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SOME METRIC PROPERTIES OF PLANAR GAUSSIAN FREE FIELD

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ABSTRACT

In this thesis we study the properties of some metrics arising from two-dimensional Gaussian free field (GFF), namely the Liouville first-passage percolation (Liouville FPP), the Liouville graph distance and an effective resistance metric.

In Chapter 1, we define these metrics as well as discuss the motivations for studying them. Roughly speaking, Liouville FPP is the shortest path metric in a planar domain D where the length of a path P is given by $\int_P \mathrm{e}^{\gamma h(z)}|dz|$ where h is the GFF on D and $\gamma > 0$. In Chapter 2, we present an upper bound on the expected Liouville FPP distance between two typical points for small values of γ (the *near-Euclidean* regime). A similar upper bound is derived in Chapter 3 for the Liouville graph distance which is, roughly, the minimal number of Euclidean balls with comparable Liouville quantum gravity (LQG) measure whose union contains a continuous path between two endpoints. Our bounds seem to be in disagreement with Watabikis prediction (1993) on the random metric of Liouville quantum gravity in this regime. The contents of these two chapters are based on a joint work with Jian Ding [32].

In Chapter 4, we derive some asymptotic estimates for effective resistances on a random network which is defined as follows. Given any $\gamma > 0$ and for $\eta = \{\eta_v\}_{v\in\mathbb{Z}^2}$ denoting a sample of the two-dimensional discