \wedge y \mapsto x \cdot \beta(y) - y \cdot \beta(x)$, where $\cdot$ denotes the action of $\mathbb{R}^2$ on $\mathbb{R}^3$. Then (like in Part II, Lemma 9.15) coboundaries account for values lying in $x \cdot \mathbb{R}^3 + y \cdot \mathbb{R}^3$, and

$$H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \mathbb{R}^3/(x \cdot \mathbb{R}^3 + y \cdot \mathbb{R}^3).$$

Using the list of actions of $\mathfrak{isom}\,\mathbb{E}^2$ on $\mathbb{R}^3$ in Lemma 15.16), one action has x and y both acting as zero. In that case, every cocycle represents its own class, and $\mathfrak{g}'$ is either isomorphic to $\mathbb{R}^5$ (zero class) or $R^2 \oplus T_{\mathbf{1}}\mathrm{Heis}^3$ (nonzero class, normalized by scaling and conjugating in $\mathrm{GL}(3, \mathbb{R}) = \mathrm{Aut}\,\mathbb{R}^3$).

Letting $\{u, v, w\}$ denote a basis of $\mathbb{R}^3$, the two nontrivial actions are

$$\left\{ \begin{array}{c} [x, w] = u \\ [y, w] = v \end{array} \right\} \text{ and } \left\{ \begin{array}{c} [x, u] = w \\ [y, v] = w \end{array} \right\}.$$

On the left, the split extension can be re-expressed as the semidirect sum of $\mathbb{R}w$ acting on the span of the other four basis elements with two 2×2 Jordan blocks of eigenvalue 0—call this $\mathbb{R}^4 \ni_{x^2, x^2} \mathbb{R}$. On the right, the split extension is the 5-dimensional Heisenberg algebra $T_{\mathbf{1}}\mathrm{Heis}_5$.

Up to changes of coordinates in the x, y and u, v planes, the non-split extensions are accounted for by, respectively,

$$[x, y] = w \qquad\qquad [x, y] = u.$$

Step 3: Eliminate $\mathfrak{m}$ whose automorphism group is either too large or too small. The maximal geometry realizing M is $M \rtimes K/K$ where K is maximal compact in $\mathrm{Aut}\,M$ (Lemma 11.3). Since the assumptions imply $\dim G = 6$, we require the maximal compact subgroup of $\mathrm{Aut}\,M$ to be S^1.

In three of the groups M produced in Step 2, the maximal compact subgroup of $\mathrm{Aut}\,M$

is too large, which makes G/G_p non-maximal.

$$\operatorname{Aut}\mathbb{R}^5 \cong \mathrm{GL}(5,\mathbb{R}) \supset \mathrm{SO}(5)$$
$$\operatorname{Aut}(\mathbb{R}^2 \times \mathrm{Heis}_3) \supseteq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \supset \mathrm{SO}(2) \times \mathrm{SO}(2)$$
$$\operatorname{Aut}\mathrm{Heis}_5 \supseteq U(2) \text{ (see Section 13.2, Step 3)}$$

If $\mathfrak{m}$ has basis $\{x, y, u, v, w\}$ with

$$[x, u] = w \qquad [y, v] = w \qquad [x, y] = u,$$

then any automorphism preserves the following filtration of characterstic ideals.

$$Z(\mathfrak{m}) = \mathbb{R}w$$
$$[\mathfrak{m}, \mathfrak{m}] = \mathbb{R}u + \mathbb{R}w$$
$$\text{The preimage of } Z(\mathfrak{m}/Z(\mathfrak{m})) = \mathbb{R}u + \mathbb{R}v + \mathbb{R}w$$
$$\text{The centralizer of } [\mathfrak{m}, \mathfrak{m}] = \mathbb{R}y + \mathbb{R}u + \mathbb{R}v + \mathbb{R}w$$

Therefore every automorphism is upper-triangular, so $\operatorname{Aut}\mathfrak{m}$ contains no compact subgroups of positive dimension. Then $M \rtimes S^1$ cannot be defined with a nontrivial action by S^1.

Step 4: Both remaining geometries are maximal. What remains are the two possibilities for M claimed in the statement of this Proposition. Their Lie algebras can be expressed with basis $\{x, y, u, v, w\}$ and

$$[x, w] = u \qquad [y, w] = v \qquad [x, y] = w \text{ or } 0.$$

Similarly to Step 3 above,

$$Z(\mathfrak{m}) = \mathbb{R}u + \mathbb{R}v$$
$$[\mathfrak{m}, \mathfrak{m}] = \mathbb{R}u + \mathbb{R}v + \mathbb{R}w \text{ if } [x, y] = w$$
$$\{a \mid \operatorname{rank} \operatorname{ad} a \leq 1\} = \mathbb{R}x + \mathbb{R}y + \mathbb{R}u + \mathbb{R}v \text{ if } [x, y] = 0.$$

Either way, every automorphism is block upper-triangular with two 2×2 blocks and one 1×1 block. The group of such block upper-triangular matrices has a torus T^2 (in each 2×2 block, something conjugate to a rotation) as its maximal compact subgroup (Part II, Lemma 9.29); so $\operatorname{Aut} \mathfrak{m}$ has maximal compact subgroup inside that.

Up to scaling x, y, and possibly w, any automorphism in this T^2 merely rotates the x, y plane. Since $\operatorname{ad} w$ takes this plane to the u, v plane, the rotation on the u, v plane is determined; so $K \cong S^1$, acting by rotation at equal rates on these planes.

Step 5: Both remaining candiates are model geometries. A nilpotent Lie algebra with rational structure constants in some basis admits a cocompact lattice Γ [Rag72, Thm 2.12]. In the bases used above, the structure constants are all integers; so in each case $\Gamma \backslash M/\{1\} \cong \Gamma \backslash M \rtimes S^1/S^1$ is a compact manifold modeled on M. □

15.4 Line bundles over $\mathbb{F}^4$ and $T^1\mathbb{E}^{1,2}$ (semidirect-product isometry groups)

Having just classified geometries G/G_p where $T_{\mathbf{1}}G$ is an extension of $\mathfrak{isom}\,\mathbb{E}^2$ by $\mathbb{R}^3$ or any non-split extension, this section classifies the next cluster of geometries—those where $T_{\mathbf{1}}G$ is a split extension, i.e. a semidirect sum $\mathfrak{h} \ni\!\!\!+ \mathfrak{q}$ with one of the remaining combinations of $\mathfrak{h}$ and $\mathfrak{q}$ that has a nontrivial action. Five combinations remain of the list from Lemma 15.16, producing the five groups in Table 15.19. in terms of which this section's classification result is stated. We will verify that these are model geometries in Prop. 15.20, but maximality is deferred to Section 15.6 to take advantage of results from the remainder of the classification (Section 15.5).

Proposition 15.17. *Let $M = G/G_p$ be a maximal model geometry such that $\tilde{G}$ is one of the semidirect products listed in Table 15.19. Let $\gamma : \mathbb{R} \to \widetilde{\mathrm{SL}_2}$ send t to a rotation by $2\pi t$ radians, and fix a nontrivial $z \in Z(\mathrm{Heis}_3)$. Then M is one of*

$$\begin{aligned}
\mathrm{SAff}\,\mathbb{R}^2 = \mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2} &\cong (\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}) \rtimes \mathrm{SO}(2)/\,\mathrm{SO}(2) \\
\mathbb{E} \times \mathbb{F}^4 &= \mathbb{R} \times \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) \\
\mathbb{F}^5_a &= \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SL}_2}/\{atz, \gamma(t)\}_{t\in\mathbb{R}}, \quad a = 0 \text{ or } 1 \\
T^1\mathbb{E}^{1,2} &= \mathbb{R}^3 \rtimes \mathrm{SO}(1,2)/\,\mathrm{SO}(2).
\end{aligned}$$

The proof requires knowing the center of the semidirect product $\tilde{G}$; the following formula computes the centers listed in Table 15.19.

Lemma 15.18. *If C_x is conjugation by x, then*

$$Z(A \rtimes_\phi B) = \{(x, y) \in A^B \times Z(B) \mid \phi(y) = C_{x^{-1}}\}.$$

In particular, if $\phi(B) \cap \mathrm{Inn}\, A = \{1\}$, then

$$Z(A \rtimes_\phi B) = Z(A)^B \times (Z(B) \cap \ker \phi).$$

Proof. Suppose $xy = (x, y) \in Z(A \rtimes_\phi B)$. Then to commute with all $b \in B$,

$$\begin{aligned}
b &= xyby^{-1}x^{-1} \\
bx = \phi(b)(x)b &= xyby^{-1};
\end{aligned}$$

so $\phi(b)(x) = x$ and $b = yby^{-1}$ for all $b \in B$, i.e. $x \in A^B$ and $y \in Z(B)$. To commute with all $a \in A$,

$$\begin{aligned} a &= xyay^{-1}x^{-1} \\ &= x\phi(y)(a)x^{-1} \\ &= C_x\phi(y)(a). \end{aligned}$$

Since A and B generate $A \rtimes_\phi B$, these conditions suffice.

If $\phi(B) \cap \operatorname{Inn} A = \{1\}$, then $\phi(y)$ is identity, and the second condition reduces to $x \in Z(A)$. □

Table 15.19: Nontrivial semidirect products covering candidate isometry groups G

$\tilde{G} = \tilde{H} \rtimes \tilde{Q}$	center	notes on action
$\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}(3)}$	$\{0\} \times \mathbb{Z}/2\mathbb{Z}$	standard representation
$\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}(2,1)}$	$\{0\} \times \mathbb{Z}$	standard representation
$\mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$	$\mathbb{R} \times \mathbb{Z}$	as $\mathrm{SL}(2,\mathbb{R})$ on x, y plane
$\mathrm{Heis}_3 \rtimes \widetilde{\mathrm{Isom}_0\, \mathbb{E}^2}$	$\mathbb{R} \times \mathbb{Z}$	family of actions, factors through $\widetilde{\mathrm{SO}(2)}$
$\mathbb{R} \times \mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$	$\mathbb{R} \times \mathbb{Z}$	trivial action on $\mathbb{R}$

Proof of Prop. 15.17 (Classification when the isometry group is a semidirect product). Suppose that $M = G/G_p$ is such a geometry (a maximal model geometry with $\tilde{G}$ listed in Table 15.19), and let $\tilde{G}_p$ be the preimage in $\tilde{G}$ of G_p.

Preparatory step: $\tilde{G}_p$ is a 1-parameter subgroup lying over a maximal torus of $\tilde{G}/Z(\tilde{G})$. Since every candidate $\tilde{G}$ in Table 15.19 is 6-dimensional, G_p and $\tilde{G}_p$ are 1-dimensional. The homotopy exact sequence for $\tilde{G}_p \to \tilde{G} \to M$ and the assumption that $\pi_1(M) \cong \pi_0(M) \cong 0$ imply that $\tilde{G}_p$ is connected. Moreover, $G_p = \tilde{G}_p/(\tilde{G}_p \cap Z(\tilde{G}))$

(Rmk. 15.4); so G_p becomes a copy of S^1 in $\tilde{G}/Z(\tilde{G})$. For every candidate $\tilde{G}$ listed, $\tilde{G}/Z(\tilde{G})$ has maximal torus $\mathrm{SO}(2) \cong S^1$.

For every candidate in Table 15.19, the quotient $\tilde{G}/Z(\tilde{G})$ so in the contractible $\tilde{G}$, where 1-parameter subgroups are isomorphic to $\mathbb{R}$, the intersection of $\tilde{G}_p$ and $Z(\tilde{G})$ is nontrivial.

Case 1: If $\dim Z(\tilde{G}) = 0$ **then** $M = T^1\mathbb{E}^{1,2}$**.** If $\dim Z(\tilde{G}) = 0$ then $\tilde{G}$ covers $\tilde{G}/Z(\tilde{G})$; so $\tilde{G}_p$ is unique up to conjugacy in $\tilde{G}$. The resulting spaces, and the names chosen for them, are

$$\mathbb{R}^3 \rtimes \mathrm{SO}(3)/\mathrm{SO}(2) = T^1\mathbb{E}^3$$
$$\mathbb{R}^3 \rtimes \mathrm{SO}(2,1)/\mathrm{SO}(2) = T^1\mathbb{E}^{1,2}.$$

The final classification omits $T^1\mathbb{E}^3$ since it is non-maximal, being subsumed by $\mathbb{E}^3 \times S^2$. An interpretation of $T^1\mathbb{E}^{1,2}$, by analogy to the Euclidean (i.e. positive-definite) case, is as one connected component of the sub-bundle of $T\mathbb{R}^3$ consisting of the vectors v where $\langle v, v\rangle = 1$ and $\langle \cdot, \cdot \rangle$ has signature $(+,-,-)$.

Case 2: If $\tilde{G} = \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{Isom}_0}\,\mathbb{E}^2$ **then** M **is a non-maximal** $\mathrm{Heis}_3 \times \mathbb{E}^2$**.** In this case, $\tilde{G}$ is an extension $1 \to \mathrm{Heis}_3 \times \mathbb{R}^2 \to \tilde{G} \to \widetilde{\mathrm{SO}(2)} \to 1$, with $\tilde{G}_p$ surjecting onto $\widetilde{\mathrm{SO}(2)}$; so

$$G/G_p \cong (\mathrm{Heis}_3 \times \mathbb{R}^2) \rtimes \widetilde{\mathrm{SO}(2)}/\widetilde{\mathrm{SO}(2)}.$$

Since $\mathrm{Heis}_3 \times \mathbb{R}^2$ admits at least a torus's worth of automorphisms (one SO(2) acting by rotation on $\mathbb{R}^2$ and another acting by rotation on Heis_3), this geometry is subsumed by $(\mathrm{Heis}_3 \times \mathbb{R}^2) \rtimes \mathrm{SO}(2)^2/\mathrm{SO}(2)^2$.

Case 3: If $\tilde{G} = \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$ then M is F_0^5 or F_1^5. Let $\gamma : \mathbb{R} \to \widetilde{\mathrm{SO}(2)} \subset \widetilde{\mathrm{SL}(2,\mathbb{R})}$ send t to a rotation by $2\pi t$, and put coordinates on Heis_3 so that

$$(x, y, z)(x'y'z') = (x + x', y + y', z + z' + xy').$$

Since $Z(\tilde{G})$ is 1-dimensional, the groups $\tilde{G}_p$ lying over $\widetilde{\mathrm{SO}(2)}$ form a 1-dimensional family: for $a \in \mathbb{R}$, let

$$F_a^5 = \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}/\{(0, 0, at), \gamma(t)\}_{t\in\mathbb{R}}.$$

Conjugating $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ by a reflection in $O(2)$ would exchange $\gamma(t)$ with $\gamma(-t)$; so up to this conjugation we may assume $a \geq 0$. If $a > 0$, conjugating Heis_3 by the automorphism

$$(x, y, z) \mapsto \left(xa^{-1/2}, ya^{-1/2}, za^{-1}\right)$$

allows assuming $a = 1$; so $M = G/G_p$ is equivariantly diffeomorphic to either F_0^5 or F_1^5. The two cases are distinguished by whether a Levi subgroup of G is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ ($a = 0$) or its universal cover ($a = 1$).

Case 4: If $\tilde{G} = \mathbb{R} \times \mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$ then M is $\mathbb{E} \times \mathbb{F}^4$ or $\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$. As in Case 3, conjugation by an element of $\mathrm{O}(2)$ and a rescaling of the $\mathbb{R} = Z(\tilde{G})^{\circ}$ factor allow assuming

$$G/G_p \cong \mathbb{R} \times \mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}/\{(at, 0, 0), \gamma(t)\}_{t\in\mathbb{R}}$$

with $a = 0$ ($M = \mathbb{E} \times \mathbb{F}^4$) or $a = 1$ ($M = \mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}$); and the two are again distinguished by Levi subgroups. □

Proposition 15.20. *All geometries listed in Prop. 15.17 are model geometries.*

Proof. The kernel Γ_3 of $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/3\mathbb{Z})$ is a torsion-free lattice in $\mathrm{SL}(2,\mathbb{R})$ [Mor15, Thm. 4.8.2 Case 1]; so its lift $\tilde{\Gamma}_3$ is a torsion-free lattice in the infinite cyclic cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Then Γ_3 (resp. $\tilde{\Gamma}_3$) acts without fixed points (i.e. freely) anywhere that $\mathrm{SL}(2,\mathbb{R})$ (resp. $\widetilde{\mathrm{SL}(2,\mathbb{R})}$) acts faithfully. So finite-volume manifolds modeled on three of the geometries from Prop. 15.17 can be constructed using these groups and are listed in Table 15.21.

Table 15.21: Finite-volume manifolds $\Gamma\backslash G/K$ modeled on G/K fibering over $\mathbb{F}^4$

Γ	G	K	G/K
$\mathbb{Z}^2 \rtimes \tilde{\Gamma}_3$	$(\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}(2,\mathbb{R})}) \rtimes \mathrm{SO}(2)$	$\mathrm{SO}(2)$	$\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}$
$\mathbb{Z} \times \mathbb{Z}^2 \rtimes \Gamma_3$	$\mathbb{R} \times \mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}$	$\mathrm{SO}(2)$	$\mathbb{E} \times \mathbb{F}^4$
$\mathrm{Heis}_3(\mathbb{Z}) \rtimes \Gamma_3$	$\mathrm{Heis}_3 \rtimes \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathbb{F}^5_0$

Special case 1: $\mathbb{F}^5_1$. There is one complication: while $\Lambda = \mathrm{Heis}_3(\mathbb{Z}) \rtimes \tilde{\Gamma}_3$ is a torsion-free lattice in $\tilde{G}$, we must verify that it descends to a torsion-free lattice in G. Observe that

$$\tilde{G}_p = \{(0,0,t), \gamma(t)\}_{t\in\mathbb{R}},$$

where $\gamma(t)$ is rotation by $2\pi t$, meets Λ in $\{(0,0,t),\gamma(t)\}_{t\in\mathbb{Z}} = \tilde{G}_p \cap Z(\tilde{G})$. Then Λ remains discrete in $G = \tilde{G}/(\tilde{G}_p \cap Z(\tilde{G}))$.

Only elliptical elements can become torsion in a quotient of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and the elliptical elements of the torsion-free $\tilde{\Gamma}_3$ all lie over the identity. Then anything in Λ that becomes torsion in G lies over the identity of $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$—i.e. in $Z(\tilde{G}) \cong \mathbb{R} \times \mathbb{Z}$. Since $Z(\tilde{G})$ has image in G isomorphic to $Z(\tilde{G})/(\tilde{G}_p \cap Z(\tilde{G})) \cong \mathbb{R}$, the image in G of Λ is torsion-free.

Then $\Lambda\backslash\mathbb{F}^5_1$ is a manifold, with finite volume since Λ is a lattice.

Special case 2: $T^1\mathbb{E}^{1,2}$. By the classification of representations of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ [FH91, 11.8], the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathrm{Sym}^2\,\mathbb{R}^2$ of symmetric 2×2 matrices given by

$$g \cdot M = gMg^T$$

factors through the standard representation of $\mathrm{SO}(2,1)$. Since elements Γ_3 have integer entries, the action by Γ_3 preserves the subgroup Λ consisting of symmetric integer matrices. Then $\Lambda \rtimes \Gamma_3$ descends to a lattice in $\mathbb{R}^3 \rtimes \mathrm{SO}(2,1)$. This lattice is torsion-free: since $\mathrm{SL}(2,\mathbb{R})$ double covers $\mathrm{SO}(2,1)$, an n-torsion element of $\mathbb{R}^3 \rtimes \mathrm{SO}(2,1)$ lifts to a $2n$-torsion element of $\mathbb{R}^3 \rtimes \mathrm{SL}(2,\mathbb{R})$, and $\Lambda \rtimes \Gamma$ was constructed to be torsion-free.

Therefore $(\Lambda \rtimes \Gamma_3)\backslash T^1\mathbb{E}^{1,2}$ is a finite-volume manifold modeled on $T^1\mathbb{E}^{1,2}$. □

Remark 15.22. Maximality will need to wait until the classification is more complete, since the proof depends on knowing all geometries with point stabilizer containing S^1_1 or $S^1_{1/2}$. The proof is eventually given in Prop. 15.41.

15.5 Product geometries and associated bundles

The last piece of the classification, necessarily a sort of catch-all for the leftovers, is this section's main result.

Proposition 15.23. *The* 5*-dimensional maximal model geometries* G/G_p *for which* $\tilde{G}$ *occurs in Table 15.24 are*

(i) products of 2*- and* 3*-dimensional geometries and*

(ii) the following "associated bundle" geometries.

$$\mathrm{Heis}_3 \times_{\mathbb{R}} S^3 \qquad \widetilde{\mathrm{SL}_2} \times_\alpha S^3, \quad 0 < \alpha < \infty$$

$$\mathrm{Heis}_3 \times_{\mathbb{R}} \widetilde{\mathrm{SL}_2} \qquad \widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2}, \quad 0 < \alpha \leq 1$$

$$L(a;1) \times_{S^1} L(b;1), \quad 0 < a \leq b \text{ coprime in } \mathbb{Z}$$

After the preceding sections, all that remain are geometries where $\tilde{G}$ is a split extension with trivial action—i.e. direct products, which Table 15.24 lists with duplicates hidden.

We start with a study the associated bundle geometries from several viewpoints in Section 15.5.1, since the vocabulary for describing them is useful in proving the classification in Prop. 15.23 (Section 15.5.2). For completeness, the product geometries are listed in Section 15.5.3. Maximality for non-products is deferred to Section 15.6.

Table 15.24: Groups covering the isometry group G of geometries in this section

$\mathfrak{h}$	$B = \mathbb{E}^2$	$B = S^2$	$B = \mathbb{H}^2$
$\mathfrak{isom}\,\mathbb{E}^2 \times \mathbb{R}$	$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \mathbb{R}$	$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times S^3 \times \mathbb{R}$	$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}$
$\mathfrak{so}_3 \times \mathbb{R}$	(duplicate)	$S^3 \times S^3 \times \mathbb{R}$	$S^3 \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}$
$\mathfrak{sl}_2 \times \mathbb{R}$	(duplicate)	(duplicate)	$\widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}$
$\mathfrak{isom}\,\mathrm{Heis}_3$	$\widetilde{\operatorname{Isom}_0 \mathrm{Heis}_3} \times \widetilde{\operatorname{Isom}_0 \mathbb{E}^2}$	$\widetilde{\operatorname{Isom}_0 \mathrm{Heis}_3} \times S^3$	$\widetilde{\operatorname{Isom}_0 \mathrm{Heis}_3} \times \widetilde{\mathrm{SL}_2}$
$\mathfrak{isom}\,\mathbb{E}^2$	$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \widetilde{\operatorname{Isom}_0 \mathbb{E}^2}$	$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times S^3$	$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \widetilde{\mathrm{SL}_2}$
$\mathfrak{so}_3$	(duplicate)	$S^3 \times S^3$	$S^3 \times \widetilde{\mathrm{SL}_2}$
$\mathfrak{sl}_2$	(duplicate)	(duplicate)	$\widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2}$
$T_{\mathbf{1}} \mathrm{Sol}^3$	$\mathrm{Sol}^3 \times \widetilde{\operatorname{Isom}_0 \mathbb{E}^2}$	$\mathrm{Sol}^3 \times S^3$	$\mathrm{Sol}^3 \times \widetilde{\mathrm{SL}_2}$
$T_{\mathbf{1}} \mathrm{Heis}_3$	$\mathrm{Heis}_3 \times \widetilde{\operatorname{Isom}_0 \mathbb{E}^2}$	$\mathrm{Heis}_3 \times S^3$	$\mathrm{Heis}_3 \times \widetilde{\mathrm{SL}_2}$
$\mathbb{R}^3$	(Prop. 15.13)	$\mathbb{R}^3 \times S^3$	$\mathbb{R}^3 \times \widetilde{\mathrm{SL}_2}$

15.5.1 Associated bundle geometries

This section is a study of the associated bundle geometries—the only geometries in this classification that have abelian isotropy and are not products. The most concise names, the classification, and the verification of maximality all use slightly different constructions of these spaces; so this section will provide the notation for and relationships between all three constructions. The first definition motivates the name "associated bundle".

Definition 15.25 (**Associated bundles**, see e.g. [Hus94, Defn. 5.1])**.** Let $E \to B$ be a principal G-bundle, and let $\rho : G \to \operatorname{Aut} F$ be an action of G on some space F. The

F-bundle $E \times_\rho F$ *associated* to $E \to B$ is the bundle over B whose total space is

$$E \times F/(e,f) \sim (eg, \rho(g^{-1})f) \text{ for } g \in G.$$

Example 15.26 (**Associated bundle geometries**). For homogeneous $E = H/H_p$ and $F = K/K_q$, suppose G is a 1-dimensional subgroup of H commuting with H_p by which it acts on E on the right, and $\rho(G)$ is central in K. Then

$$E \times_\rho F = (H \times K)/\left(H_p \cdot K_q \cdot \left\{(g, \rho(g)^{-1}) \mid g \in G\right\}\right).$$

In particular we may take E and F each to be one of the homogeneous spaces

$$\begin{aligned} \mathrm{Heis}_3 &\cong \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SO}(2)}/\widetilde{\mathrm{SO}(2)} \\ L(a;1) &\cong S^3 \rtimes S^1/\left(e^{(2\pi i/a)\mathbb{Z}} \times S^1\right) \\ \widetilde{\mathrm{SL}_2} &\cong \widetilde{\mathrm{SL}_2} \rtimes \widetilde{\mathrm{SO}(2)}/\widetilde{\mathrm{SO}(2)}, \end{aligned}$$

with G or $\rho(G)$ the identity component of the center in each case.

The lens space $L(a;1)$ is $S^3/e^{(2\pi i/a)\mathbb{Z}}$ where S^3 is the unit quaternions and $e^{(2\pi i/a)\mathbb{Z}} = \mathbb{Z}/a\mathbb{Z} \subset S^1 \subset \mathbb{R} + \mathbb{R}i$. In paticular, $L(1;1) \cong S^3$ and $L(2;1) \cong \mathbb{R}P^3$.

Our notation is modeled on that in [Sha00, §1.3] surrounding the "Vector Bundles" subsection—we denote such a bundle by $E \times_G F$ when there is a natural choice of G-action on F, otherwise $E \times_* F$ where $*$ is enough data to specify the G-action. Since G is 1-dimensional, a real number suffices to express this data once some conventions are chosen.

The next definition is a description of the associated bundle geometries as homogeneous spaces in terms of some data arising from the classification strategy.

Definition 15.27 (**Point stabilizers of associated bundle geometries**). Let $\tilde{G}$ be a connected, simply-connected Lie group, and let π denote the quotient map $\tilde{G} \to \tilde{G}/Z(\tilde{G})$.

Suppose $Z(\tilde{G})$ is 1-dimensional, $\tilde{G}/Z(\tilde{G})$ has a 2-dimensional maximal torus T, and $\tau_{0,0} : \mathbb{R}^2 \to \tilde{G}$ is a homomorphism such that $\pi \circ \tau_{a,b}$ has image T and kernel the standard $\mathbb{Z}^2$. For a and b in $\mathbb{R} \cong Z(\tilde{G})^0$, define[2]

$$\begin{aligned} \tau_{a,b} : \mathbb{R}^2 &\to \tilde{G} \\ x, y &\mapsto (xa + yb)\tau_{0,0}(x, y). \end{aligned}$$

The final definition is a notation for line bundles and circle bundles over products of 2-dimensional geometries. Its resemblance to the 3-dimensional geometries with SO(2) isotropy may provide some intuition and will be used to distinguish the associated bundle geometries from each other and from other geometries.

Definition 15.28 (**Associated bundle geometries as bundles over products**)**.** Let X and Y be $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$, scaled to have curvature 0 or ± 1; and let H be $\mathbb{R}$ or $S^1 \cong \mathbb{R}/\mathbb{Z}$. Given a and b in $T_{\mathbf{1}}H$, let $\xi^{a,b} \to X \times Y$ denote the simply-connected principal H-bundle over $X \times Y$ with a connection whose curvature form is

$$\Omega_{a,b} = \frac{1}{2\pi} \left(\mathrm{vol}_X \otimes a + \mathrm{vol}_Y \otimes b \right).$$

The correspondence between these three definitions is summarized in Table 15.29. The proof of the correspondence for the "associated bundle" column is written out only for the first row for illustrative purposes, since this part of the correspondence is only used in the hope of selecting an evocative name. The proof for the rest of the correspondence is Prop. 15.34.

Proposition 15.30. *Let $L(a;1) \times_{S^1,d} L(b;1)$ be the associated bundle geometry as defined in Defn. 15.25, where the fiber S^1 of $L(a;1) \to S^2$ by translating along fibers of $L(b;1) \to S^2$,*

2. $Z(\tilde{G})^0 \cong \mathbb{R}$ follows from computing its π_1 using the homotopy exact sequence for $Z(\tilde{G}) \to \tilde{G} \to \tilde{G}/Z(\tilde{G})$ and the fact that $\pi_2 = 0$ for Lie groups [BD85, Prop. V.7.5].

Table 15.29: Descriptions of associated bundle geometries, assuming nonzero a and b

$\tilde{G}$ for $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$	$\tau_{0,0}(\mathbb{R}^2)$	Associated bundle	$\xi^{x,y} \to X \times Y$
$S^3 \times S^3 \times \mathbb{R}$	$S^1 \times S^1$	$L(a;1) \times_{S^1} L(b;1)$	$\xi^{a,b} \to S^2 \times S^2$
$\widetilde{\mathrm{SL}_2} \times S^3 \times \mathbb{R}$	$\widetilde{\mathrm{SO}(2)} \times S^1$	$\widetilde{\mathrm{SL}_2} \times_{a/b} S^3$	$\xi^{-a,b} \to \mathbb{H}^2 \times S^2$
$\widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}$	$\widetilde{\mathrm{SO}(2)} \times \widetilde{\mathrm{SO}(2)}$	$\widetilde{\mathrm{SL}_2} \times_{a/b} \widetilde{\mathrm{SL}_2}$	$\xi^{-a,-b} \to \mathbb{H}^2 \times \mathbb{H}^2$
$(\widetilde{\mathrm{Isom}\,\mathrm{Heis}_3})^0 \times \widetilde{\mathrm{SL}_2}$	$\widetilde{\mathrm{SO}(2)} \times \widetilde{\mathrm{SO}(2)}$	$\mathrm{Heis}_3 \times_{\mathbb{R}} \widetilde{\mathrm{SL}_2}$	$\xi^{1,1} \to \mathbb{E}^2 \times \mathbb{H}^2$
$(\widetilde{\mathrm{Isom}\,\mathrm{Heis}_3})^0 \times S^3$	$\widetilde{\mathrm{SO}(2)} \times S^1$	$\mathrm{Heis}_3 \times_{\mathbb{R}} S^3$	$\xi^{1,1} \to \mathbb{E}^2 \times S^2$

with kernel $\mathbb{Z}/d\mathbb{Z}$. Then the homogeneous space

$$S^3 \times S^3 \times \mathbb{R}/\tau_{ad,b}(\mathbb{R}^2) \cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, ads + bt\}_{s,t\in\mathbb{R}}.$$

is a $\gcd(ad, b)$*-fold cover of* $L(a;1) \times_{S^1,d} L(b;1)$.

The notation interprets S^3 as the unit quaternions and any complex numbers as lying inside the quaternions. It follows from the above (Prop. 15.30) that

1. $L(a;1) \times_{S^1,d} L(b;1) \cong L(ad;1) \times_{S^1} L(b;1)$;
2. $L(a;1) \times_{S^1} L(b;1) \cong L(b;1) \times_{S^1} L(a;1)$; and
3. $L(a;1) \times_{S^1} L(b;1)$ is simply-connected if and only if $\gcd(a,b) = 1$. (Use the homotopy exact sequence for $\tau_{a,b}(\mathbb{R}^2) \to S^3 \times S^3 \times \mathbb{R} \to L(a;1) \times_{S^1} L(b;1)$.)

Just one key Lemma is required for its proof.

Lemma 15.31. *Let $\gamma : H \to G$ be a homomorphism, and let $C : G \to \mathrm{Inn}\, G$ denote conjugation. Then*

$$G \times H/\{(\gamma(h), h)\}_{h\in H} \cong G \rtimes_{C\circ\gamma} H/H.$$

Proof. Verify from the definition of a semidirect product that

$$\begin{aligned}\phi : G \times H &\to G \rtimes_{C\circ\gamma} H \\ g, h &\mapsto g\gamma(h)^{-1}, h.\end{aligned}$$

is an isomorphism sending $\{(\gamma(h), h)\}_{h\in H}$ onto $H \subseteq G \rtimes_{C\circ\gamma} H$. $\square$

Proof of Prop. 15.30. Interpret S^3 as the unit quaternions, and all complex numbers as lying in the same copy of $\mathbb{C}$ in the quaternions. Since semidirect products with inner action are isomorphic to direct products (Lemma 15.31),

$$\begin{aligned}L(a;1) &= S^3 \rtimes S^1/\{e^{(2\pi i/a)\mathbb{Z}}, t \bmod 2\}_{t\in\mathbb{R}} \\ &\cong S^3 \times \mathbb{R}/\{e^{\pi it+(2\pi i/a)\mathbb{Z}}, t\}_{t\in\mathbb{R}} \\ &\cong S^3 \times \mathbb{R}/\{e^{\pi it}, at + 2\mathbb{Z}\}_{t\in\mathbb{R}}.\end{aligned}$$

So with the formula for homogeneous associated bundles (Example 15.26),

$$L(a;1) \times_{S^1,d} L(b;1) = S^3 \times \mathbb{R} \times S^3 \times \mathbb{R}/\{e^{\pi is}, as + 2\mathbb{Z} + u, e^{\pi it}, bt + 2\mathbb{Z} - du\}_{s,t,u\in\mathbb{R}}.$$

Since u ranges over all of $\mathbb{R}$, the entire first $\mathbb{R}$ factor in $S^3 \times \mathbb{R} \times S^3 \times \mathbb{R}$ is part of the point stabilizer. Since this $\mathbb{R}$ factor is normal, the homogeneous space is equivariantly diffeomorphic to a coset space of $S^3 \times S^3 \times \mathbb{R}$. To see exactly which coset space, let u' be the coordinate in this $\mathbb{R}$ factor. Then

$$\begin{aligned}L(a;1) \times_{S^1,d} L(b;1) &\cong S^3 \times \mathbb{R} \times S^3 \times \mathbb{R}/\{e^{\pi is}, u', e^{\pi it}, bt + 2\mathbb{Z} - d(u' - as - 2\mathbb{Z})\}_{s,t,u'\in\mathbb{R}} \\ &\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, bt + 2\mathbb{Z} + d(as + 2\mathbb{Z})\}_{s,t\in\mathbb{R}} \\ &\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, ads + bt + 2\mathbb{Z}\}_{s,t\in\mathbb{R}}.\end{aligned}$$

The resemblance to $S^3 \times S^3 \times \mathbb{R}/\tau_{ad,b}(\mathbb{R}^2)$ is now apparent. To finish the proof, observe that since $e^{2\pi i} = 1$,

$$\begin{aligned} S^3 \times S^3 \times \mathbb{R}/\tau_{ad,b}(\mathbb{R}^2) &\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, ads + bt\}_{s,t\in\mathbb{R}} \\ &\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, ad(s + 2\mathbb{Z}) + b(t + 2\mathbb{Z})\}_{s,t\in\mathbb{R}} \\ &\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, ads + bt + 2\gcd(ad, b)\mathbb{Z}\}_{s,t\in\mathbb{R}}. \qquad \square \end{aligned}$$

Proposition 15.32. *$\tau_{a,b}(\mathbb{R}^2)$ is closed in $\tilde{G} = S^3 \times S^3 \times \mathbb{R}$ if and only if a and b are linearly dependent over $\mathbb{Q}$.*

Proof. Let H be the preimage in $\tilde{G}$ of the maximal torus in $\tilde{G}/Z(\tilde{G})$ over which $\tau_{a,b}(\mathbb{R}^2)$ lies. As the continuous preimage of a closed set, H is closed; so it suffices to consider closedness in H.

If $\tilde{G} = S^3 \times S^3 \times \mathbb{R}$, then $H \cong S^1 \times S^1 \times \mathbb{R} \cong \mathbb{R}^3/(\mathbb{Z}^2 \times \{0\})$; and $\tau_{a,b}(\mathbb{R}^2)$ is the image in H of a vector subspace $V \subseteq \mathbb{R}^3$ given in coordinates (x, y, z) by $ax + by - z = 0$. Since V and $\mathbb{Z}^2$ are both closed in $\mathbb{R}^3$, all of the following inclusions are closed if any one of them is.

$$\tau_{a,b}(\mathbb{R}^2) = V/(V \cap \mathbb{Z}^2) \subseteq \mathbb{R}^3/\mathbb{Z}^2 \cong H$$

$$V\mathbb{Z}^2 \subseteq \mathbb{R}^3$$

$$\mathbb{Z}^2/(V \cap \mathbb{Z}^2) \subseteq \mathbb{R}^3/V$$

Modulo $ax+by-z$, the generators $(1, 0, 0)$ and $(0, 1, 0)$ of $\mathbb{Z}^2$ become $(0, 0, -a)$ and $(0, 0, -b)$, which generate a closed subgroup of $\mathbb{R}^3/V \cong \mathbb{R}$ if and only if $\gcd(a, b)$ exists. $\square$

Remark 15.33. The rational dependence constraint does not appear if the rank k of $\pi_1(H) \cong \mathbb{Z}^k$ is less than 2, since a rank-at-most-one subgroup of a vector space $\tilde{H}/V$ is closed.

In particular, for all $\tilde{G}$ in this part of the classification other than $S^3 \times S^3 \times \mathbb{R}$, the ratio a/b for $\tau_{a,b}$ is allowed to be any real number (really, any point of $\mathbb{R}P^1$).

Proposition 15.34 (**Correctness of Table 15.29**). *Let* $\tilde{G}$ *and* $\tau_{0,0}$ *be as in one of the rows of Table 15.29. Then*

(i) $M = \tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ *(Defn. 15.27) fibers over the product* $X \times Y$ *in the last column.*

(ii) For some invariant Riemannian metrics on X *and* Y *with curvature* 0 *or* ± 1*, the connection on* $M \to X \times Y$ *given by* $(TM^{\tau_{a,b}(\mathbb{R}^2)})^{\perp}$ *has the same curvature as the bundle in the last column, provided that both* a *and* b *are nonzero.*

Proof of Prop. 15.34(i). The quotient map $\pi : \tilde{G} \to \operatorname{Inn} G = \tilde{G}/Z(\tilde{G})$ sends $\tau_{a,b}(\mathbb{R}^2)$ to a 2-dimensional maximal torus T. Then $(\operatorname{Inn} G)/T$ is a 4-dimensional geometry with point stabilizer T, which is a product of 2-dimensional geometries $X \times Y$ by [Fil83, Thm. 3.1.1(c)]. Computing $\tilde{G}/Z(\tilde{G})$ determines X and Y. □

Proof of Prop. 15.34(ii). Suppose $a, b \in Z(\tilde{G})^0 \cong \mathbb{R}$ are both nonzero. For the connection on $M = \tilde{G}/\tau_{a,b}(\mathbb{R}^2) \to X \times Y$ given by $(TM^{\tau_{a,b}(\mathbb{R}^2)})^{\perp}$, the curvature Ω is an invariant 2-form on $X \times Y$ with values in the Lie algebra of the fiber.

Step 1: Curvature is nonzero only along X **and** Y**.** If V is the standard representation of SO(2), then a tangent space to $X \times Y$ decomposes as the $\mathrm{SO}(2) \times \mathrm{SO}(2)$ representation

$$T_x X \oplus T_y Y \cong (V \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes V),$$

whose second exterior power is

$$\Lambda^2(V \otimes \mathbb{R}) \oplus (V \otimes V) \oplus \Lambda^2(\mathbb{R} \otimes V) \cong 2\mathbb{R} \oplus (V \otimes V).$$

So the 2-form Ω, being $\mathrm{SO}(2) \times \mathrm{SO}(2)$-invariant, is determined by its values on the $2\mathbb{R}$ summand—that is, on $\Lambda^2(T_x X)$ and $\Lambda^2(T_y Y)$.

Step 2: Express the preimage of X as a homogeneous space. Fix a copy of X, and let E be its preimage in M. Then choosing a factor of $\tilde{G}$ acting transitively on X and containing $Z(\tilde{G})^0$ expresses E as covered by one of the following homogeneous spaces H/H_p.

$$\mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SO}(2)}/\{(0,0,at),\gamma(t)\}_{t\in\mathbb{R}} \cong \mathrm{Heis}_3$$
$$S^3 \times \mathbb{R}/\{e^{\pi it}, at\}_{t\in\mathbb{R}} \cong S^3 \text{ (or } S^2 \times \mathbb{R} \text{ if } a=0)$$
$$\widetilde{\mathrm{SL}_2} \times \mathbb{R}/\{\gamma(t), at\}_{t\in\mathbb{R}} \cong \widetilde{\mathrm{SL}_2} \text{ (or } \mathbb{H}^2 \times \mathbb{R} \text{ if } a=0)$$

The notation conventions are that:

1. Heis_3 is $\mathbb{R}^3$ with the composition law $(x,y,z)(x',y',z') = (x+x', y+y', z+z'+xy')$;
2. $\gamma : \mathbb{R} \xrightarrow{\sim} \widetilde{\mathrm{SO}(2)}$ sends $t \in \mathbb{R}$ to a rotation by $2\pi t$; and
3. S^3 and $\mathbb{C}$ are interpreted as subsets of the quaternions.

Step 3: If X has nonzero curvature K, then the curvature along X is $\frac{K}{2\pi}\mathrm{vol}_X \otimes a$. The curvature of $(TM^{\tau_{a,b}(\mathbb{R}^2)})^\perp$ restricted to X is the curvature of TE^H as a connection on $E \to X$.

Equivariantly, S^3 covers $\mathrm{SO}(3) \cong T^1S^2$ and $\widetilde{\mathrm{SL}_2}$ covers $\mathrm{PSL}(2,\mathbb{R}) \cong T^1\mathbb{H}^2$; which takes TE^H to the connection induced by the Levi-Civita connection on S^2 or $\mathbb{H}^2$. If the unit tangent circles are declared to have length 2π, then this connection's curvature is the surface's scalar curvature K times its area form vol_X. (This is a version of the Gauss-Bonnet Theorem; see e.g. [DC76, §4.5 p. 274].) In $E = H/H_p$, such a circle is most naturally assigned the length of the interval $[0,a) \subset \mathbb{R} \cong Z(\tilde{G})^0 \subset H$ that maps bijectively onto it. Then identifying $T_0\mathbb{R}$ with $\mathbb{R} \cong Z(\tilde{G})^0$ by the exponential map gives the expression $\frac{K}{2\pi}\mathrm{vol}_X \otimes a$ for the curvature over X.

The same argument applies to Y, which establishes the claim (ii) for the first three rows of Table 15.29.

Step 4: Curvature along $X = \mathbb{E}^2$ is any nonzero multiple of the volume form. First, observe that

$$\begin{aligned} \mathrm{Heis}_3 \rtimes \widetilde{\mathrm{SO}(2)}/\{(0,0,at), \gamma(t)\}_{t\in\mathbb{R}} &\cong \mathrm{Heis}_3 \rtimes \mathrm{SO}(2)/\,\mathrm{SO}(2) \\ x, y, z, \gamma(t) &\mapsto x, y, z - at, \gamma(t). \end{aligned}$$

Under the usual metrics, $\mathrm{Heis}_3 \to \mathbb{E}^2$ with the connection $T\mathrm{Heis}_3^{\mathrm{SO}(2)}$ has curvature 1 (i.e. 1 times the area form on $\mathbb{E}^2$) [Thu97, discussion after Exercise 3.7.1]. Through an appropriate (possibly orientation-reversing) conformal automorphism of $\mathbb{E}^2$, the area form can be pulled back to any nonzero constant multiple of itself.

To finish, apply Step 3 to Y, choosing a length scale on $Z(\tilde{G})^0 = Z(\mathrm{Heis}_3)$ that makes the curvature along Y equal to $\frac{1}{2\pi}\mathrm{vol}_Y$. Then choose an orientation and scale on $X = \mathbb{E}^2$ that makes the curvature along X equal to $\frac{1}{2\pi}\mathrm{vol}_X$. This establishes claim (ii) for the remaining (last two) rows of Table 15.29. □

15.5.2 Proof of the classification

This section proves Prop. 15.23, the classification of geometries G/G_p when $\tilde{G}$ is a direct product listed in Table 15.24. The recurring method is to relate G_p to a maximal torus of some group; each individual step is merely whatever happens to decrease the number of remaining cases. Maximality is deferred to Section 15.6.

As part of the classification, we also prove that the associated bundle geometry $\widetilde{\mathrm{SL}_2} \times_\alpha S^3$ is a model geometry (Prop. 15.35) but has compact quotients if and only if α is rational (Prop. 15.36).

Proof of Prop. 15.23 (except for maximality). Let G/G_p be a 5-dimensional maximal model geometry where G is covered by one of the groups in Table 15.24.

Step 1: If $\dim G = 6$ and $\tilde{G} \not\cong * \times \mathrm{Sol}^3$, then G/H is non-maximal. We will show that if $\dim G = 6$ and the maximal torus of $\operatorname{Aut}\tilde{G}$ has dimension at least 2, then any 5-dimensional geometry with isometry group G is non-maximal.

Since $\dim G = 6$, the point stabilizer G_p is 1-dimensional and therefore isomorphic to S^1. Let H be the lift of G_p to $\tilde{G}$, i.e. the group such that $\tilde{G}/H \cong G/G_p$. Since $\operatorname{Aut}\tilde{G}$ has a maximal torus of dimension at least 2, there is an $S^1 \subset \operatorname{Aut}\tilde{G}$ that commutes with the inner action of H and is independent—that is, $H \times S^1$ maps to a 2-dimensional subgroup of $\operatorname{Aut}\tilde{G}$. Then

$$(\tilde{G} \rtimes S^1)/(H \times S^1)$$

is a homogeneous space subsuming G/G_p; and a geometry subsuming G/G_p is realized by passing to the quotient by $Z(\tilde{G} \rtimes S^1) \cap (H \times S^1)$ (Rmk. 15.4).

In particular, if $\tilde{G}$ is a product of any two of the following groups—each with dimension 3 and an S^1 in its automorphism group—then any 5-dimensional geometry G/G_p is non-maximal.

$$\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \qquad S^3 \qquad \widetilde{\mathrm{PSL}(2,\mathbb{R})} \qquad \mathrm{Heis}_3 \qquad \mathbb{R}^3$$

Step 2: G_p maps injectively to $\operatorname{Inn} G = \tilde{G}/Z(\tilde{G})$ as a maximal torus. In the remaining cases, $\tilde{G}$ is one of the following products, where each $\bullet$ denotes the identity component of the isometry group of $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$.

$$\tilde{\bullet} \times \tilde{\bullet} \times \mathbb{R} \qquad \tilde{\bullet} \times \widetilde{\operatorname{Isom}_0 \mathrm{Heis}_3} \qquad \tilde{\bullet} \times \mathrm{Sol}^3$$

The dimensions of these $\tilde{G}$ are respectively 7, 7, and 6. The corresponding $\operatorname{Inn} G$ are

$$\bullet \times \bullet \qquad \bullet \times \operatorname{Isom}_0 \mathbb{E}^2 \qquad \bullet \times \mathrm{Sol}^3 .$$

The maximal torus of $\bullet$ has dimension 1, so the maximal torus T of $\operatorname{Inn} G$ has dimension 2, 2, or 1, respectively. Since G_p has faithful conjugation action in a geometry (Lemma 15.3), which is equivalent to injectivity of $G_p \to \operatorname{Inn} G$, the claim in this step follows from $\dim T = \dim G - 5$.

Step 3: If $\tilde{G} = * \times \mathrm{Sol}^3$ then G/G_p is a product with Sol^3. In this case, $\tilde{G} \to \operatorname{Inn} G$ is a covering map; so if H is the identity component of the preimage of G_p in $\tilde{G}$, then H covers a maximal torus of $\operatorname{Inn} G$. Then all possible H are conjugate in $\tilde{G}$, so

$$G/G_p \cong \mathrm{Sol}^3 \times \mathrm{Isom}_0 B / \mathrm{SO}(2) \cong \mathrm{Sol}^3 \times B$$

where B is $\mathbb{E}^2$, S^2, or $\mathbb{H}^2$.

Step 4: $G/G_p \cong \tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ for some a and b in $Z(\tilde{G})^0$. In the remaining cases, $\dim Z(\tilde{G}) = 1$ and $\operatorname{Inn} G = \tilde{G}/Z(\tilde{G})$ has 2-dimensional maximal torus T. Since $\pi : \tilde{G} \to \operatorname{Inn} G$ descends to a map from G sending G_p isomorphically to T (Step 2), the preimage of G_p in $\tilde{G}$ is the image of a homomorphism $\tau : \mathbb{R}^2 \to \pi^{-1}(T)$. Any two such τ that compose with π to the same map $\mathbb{R}^2 \to T$ differ only by some $\mathbb{R}^2 \to Z(\tilde{G})$; so τ can be identified with some $\tau_{a,b}$ as defined in Defn. 15.27 after choosing $\tau_{0,0}$.

In $\widetilde{\mathrm{Isom}_0 \mathbb{E}^2} \times S^3 \times \mathbb{R}$, choose $\tau_{0,0}(\mathbb{R}^2) = \widetilde{\mathrm{SO}(2)} \times S^1 \times \{0\}$. In the three other $\tilde{G}$ that have $\widetilde{\mathrm{Isom}_0 \mathbb{E}^2}$ as a factor, let $\tau_{0,0}(\mathbb{R}^2) = \widetilde{\mathrm{SO}(2)} \times \widetilde{\mathrm{SO}(2)}$. All remaining $\tilde{G}$ occur in Table 15.29, where the choice of $\tau_{0,0}$ is also recorded.

Step 5: If $\tilde{G} = \widetilde{\mathrm{Isom}_0 \mathbb{E}^2} \times K$ for some group K, then G/G_p is a product with $\mathbb{E}^2$. Express $\widetilde{\mathrm{Isom}_0 \mathbb{E}^2}$ as $\mathbb{C} \rtimes \widetilde{\mathrm{SO}(2)}$, i.e. as $\mathbb{C} \times \mathbb{R}$ with the composition law

$$(x + iy, z)(x' + iy', z') = (x + iy + e^{iz}(x' + iy'), z + z').$$

Then in $\tilde{G}/\tau_{a,b}(\mathbb{R}^2) \to \mathbb{E}^2 \times Y$, the preimage E of $\mathbb{E}^2$ has a transitive action by a subgroup of $\tilde{G}$ isomorphic to $\widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \mathbb{R}$; and there is an equivariant diffeomorphism

$$E = \widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times \mathbb{R}/\{(0+0i, z), az\}_{z\in\mathbb{R}} \xrightarrow{\sim} \operatorname{Isom}_0 \mathbb{E}^2 \times \mathbb{R}/\operatorname{SO}(2)$$
$$x+iy, z, t \mapsto x+iy, z, t-az.$$

Extending this to $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ yields[3]

$$\tilde{G}/\tau_{a,b}(\mathbb{R}^2) \cong \widetilde{\operatorname{Isom}_0 \mathbb{E}^2} \times K/\tau_{0,b}(\mathbb{R}^2) \cong \mathbb{E}^2 \times (K/K_q).$$

Step 6: If $\tilde{G} = (\widetilde{\operatorname{Isom} \operatorname{Heis}_3})^0 \times K$, then G/G_p is one of four geometries. There are only two cases remaining for this step to handle: $K = S^3$ and $K = \widetilde{\operatorname{SL}_2}$. Step 4 of Prop. 15.34 showed that the equivariant diffeomorphism type of $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ is independent of a, so set $a = 0$. Then if Heis_3 is $\mathbb{R}^3$ with the composition law

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$$

and $\gamma : \mathbb{R} \to K$ is a 1-parameter subgroup of K with $\gamma(\mathbb{Z}) = \gamma(\mathbb{R}) \cap Z(K)$,

$$\tilde{G}/\tau_{0,b}(\mathbb{R}^2) = \operatorname{Heis}_3 \rtimes \operatorname{SO}(2) \times K/\{(0, 0, bt), s, \gamma(t)\}_{s,t\in\mathbb{R}}$$

is one of

1. a product $\operatorname{Heis}_3 \times S^2$ or $\operatorname{Heis}_3 \times \mathbb{H}^2$ if $b = 0$; or
2. equivariantly diffeomorphic to $\tilde{G}/\tau_{0,1}(\mathbb{R}^2)$ by a map sending $(x, y, z, r) \in \operatorname{Heis}_3 \rtimes \operatorname{SO}(2)$

3. That the diffeomorphism extends may be easier to see on the Lie algebra level. It corresponds to an automorphism $\mathfrak{isom}\,\mathbb{E}^2 \times \mathbb{R}$ that sends a basis $(\hat{x}, \hat{y}, \hat{z}, \hat{t})$ to $(\hat{x}, \hat{y}, \hat{z} - a\hat{t}, \hat{t})$. This extends to the appropriate automorphism of $T_1 G = \mathfrak{isom}\,\mathbb{E}^2 \oplus \mathfrak{k}$ by the identity on $\mathfrak{k}$.

to

$$\left(x|b|^{-1/2}, sy|b|^{-1/2}, z|b|^{-1}, r^s\right)$$

where $s = b/|b|$, if $b \neq 0$. So the names $\mathrm{Heis}_3 \times_{\mathbb{R}} S^3$ and $\mathrm{Heis}_3 \times_{\mathbb{R}} \widetilde{\mathrm{SL}_2}$ from Table 15.29 can be used without ambiguity.

Step 7: Parametrize the isomorphism types of $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ by $\big[|a| : |b|\big] \in \mathbb{R}P^1$. In the remaining cases, $M = \tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ has a canonical fibering over

$$X \times Y \cong \tilde{G}/(Z(\tilde{G}) \cdot \tau_{a,b}(\mathbb{R}^2)),$$

where X and Y are surfaces of nonzero constant curvature. Decomposing $G_p \curvearrowright T_pM$ into irreducible subrepresentations, every invariant inner product on T_pM (and hence every invariant metric on M) is determined by a scale factor along the fiber (i.e. on $T_pM^{G_p}$), a scale factor on X, and a scale factor on Y (Lemmas 12.10 and 12.14). The scale factors on X and Y can be chosen by normalizing their curvatures to be ± 1.

Given any member of this family of normalized metrics, the ratio of curvatures for the connection $(TM^G)^\perp$ on $M \to X \times Y$ (listed in Table 15.29) represents, up to finite choices, an invariant for the family. Explicitly, an invariant number can be recovered as the ratio of displacements along a fiber that result from horizontal lifts of loops enclosing the same small area in X and Y. The choices that need to be made are

1. assigning X and Y to the two factors in the base space, and
2. orientations on X and Y.

These reflect the following symmetries.

0. Rescaling $\mathbb{R} \subset \tilde{G}$ induces $\tilde{G}/\tau_{a,b}(\mathbb{R}^2) \cong \tilde{G}/\tau_{at,bt}(\mathbb{R}^2)$ for nonzero t.
1. If $X \cong Y$, then exchanging X and Y allows assuming $|a| \leq |b|$.

2. Conjugating by j in S^3 reverses the 1-parameter subgroup e^{it}; and conjugating $\widetilde{\mathrm{SL}_2}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reverses the 1-parameter subgroup $\widetilde{\mathrm{SO}(2)}$. This allows assuming $a > 0$ and $b > 0$.

There are two more considerations that affect the parametrization:

3. If $a = 0$ or $b = 0$, then products result as in Step 6.

4. If $\tilde{G} = S^3 \times S^3 \times \mathbb{R}$, then a and b need to be rationally dependent (Prop. 15.32), which allows rescaling them both to be coprime integers. This constraint is not present for other $\tilde{G}$ (Rmk. 15.33).

So excluding products, the geometries remaining to be classified—those fibering over products of S^2 and $\mathbb{H}^2$—are specified exactly once each by the following.

$$\widetilde{\mathrm{SL}_2} \times_{a/b} \widetilde{\mathrm{SL}_2} = \widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}/\tau_{a,b}(\mathbb{R}^2), \quad a/b \in (0, 1]$$
$$\widetilde{\mathrm{SL}_2} \times_{a/b} S^3 = \widetilde{\mathrm{SL}_2} \times S^3 \times \mathbb{R}/\tau_{a,b}(\mathbb{R}^2), \quad a/b \in (0, \infty)$$
$$L(a; 1) \times_{S^1} L(b; 1) = S^3 \times S^3 \times \mathbb{R}/\tau_{a,b}(\mathbb{R}^2), \quad 0 < a \leq b \text{ coprime in } \mathbb{Z}.$$

Step 8: All of the above are model geometries. Products of model geometries are model geometries, since they model products of manifolds modeled on the factors. The $L(a; 1) \times_{S^1} L(b; 1)$ geometry is a model geometry since it is already compact. So it only remains to show that bundles associated to Heis_3 and $\widetilde{\mathrm{SL}_2}$ are also model geometries.

The construction is: if E and F model compact circle bundles M and N, then $M \times_{S^1} N$ is modeled on some $E \times_\rho F$. In particular, Heis_3 models the circle bundle $\mathrm{Heis}_3/\mathrm{Heis}_3(\mathbb{Z})$ over a torus modeled on $\mathbb{E}^2 \cong \mathrm{Heis}_3/Z(\mathrm{Heis}_3)$; S^3 models the Hopf fibration over S^2; and $\widetilde{\mathrm{SL}_2}$ models the unit tangent bundle of any compact hyperbolic surface.

However, only some $E \times_\rho F$ can be recovered as the universal cover of an $M \times_{S^1} N$: combinations of these three bundles are modeled only on $\tilde{G}/\tau_{1,1}(\mathbb{R}^2)$. This is enough for the bundles associated to Heis_3. For bundles associated to $\widetilde{\mathrm{SL}_2}$, compact bundles modeled on $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ can be obtained, for integers a and b, by starting with quotients of the initial circle bundles by rotations of $2\pi/a$ and $2\pi/b$. This, however, still leaves the case when a/b is irrational, which is a somewhat different construction, given below in Prop. 15.35. □

Proposition 15.35. *If $\tilde{G} = \widetilde{\mathrm{SL}_2} \times * \times \mathbb{R}$, then $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ is a model geometry.*

Proof. More honestly, since a geometry $M = G/G_p$ must have G acting faithfully, we need to use $G = \tilde{G}/(Z(\tilde{G}) \cap \tau_{a,b}(\mathbb{R}^2)) \cong \tilde{G}/\tau_{a,b}(\mathbb{Z}^2)$ (Rmk. 15.4).

Step 1: Construct a subgroup $\tilde{\Gamma} \times \tilde{\Lambda}$ of $\tilde{G}$. Assume $b \neq 0$ since otherwise G/G_p is a product with S^2 or $\mathbb{H}^2$. Let $\Gamma = \pi_1(S) \subset \mathrm{PSL}(2,\mathbb{R})$ for some orienable[4] punctured surface S of genus at least 2, and let $\tilde{\Gamma}$ be its preimage in $\widetilde{\mathrm{SL}_2}$. Then $\tilde{\Gamma}$ is central extension of a free group by $Z(\widetilde{\mathrm{SL}_2}) \cong \mathbb{Z}$; so it splits as a semidirect product, and $Z(\widetilde{\mathrm{SL}_2})$ maps to a copy of $\mathbb{Z}$ in the abelianization of $\tilde{\Gamma}$. Then there is a homomorphism $f : \tilde{\Gamma} \to \mathbb{R}$ that sends $Z(\widetilde{\mathrm{SL}_2}) \subset \widetilde{\mathrm{SO}(2)}$ to $\mathbb{Z}$. Let Λ be the fundamental group of S^2 or a compact orientable surface modeled on $\mathbb{H}^2$, let $\tilde{\Lambda}$ be its lift to the group $*$, and define

$$
\begin{aligned}
i : \tilde{\Gamma} \times \tilde{\Lambda} &\to \widetilde{\mathrm{SL}_2} \times * \times \mathbb{R} \\
g, h &\mapsto (g, h, af(g)).
\end{aligned}
$$

Step 2: $\tilde{\Gamma} \times \tilde{\Lambda}$ descends to a discrete subgroup Δ of G. Let γ be a 1-parameter subgroup of $*$ sending $\mathbb{Z}$ to the center. The image of i does not accumulate on $\tau_{a,b}(\{0\}\times\mathbb{Z}) = \{1, \gamma(n), nb\}_{n\in\mathbb{Z}}$: if the first coordinate of $i(g,h)$ is near 1, then g is near—hence equal to—1, which makes the last coordinate 0; so with $b \neq 0$ the only nearby point of $\{1, \gamma(n), nb\}_{n\in\mathbb{Z}}$ is the identity in $\tilde{G}$. Therefore $i(\tilde{\Gamma} \times \tilde{\Lambda})$ remains discrete in $\tilde{G}/\tau_{a,b}(\{0\} \times \mathbb{Z})$. Since it was constructed to contain $\tilde{G}/\tau_{a,b}(\mathbb{Z} \times \{0\})$, it remains discrete in $G = \tilde{G}/\tau_{a,b}(\mathbb{Z}^2)$.

4. Requiring orientability permits assuming that $\pi_1(S)$ embeds in $\mathrm{PSL}(2,\mathbb{R})$, the identity component of $\mathrm{Isom}\,\mathbb{H}^2$.

Step 3: Δ is a lattice in G. With discreteness established, it suffices to show that for the image Δ of $\tilde{\Gamma} \times \tilde{\Lambda}$ in G, the volume of $G/\Delta \cong \tilde{G}/\left(i(\tilde{\Gamma} \times \tilde{\Lambda}) \cdot \tau_{a,b}(\mathbb{Z}^2)\right)$ is finite.

So let $H = i(\tilde{\Gamma}) \times \tilde{\Lambda}) \cdot \tau_{a,b}(\mathbb{Z}^2)$, and observe that

$$\tilde{G}/(H \cdot (* \times \mathbb{R})) \cong \widetilde{\mathrm{SL}_2}/\tilde{\Gamma} \cong S \quad \text{(chosen at the start of Step 1)}$$

$$H \cdot (* \times \mathbb{R})/(H \cdot \mathbb{R}) \cong */\tilde{\Lambda} \quad \text{(the compact surface chosen at the end of Step 1)}$$

$$H \cdot \mathbb{R}/H \cong \mathbb{R}/\{nb\}_{n \in \mathbb{Z}} \cong S^1.$$

In a situation involving only closed subgroups $E \subseteq F \subseteq G$ of a locally compact G, an invariant measure on G/E is constructed as a product of invariant measures on G/F and F/E as in [Mos62b, 2.4 Case 2]; so the volume of G/H is finite since all three intermediate spaces are.

Step 4: $\Delta\backslash G/G_p$ is a finite-volume manifold. The space $\Delta\backslash G/G_p$ has finite volume and is modeled on G/G_p but might be an orbifold—ruling out orbifold points requires checking that Δ acts freely, i.e. that Δ meets each point stabilizer in only the identity. Since Δ is discrete, its intersection with any compact point stabilizer has finite order. So it suffices to check that a subgroup of $\tilde{G} = \widetilde{\mathrm{SL}_2} \times * \times \mathbb{R}$ surjecting onto Δ—specifically, $i(\tilde{\Gamma} \times \tilde{\Lambda})$—contains no element (g, h, t) outside of $\tau_{a,b}(\mathbb{Z}^2)$ with a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$.

Since $\tau_{a,b}(\mathbb{Z}^2)$ is central, any $i(g, h) = (g, h, af(g))$ with a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$ has finite-order image in $\tilde{G}/Z(\tilde{G})$. Since g is in the lift $\tilde{\Gamma}$ of $\pi_1(S)$ where S is a surface (in particular, with no orbifold points), g has finite-order image in $\mathrm{PSL}(2, \mathbb{R})$ only if this image is the identity. Similarly, h lies over the identity of $\mathrm{SO}(3)$ or $\mathrm{PSL}(2, \mathbb{R})$; so some $\tau_{a,b}(m, n)$ has the same first two coordinates g and h.

If $* = S^3$, then $h = 1$ and $(g, 1, af(g)) = \tau_{a,b}(m, 0)$. If $* = \widetilde{\mathrm{SL}_2}$, then $Z(\tilde{G})$ has no finite-order elements, which makes nth roots unique; so $(g, h, af(g))$ has a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$ if and only if it lies in $\tau_{a,b}(\mathbb{Z}^2)$ itself. Either way, $i(\tilde{\Gamma} \times \tilde{\Lambda})$ contains nothing outside of $\tau_{a,b}(\mathbb{Z}^2)$ with a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$; so by the first paragraph, $\Delta\backslash G/G_p$ has no orbifold points and is a finite-volume manifold modeled on G/G_p. $\square$

The above construction always produces noncompact $\Delta\backslash G/G_p$, with fundamental group independent of a and b. In compact manifolds, the story is different—for instance, one can prove the following.

Proposition 15.36. *Let $\tilde{G} \cong \widetilde{\mathrm{SL}_2} \times S^3 \times \mathbb{R}$, If there is a compact manifold N modeled on $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ and $b \neq 0$, then $a/b \in \mathbb{Q}$.*

Proof. As above in Prop. 15.35, $G = \tilde{G}/\tau_{a,b}(\mathbb{Z}^2)$, in which $\pi_1(N)$ is a cocompact lattice. Its preimage $\tilde{\Gamma}$ in $\tilde{G}$ is also a cocompact lattice. We will study the projection of $\tilde{\Gamma}$ to $\Gamma \subset \widetilde{\mathrm{SL}_2} \times (\mathbb{R}/2b\mathbb{Z})$ and show in particular that $\tau_{a,b}(\mathbb{Z} \oplus \{0\})$ projects to a finite subgroup of $\mathbb{R}/2b\mathbb{Z}$.

Step 1: Γ is discrete in $\widetilde{\mathrm{SL}_2} \times (\mathbb{R}/2b\mathbb{Z})$. Since $\tilde{\Gamma}$ is discrete and S^3 is compact, $\tilde{\Gamma}$ cannot accumulate on a coset of S^3. Then $\tilde{\Gamma}S^3$ is closed, so $\tilde{\Gamma}/(\tilde{\Gamma}\cap S^3)$ is discrete in $\tilde{G}/S^3 \cong \widetilde{\mathrm{SL}_2}\times\mathbb{R}$. Moreover,

$$\tilde{\Gamma} \supset \tau_{a,b}(\mathbb{Z}^2) \supset \tau_{a,b}(\{0\} \oplus \mathbb{Z}) = \{1, e^{\pi i n}, bn\}_{n\in\mathbb{Z}} \subset \widetilde{\mathrm{SL}_2} \times S^3 \times \mathbb{R},$$

so $\tilde{\Gamma}$ contains $2b\mathbb{Z} \subset \mathbb{R}$. Then $\Gamma = \tilde{\Gamma}/(S^3 \times 2b\mathbb{Z})$ is discrete in $\widetilde{\mathrm{SL}_2} \times (\mathbb{R}/2b\mathbb{Z})$.

Step 2: Γ descends to a cocompact lattice Λ in $\mathrm{PSL}(2,\mathbb{R})$. This proceeds similarly to Step 1. Let $\tilde{\Lambda}$ be the projection of Γ to $\widetilde{\mathrm{SL}_2}$. Since $\mathbb{R}/2b\mathbb{Z}$ is compact, $\tilde{\Lambda}$ is discrete in $\widetilde{\mathrm{SL}_2}$. Furthermore, if k is a generator of $Z(\widetilde{\mathrm{SL}_2})$, then the image in $\widetilde{\mathrm{SL}_2} \times (\mathbb{R}/2b\mathbb{Z})$ of $\tau_{a,b}(\mathbb{Z}^2) \subset \tilde{\Gamma}$ is

$$\{k^n, an \bmod 2b\}_{n\in\mathbb{Z}} \subset \Gamma.$$

Then $\tilde{\Lambda}$ contains $Z(\widetilde{\mathrm{SL}_2})$, so its image Λ in $\mathrm{PSL}(2,\mathbb{R})$ is discrete; and Λ is cocompact since $\Lambda\backslash \mathrm{PSL}(2,\mathbb{R})$ is a quotient space of $\tilde{\Gamma}\backslash\tilde{G}$.

Step 3: k becomes torsion in the abelianization of $\tilde{\Lambda}$. Since $\Lambda \subset \mathrm{PSL}(2,\mathbb{R})$ is a cocompact lattice, it is the orbifold fundamental group of a compact orbifold O modeled on $\mathbb{H}^2$; and $\tilde{\Lambda} = \pi_1(T^1 O)$ (see e.g. [Thu02, §13.4] for some discussion of unit tangent bundles of orbifolds). Since a hyperbolic 2-orbifold admits a finite cover by a hyperbolic surface [Sco83, Thm. 2.3 and 2.5], $\tilde{\Lambda}$ contains a subgroup isomorphic to the unit tangent bundle of a closed hyperbolic surface S. Its center is generated by k, and $k^{\chi(S)}$ is a product of commutators [Sco83, discussion surrounding Lemma 3.5]. So k becomes finite order in the abelianization of $\tilde{\Lambda}$.

Step 4: $\tau_{a,b}(\mathbb{Z}^2)$ becomes torsion in the abelianization of Γ. The intersection of the compact $\mathbb{R}/2b\mathbb{Z}$ with the discrete Γ is a finite cyclic group C, which makes Γ a central extension

$$1 \to C \to \Gamma \to \tilde{\Lambda} \to 1.$$

Using the Stallings exact sequence [Bro82, II.5 Exercise 6(a)], the induced

$$C \to \Gamma^{\mathrm{Ab}} \to \tilde{\Lambda}^{\mathrm{Ab}}$$

is exact in the middle. Then $(k, a \bmod 2b) \in \Gamma$, which lies over $k \in \tilde{\Lambda}$, becomes finite order in the abelianization; so its image a in the abelian $\mathbb{R}/2b\mathbb{Z}$ has finite order. Therefore a is a rational multiple of b. □

15.5.3 Explicit enumeration of product geometries

This section collects a list of the product geometries with nontrivial abelian isotropy—i.e. those with the 2-dimensional base in the fibering description (Prop. 12.3(iii)).

Proposition 15.37. *The maximal model product geometries with nontrivial abelian isotropy are:*

(i) 4*-by-*1*:*

$$\mathbb{F}^4 \times \mathbb{E} \qquad\qquad \mathrm{Sol}_0^4 \times \mathbb{E}$$

(ii) 2*-by-*2*-by-*1*:*

$$S^2 \times S^2 \times \mathbb{E} \qquad S^2 \times \mathbb{H}^2 \times \mathbb{E} \qquad \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{E}$$

(iii) 3*-by-*2*:*

$$\begin{array}{ccc} \mathrm{Heis}_3 \times \mathbb{E}^2 & \mathrm{Heis}_3 \times S^2 & \mathrm{Heis}_3 \times \mathbb{H}^2 \\ \mathrm{Sol}^3 \times \mathbb{E}^2 & \mathrm{Sol}^3 \times S^2 & \mathrm{Sol}^3 \times \mathbb{H}^2 \\ \widetilde{\mathrm{SL}_2} \times \mathbb{E}^2 & \widetilde{\mathrm{SL}_2} \times S^2 & \widetilde{\mathrm{SL}_2} \times \mathbb{H}^2 \end{array}$$

Proof. The list can be built up by looking up the factors in previous classifications by Thurston [Thu97, Thm. 3.8.4] and Filipkiewicz [Fil83]. Products with multiple Euclidean factors are non-maximal and omitted.

As products of model geometries, these are all model geometries (Prop. 11.2). All are products with at most one Euclidean factor; and except for $\mathrm{Sol}_0^4 \times \mathbb{E}$, all are products of two maximal geometries where one has no trivial subrepresentation in its isotropy, which makes them maximal (Prop. 12.12). Maximality of $\mathrm{Sol}_0^4 \times \mathbb{E}$ was proven in Step 6 of Prop. 15.7(iii). □

15.6 Deferred maximality proofs for isometrically fibering geometries

Having just handled the products in Section 15.5.3 above, maximality remains to be proven only for the five associated bundle geometries or families and the six geometries from semidirect products in Prop. 15.17.

The strategy in many cases (perhaps because the author failed to think of anything less ad-hoc) is to show that G'/H' cannot subsume G/H by showing that some subgroup of G does not appear in G'. The most useful in what follows is a restriction on embeddings of Heis_3.

Lemma 15.38. *Let $\mathfrak{g}$ be a direct product of algebras of the form $T_{\mathbf{1}} \operatorname{Isom} M$ ($M = S^k$, $\mathbb{E}^k$, or $\mathbb{H}^k$). Then $\mathfrak{g}$ contains no nonabelian nilpotent subalgebra.*

Proof. If a subalgebra has an abelian projection to each factor in the product, then it is itself abelian; so it suffices to prove the Lemma when $\mathfrak{g} = T_{\mathbf{1}} \operatorname{Isom} M$ for $M = S^k$, $\mathbb{E}^k$, and $\mathbb{H}^k$.

Case 1: $M = S^k$. In this case one can in fact show that every solvable subalgebra is abelian. A solvable subalgebra of $T_{\mathbf{1}} \operatorname{Isom}_0 S^k \cong \mathfrak{so}_{k+1}\mathbb{R}$ is tangent to a connected solvable group, whose closure in $\mathrm{SO}(k+1)$ is solvable since solvability is a closed condition. A connected solvable Lie group is abelian if it is compact [Kna02, Cor. IV.4.25].

Case 2: $M = \mathbb{E}^k$. Suppose x and y generate a nonabelian subalgebra $\mathfrak{g}$ of $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^k \cong \mathbb{R}^k \rtimes \mathfrak{so}_k$. If their images in $\mathfrak{so}_k$ do not commute, then Case 1 implies $\mathfrak{g}$ is not solvable.

Otherwise, write $x = (t_x \in \mathbb{R}^k, r_x \in \mathfrak{so}_k)$ and $y = (t_y, r_y)$. Then

$$\begin{aligned}[x,y] &= r_x t_y - r_y t_x \\ [x,[x,y]] &= r_x^2 t_y \\ [y,[y,x]] &= r_y^2 t_x.\end{aligned}$$

Since $[x,y] \neq 0$, at least one of $r_x t_y$ and $r_y t_x$ is nonzero. Then since r_x and r_y act semisimply on $\mathbb{R}^k$, at least one of $[x,[x,y]]$ and $[y,[y,x]]$ is nonzero. Recursing, the lower central series of $\mathfrak{g}$ is never zero, so $\mathfrak{g}$ is not nilpotent.

Case 3: $M = \mathbb{H}^k$. Suppose $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{so}_{1,k} \cong T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{H}^k$, with corresponding group G. Every connected solvable subgroup of $\operatorname{Isom}_0 \mathbb{H}^k$ fixes a point either in $\mathbb{H}^k$ or in its boundary at infinity $\partial_\infty \mathbb{H}^k \cong S^{k-1}$ [Rat06, Thm. 5.5.10]. Then:

- If this fixed point is in $\mathbb{H}^k$, then $G \subseteq \mathrm{SO}(k)$, which makes G abelian by Case 1.
- Otherwise, the fixed point is in the boundary at infinity $\partial_\infty \mathbb{H}^k$. Since $\operatorname{Isom}_0 \mathbb{H}^k \cong \operatorname{Conf}^+ S^{k-1}$ [BP92, Prop. A.5.13(4)] acts on S^{k-1} with point stabilizer $\operatorname{Conf}^+ \mathbb{E}^{k-1}$ [BP92, Cor. A.3.8], this implies that $G \subseteq \operatorname{Conf}^+ \mathbb{E}^{k-1} \cong \mathbb{R} \times \operatorname{Isom}_0 \mathbb{E}^{k-1}$, which makes G is abelian by Case 2. □

Proposition 15.39. *The five associated bundle geometries classified in Prop. 15.23 are maximal.*

Proof. Let $M = G/G_p$ be one of the associated bundle geometries.

Step 1: The isotropy of any subsuming geometry is $\mathrm{SO}(5)$ **or** $\mathrm{SO}(3) \times \mathrm{SO}(2)$**.** In any geometry G'/G'_p properly subsuming M, the isotropy G'_p must contain $\mathrm{SO}(2)^2$—so consulting Figure 11.4, G'_p is one of $\mathrm{SO}(3) \times \mathrm{SO}(2)$, $\mathrm{U}(2)$, $\mathrm{SO}(4)$, or $\mathrm{SO}(5)$.

In fact, G'_p cannot be U(2) or SO(4). If this were the case, G' would preserve TM^G, inducing a G-equivariant diffeomorphism $M/\mathcal{F}^G \to M/\mathcal{F}^{G'}$. But $M/\mathcal{F}^G$ is a product of 2-dimensional maximal model geometries with G acting by the isometry group, whereas (consulting the classification in Prop. 13.1) a base space of a geometry with U(2) or SO(4) isotropy can only be S^4, $\mathbb{E}^4$, $\mathbb{H}^4$, $\mathbb{C}P^2$, $\mathbb{C}^2$, and $\mathbb{CH}^2$.

Step 2: Geometries involving Heis_3 **are maximal.** In the isometry group of a constant-curvature geometry, every connected nilpotent subgroup is abelian (Lemma 15.38); so this holds for products of such groups too. Since all geometries with isotropy SO(5) or SO(3) $\times$ SO(2) are constant-curvature geometries or products thereof, their isometry groups do not contain subgroups covered by Heis_3. Therefore the geometries $\mathrm{Heis}_3 \times_{\mathbb{R}} S^3$ and $\mathrm{Heis}_3 \times_{\mathbb{R}} \widetilde{\mathrm{SL}_2}$ are maximal.

Step 3: SO(5)**-isotropy geometries do not subsume.** Since $\widetilde{\mathrm{SL}_2} \times_\alpha S^3$ is an S^3 bundle over $\mathbb{R}^2$, its nonzero π_3 distinguishes it from the SO(5)-isotropy geometries. Similarly, $L(a;1) \times_{S^1} L(b;1)$ is an S^3 bundle over S^2, so it is distinguished from the SO(5) geometries by having nontrivial π_2.

$\widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2}$ cannot be subsmed by S^5 (which is compact) or $\mathbb{E}^5$ (whose isometry group has compact semisimple part). To it distinguish from $\mathbb{H}^5$, observe that the image in $\widetilde{\mathrm{SL}_2} \times_\alpha \widetilde{\mathrm{SL}_2}$ of $\widetilde{\mathrm{SL}_2} \times \{1\} \times \mathbb{R} \subseteq \widetilde{\mathrm{SL}_2} \times \widetilde{\mathrm{SL}_2} \times \mathbb{R}$ is a copy of $\widetilde{\mathrm{SL}_2} \rtimes \mathrm{SO}(2)/\mathrm{SO}(2)$, fixed by an SO(2) with trivial projection to the first $\widetilde{\mathrm{SL}_2}$ factor; whereas in $\mathbb{H}^5$, every group of isometries conjugate to SO(2) fixes a copy of $\mathbb{H}^3$.

Step 4: SO(3) $\times$ SO(2) **geometries do not subsume.** If $\mathbb{R}$ and V denote the 1-dimensional trivial and 2-dimensional standard representations of SO(2), then the tangent space T_pM decomposes into three nonisomorphic representations of $G_p = \mathrm{SO}(2) \times \mathrm{SO}(2)$ as[5]

$$T_pM \cong (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes V) \oplus (V \otimes \mathbb{R}),$$

which determines two invariant 2-dimensional distributions. From the description of M as a bundle over 2-by-2 product geometries (Table 15.29, Prop. 15.34), neither of these is integrable. Then M cannot be subsumed by a 3-by-2 product geometry, since every geometry with $\mathrm{SO}(3) \times \mathrm{SO}(2)$ isotropy has an invariant integrable 2-dimensional distribution. □

Remark 15.40. Proposition 15.39 provides a negative answer to the question raised by Filipkiewicz in the discussion after [Fil83, Prop. 1.1.2]: is every non-maximal geometry subsumed by a *unique* maximal geometry?

The counterexample is $T^1S^3 \cong \mathrm{SO}(4)/\mathrm{SO}(2) \cong S^3 \times S^3/\Delta(S^1)$ where $\Delta : S^3 \to S^3 \times S^3$ is the diagonal map. It is subsumed:

1. by $S^3 \times S^2$ since S^3 is parallelizable by left-invariant vector fields, and

2. by $L(1;1) \times_{S^1} L(1;1) = S^3 \times S^3 \times \mathbb{R}/\tau_{1,1}(\mathbb{R}^2)$ since $\tau_{1,1}(\mathbb{R}^2)$ meets $S^3 \times S^3$ in an antidiagonal (which is conjugate to diagonal) copy of S^1.

Since $S^3 \times S^2$ is maximal as a product and $L(1;1) \times_{S^1} L(1;1)$ is maximal due to Prop. 15.39, a subsuming geometry for $\mathrm{SO}(4)/\mathrm{SO}(2)$ is not unique.

One may alternatively check that $S^3 \times S^2$ does not subsume $L(1;1) \times_{S^1} L(1;1)$ by classifying all homomorphisms

$$S^3 \times S^3 \times \mathbb{R} \to (S^3)^3 \cong \left(\mathrm{Isom}\,\widetilde{(S^3 \times S^2)}\right)^0$$

using the fact that $\mathrm{Aut}\, S^3 = \mathrm{Inn}\, S^3$.

5. Over $\mathbb{C}$, the irreducible representations of a direct product of groups are the tensor products of their irreducible representations; see e.g. [BD85, Prop. II.4.14]

Table 15.42: 3-dimensional fixed sets of order 2 isometries

Geometry	Fixed set
(product)	(product)
$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$	$\mathbb{H}^2 \times \mathbb{E}$
$\mathbb{F}^5_0 = \mathrm{Heis}_3 \rtimes \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$	$\mathbb{H}^2 \times \mathbb{E}$
$\mathbb{F}^5_1$	$\widetilde{\mathrm{SL}_2}$
$\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}$	$\widetilde{\mathrm{SL}_2}$

Proposition 15.41. *All geometries listed in Prop. 15.17 are maximal.*

Proof. Let $M = G/H$ be one of the geometries named, and suppose G'/H' subsumes it.

Step 1: List restrictions on subsuming geometries. Since G/H is contractible, so is G'/H'. Since G contains a group covered by $\widetilde{\mathrm{SL}_2}$, so does G'. Finally, H' contains H ($S^1_{1/2}$, or S^1_1 in the case of $T^1\mathbb{E}^{1,2}$) and cannot preserve TM^G—since if it did, then it would preserve the fibering $M \to M/\mathcal{F}^G$; but these geometries are the only ones encountered in the classification for which $M/\mathcal{F}^G$ is $\mathbb{F}^4$ or $T\mathbb{H}^2$. Consulting Figure 11.4, this last restriction implies H is $\mathrm{SO}(5)$, $\mathrm{SO}(3) \times \mathrm{SO}(2)$, or $\mathrm{SO}(3)_5$.

Then by the classification so far, G'/H' is either $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ or a product of a hyperbolic space with zero or more Euclidean or hyperbolic spaces.

Step 2: Find 3-dimensional fixed sets of order 2 isometries. Such a fixed set in G/H must also appear in G'/H', so we list these. See Table 15.42 for a summary.

- Using the classification of isometries of $\mathbb{H}^n$ [BP92, A.5.14] and the fact that $\mathrm{SO}(k)$ is maximal compact in $\mathrm{Isom}_0\,\mathbb{E}^k$, these fixed sets in products of Euclidean and hyperbolic spaces are also products of Euclidean and hyperbolic spaces.

- In $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, an order 2 isometry is a 3×3 matrix with a positive even number of -1 eigenvalues. Its centralizer is conjugate (by diagonalization) to the subgroup of

$2+1$ block matrices; and the orbit of this subgroup is the isometry's fixed set,

$$\mathrm{S}\left(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})\right)^0 / \mathrm{SO}(2) \cong \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R} / \mathrm{SO}(2) \cong \mathbb{H}^2 \times \mathbb{E}.$$

- In $\mathrm{Heis}_3 \rtimes \mathrm{SL}(2,\mathbb{R}) / \mathrm{SO}(2)$, the order 2 isometries are all conjugate to the order 2 element of $\mathrm{SO}(2)$. Its centralizer is $Z(\mathrm{Heis}_3) \times \mathrm{SL}(2,\mathbb{R})$, with orbit $\mathbb{E} \times \mathbb{H}^2$.
- Following the same recipe for $\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}$ and $\mathbb{F}_1^5$ produces a centralizer of $\widetilde{\mathrm{SL}_2} \rtimes \mathrm{SO}(2)$, with orbit $\widetilde{\mathrm{SL}_2}$.

This last result implies $\mathbb{R}^2 \rtimes \widetilde{\mathrm{SL}_2}$ and $\mathbb{F}_1^5$ are maximal, since none of the candidates for G'/H' have $\widetilde{\mathrm{SL}_2}$ as a fixed set of an order 2 isometry.

Step 3: $\mathrm{Heis}_3 \rtimes \mathrm{SL}(2,\mathbb{R}) / \mathrm{SO}(2)$ **is maximal.** If G contains Heis_3, then so does G'. A copy of Heis_3 in a direct product must have nonabelian—hence locally injective—image in at least one factor. Since Heis_3 covers no subgroup of isometries of $\mathbb{E}^k$ or $\mathbb{H}^k$ (Lemma 15.38), only $\mathrm{SL}(3,\mathbb{R}) / \mathrm{SO}(3)$ can subsume.

By the classification of irreducible representations of $\mathrm{SL}(2,\mathbb{R})$—one in each dimension [FH91, 11.8]—all $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(3,\mathbb{R})$ are conjugate to a standard copy. Denoting the standard representation of $\mathrm{SL}(2,\mathbb{R})$ by V, counting weights (see e.g. [FH91, §11.2]) produces the decomposition

$$\mathfrak{sl}_3\mathbb{R} \cong_{\mathrm{SL}(2,\mathbb{R})} \mathbb{R} \oplus 2V \oplus \mathfrak{sl}_2\mathbb{R}.$$

The two copies of V are $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$ and $\mathrm{Hom}(\mathbb{R}, \mathbb{R}^2)$, which are abelian subalgebras of $\mathfrak{sl}_3\mathbb{R}$ and therefore cannot generate a copy of Heis_3. Therefore $\mathrm{SL}(3,\mathbb{R}) / \mathrm{SO}(3)$ does not subsume $\mathrm{Heis}_3 \rtimes \mathrm{SL}(2,\mathbb{R}) / \mathrm{SO}(2)$.

Step 4: $T^1\mathbb{E}^{1,2}$ **is maximal.** Since the isotropy of $\mathrm{SL}(3,\mathbb{R}) / \mathrm{SO}(3)$ does not contain S^1_1 (Fig. 11.4; see also (Prop. 7.1 footnote)), it cannot subsume $T^1\mathbb{E}^{1,2}$. So it only remains to eliminate the products of hyperbolic and Euclidean spaces, as follows.

In $\mathfrak{so}_{1,2}\mathbb{R}$, there is a matrix

$$A = \begin{pmatrix} & & 1 \\ & & -1 \\ 1 & 1 & 0 \end{pmatrix},$$

which sends the third standard basis vector e_3 to $e_1 - e_2$ and sends $e_1 - e_2$ to zero. Then A, e_3, and $e_1 - e_2$ span a copy of the 3-dimensional Heisenberg algebra in $\mathbb{R}^3 \ni \mathfrak{so}_{1,2}\mathbb{R}$. Since $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{H}^k$ and $T_{\mathbf{1}} \operatorname{Isom}_0 \mathbb{E}^k$ cannot contain the Heisenberg algebra (Lemma 15.38), neither can a product of them, since the projection to at least one factor would have to be nonabelian and thereby injective. Then none of the products of hyperbolic and Euclidean spaces can subsume $T^1\mathbb{E}^{1,2}$. $\square$

REFERENCES

[AFF15] I. Agricola, A. C. Ferreira, and T. Friedrich, *The classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$*, Differential Geometry and its Applications **39** (2015), 59–92.

[Ahl10] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, AMS Chelsea, Providence, R.I, 2010.

[AMR00] D. Alekseevsky, P. W. Michor, and W. Ruppert, *Extensions of Lie algebras*, preprint, arXiv:math/0005042, May 2000.

[AMR05] ———, *Extensions of super Lie algebras*, J. Lie Theory **15** (2005), no. 1, 125–134.

[Bar65] D. Barden, *Simply connected five-manifolds*, Annals of Mathematics **82** (1965), no. 3, 365–385.

[BD85] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Springer Science & Business Media, January 1985 (en).

[BFNT13] L. Boza, E. M. Fedriani, J. Nunez, and A. F. Tenorio, *A historical review of the classifications of Lie algebras*, Rev. Un. Mat. Argentina **54** (2013), no. 2.

[BG02] C. P. Boyer and K. Galicki, *Rational homology 5-spheres with positive Ricci curvature*, Mathematical Research Letters **9** (2002), no. 4, 521–528, arXiv: math/0203048.

[BGS85] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, vol. 61, Birkäuser, 1985.

[BLM15] W. Batat, M. C. López, and E. R. María, *Four-dimensional naturally reductive pseudo-Riemannian spaces*, Differential Geometry and its Applications **41** (2015), 48–64.

[Bor63] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), no. 1, 111–122.

[BP92] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Springer, 1992.

[BR86] A. O. Barut and R. Rączka, *Theory of group representations and applications*, World Scientific, 1986 (en).

[Bro82] Kenneth S. Brown, *Cohomology of groups*, no. 87, Springer Science & Business Media, 1982.

[DC76] M. P. Do Carmo, *Differential geometry of curves and surfaces*, vol. 2, Prentice-Hall, 1976.

[DK08] W. Dickinson and M. M. Kerr, *The geometry of compact homogeneous spaces with two isotropy summands*, Annals of Global Analysis and Geometry **34** (2008), no. 4, 329–350.

[Doz63] J. Dozias, *Sur les algèbres de Lie résolubles réelles de dimension inférieure ou égale à 5*, Ph.D. thesis, Faculté des Sciences de Paris, 1963.

[Dyn00a] E. B. Dynkin, *Maximal subgroups of the classical groups*, Selected papers of E. B. Dynkin with commentary, Amer. Math. Soc., Providence, RI, 2000, pp. 37–170.

[Dyn00b] ______, *Semisimple subalgebras of semisimple Lie algebras*, Selected papers of E. B. Dynkin with commentary, Amer. Math. Soc., Providence, RI, 2000, pp. 175–308.

[DZ10] S. Dumitrescu and A. Zeghib, *Géométries Lorentziennes de dimension 3: classification et complétude*, Geometriae Dedicata **149** (2010), no. 1, 243–273.

[Fer96] J. Ferrand, *The action of conformal transformations on a Riemannian manifold*, Mathematische Annalen **304** (1996), no. 1, 277–291 (en).

[FH91] W. Fulton and J. Harris, *Representation theory: a first course*, vol. 129, Springer, 1991.

[Fil83] R. Filipkiewicz, *Four dimensional geometries*, Ph.D. thesis, University of Warwick, 1983.

[Gen16a] A. Geng, *5-dimensional geometries ii: the non-fibering geometries*, In preparation, 2016.

[Gen16b] ______, *5-dimensional geometries iii: the fibering geometries*, In preparation, 2016.

[Gor77] V. V. Gorbatsevich, *Three-dimensional homogeneous spaces*, Siberian Mathematical Journal **18** (1977), no. 2, 200–210 (en).

[Gor12] ______, *On compact aspherical homogeneous manifolds of dimension* ≤ 7, Mathematical Notes **92** (2012), no. 1-2, 186–196 (en).

[GOV93] V. V. Gorbatsevich, A. L. Onishchik, and E. B. Vinberg, *Lie groups and Lie algebras I: Foundations of Lie theory, Lie transformation groups*, Springer-Verlag, 1993.

[GOV94] ______, *Lie groups and Lie algebras III: Structure of Lie groups and Lie algebras*, Springer Science & Business Media, June 1994 (en). MR 1056486 (91j:17004)

[GW88] C. S. Gordon and E. N. Wilson, *Isometry groups of Riemannian solvmanifolds*, Transactions of the American Mathematical Society **307** (1988), no. 1, 245–269.

[Hel78] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, vol. 80, Academic press, 1978.

[Hil02] J. A. Hillman, *Four-manifolds, geometries and knots*, University of Warwick, Mathematics Institute, 2002.

[Hum72] J. Humphreys, *Introduction to Lie algebras and representation theory*, vol. 9, Springer, 1972.

[Hus94] D. Husemöller, *Fibre bundles*, 3rd ed., Graduate texts in mathematics, no. 20, Springer-Verlag, New York, 1994.

[HZ96] E. Heintze and W. Ziller, *Isotropy irreducible spaces and s-representations*, Differential Geometry and its Applications **6** (1996), no. 2, 181–188.

[Ish55] Shigeru Ishihara, *Homogeneous Riemannian spaces of four dimensions.*, Journal of the Mathematical Society of Japan **7** (1955), no. 4, 345–370 (EN). MR 82717

[Jac62] N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Applied Mathematics, no. 10, Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962. MR 0143793 (26 #1345)

[KK13] M. M. Kerr and A. Kollross, *Nonnegatively curved homogeneous metrics in low dimensions*, Annals of Global Analysis and Geometry **43** (2013), no. 3, 273–286, arXiv: 1201.4744.

[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1, Interscience Tracts in Pure and Applied Mathematics, no. 15, Interscience Publishers (a division of John Wiley & Sons), New York-London, 1963.

[KN69] ———, *Foundations of differential geometry*, Interscience Tracts in Pure and Applied Mathematics, vol. 2, Interscience Publishers (a division of John Wiley & Sons), New York-London, 1969.

[Kna02] A. W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, Birkhäuser, 2002.

[KPV96] O. Kowalski, F. Prüfer, and L. Vanhecke, *D'Atri spaces*, Topics in geometry, Springer, 1996, pp. 241–284.

[Krä75] M. Krämer, *Eine klassifikation bestimmter untergruppen kompakter zusammenhängender Liegruppen*, Communications in Algebra **3** (1975), no. 8, 691–737.

[KV85] O. Kowalski and L. Vanhecke, *Classification of five-dimensional naturally reductive spaces*, Math. Proc. Camb. Phil. Soc, vol. 97, Cambridge Univ Press, 1985, pp. 445–463.

[KV91] O. Kowalski and L. Vanhecke, *Riemannian manifolds with homogeneous geodesics*, Bollettino della Unione Matemàtica Italiana. Serie VII. B **5** (1991), no. 1, 189–246.

[Laf88] J. Lafontaine, *The theorem of Lelong-Ferrand and Obata*, Conformal geometry, Springer, 1988, pp. 93–103.

[Mac99] M. a. H. MacCallum, *On the classification of the real four–dimensional Lie algebras*, On Einstein's Path (Alex Harvey, ed.), Springer New York, 1999, pp. 299–317 (en).

[Man61a] O. V. Manturov, *Homogeneous asymmetric Riemannian spaces with an irreducible group of rotations*, Dokl. Akad. Nauk. SSSR **141** (1961), no. 4, 792–795.

[Man61b] ———, *Riemannian spaces with orthogonal and symplectic groups of motions and an irreducible group of motions*, Dokl. Akad. Nauk. SSSR **141** (1961), no. 5, 1034–1037.

[Man66] ———, *Homogeneous Riemannian manifolds with irreducible isotropy group*, Trudy Sem. Vector and Tensor Analysis **13** (1966), 68–145.

[Man98] ———, *Homogeneous Riemannian spaces with irreducible rotation group*, Tensor and Vector Analysis. Geometry, Mechanics and Physics, Gordon and Breach, Amsterdam (1998), 101–192.

[Mat02] Y. Matsumoto, *An introduction to morse theory*, vol. 208, American Mathematical Soc., 2002.

[Mor15] D. W. Morris, *Introduction to arithmetic groups*, 1.0 ed., Deductive Press, 2015, arXiv: math/0106063.

[Mos50] G. D. Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, Annals of Mathematics **52** (1950), no. 3, 606–636.

[Mos55] ———, *On covariant fiberings of Klein spaces*, American Journal of Mathematics **77** (1955), no. 2, 247–278.

[Mos62a] ———, *Covariant fiberings of Klein spaces, ii*, American Journal of Mathematics **84** (1962), no. 3, 466–474.

[Mos62b] ———, *Homogeneous spaces with finite invariant measure*, Annals of Mathematics **75** (1962), no. 1, 17–37. MR 0145007 (26 #2546)

[Mos71] ———, *Arithmetic subgroups of groups with radical*, Annals of Mathematics **93** (1971), no. 3, 409–438. MR 289713 (44 #6901)

[Mos05] ———, *A structure theorem for homogeneous spaces*, Geometriae Dedicata **114** (2005), no. 1, 87–102 (en).

[Mub63] G. M. Mubarakzyanov, *Classification of real structures of Lie algebras of fifth order*, Izv. Vyssh. Uchebn. Zaved. Mat. (1963), no. 3, 99–106. MR 155871

[Nag59] T. Nagano, *Homogeneous sphere bundles and the isotropic Riemann manifolds*, Nagoya Mathematical Journal **15** (1959), 29–55. MR MR0108810

[Neu99] J. Neukirch, *Algebraic number theory*, Grundlehren der mathematischen Wissenschaften, vol. 322, Springer, Berlin, Heidelberg, 1999.

[Oba73] M. Obata, *A non-compact Riemannian manifold admitting a transitive group of conformorphisms*, Tohoku Mathematical Journal **25** (1973), no. 4, 553–556 (EN). MR 0346704

[O'N66] B. O'Neill, *The fundamental equations of a submersion.*, The Michigan Mathematical Journal **13** (1966), no. 4, 459–469. MR 0200865

[Ott09] S. Ottenburger, *A diffeomorphism classification of 5-and 7-dimensional non-simply-connected homogeneous spaces*, Ph.D. thesis, Bonn, Univ., Diss., 2009, 2009.

[OV00] A. L. Onishchik and E. B. Vinberg, *Lie groups and Lie algebras II: Discrete subgroups of Lie groups and cohomologies of Lie groups and Lie algebras*, Encyclopaedia of Mathematical Sciences, vol. 21, Springer-Verlag, 2000.

[Pet06] Peter Petersen, *Riemannian geometry*, vol. 171, Springer Science & Business Media, 2006.

[PSWZ76] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, *Invariants of real low dimension Lie algebras*, Journal of Mathematical Physics **17** (1976), no. 6, 986–994.

[Rag72] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, 1972.

[Rat06] J. Ratcliffe, *Foundations of hyperbolic manifolds*, vol. 149, Springer Science & Business Media, 2006.

[Sco83] P. Scott, *The geometries of 3-manifolds*, Bulletin of the London Mathematical Society **15** (1983), no. 5, 401–487. MR 705527 (84m:57009)

[Sha00] R. W. Sharpe, *Differential geometry: Cartan's generalization of Klein's Erlangen program*, Springer New York, December 2000 (en).

[ŠW12] L. Šnobl and P. Winternitz, *Solvable Lie algebras with Borel nilradicals*, Journal of Physics A: Mathematical and Theoretical **45** (2012), no. 9, 095202.

[Thu97] W. P. Thurston, *Three-dimensional geometry and topology*, vol. 1, Princeton University Press, 1997.

[Thu02] ———, *The geometry and topology of three-manifolds*, 1.1 electronic ed., Princeton University, 2002, http://library.msri.org/books/gt3m/.

[Wag10] F. Wagemann, *Introduction to Lie algebra cohomology with a view towards BRST cohomology*, August 2010.

[Wal86] C. T. C. Wall, *Geometric structures on compact complex analytic surfaces*, Topology **25** (1986), no. 2, 119–153.

[Wil82] E. N. Wilson, *Isometry groups on homogeneous nilmanifolds*, Geometriae Dedicata **12** (1982), no. 3, 337–346.

[Win12] H. Winther, *Almost complex homogeneous spaces with semi-simple isotropy*, Master's thesis, University of Tromsø, 2012.

[Wol68] J. A. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Mathematica **120** (1968), no. 1, 59–148 (en).

[Wol84] ———, *Erratum to: The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Mathematica **152** (1984), no. 1, 141–142.

[Wol11] ———, *Spaces of constant curvature*, sixth ed., American Mathematical Soc., 2011 (en).

[WZ91] M. Wang and W. Ziller, *On isotropy irreducible Riemannian manifolds*, Acta Mathematica **166** (1991), no. 1, 223–261.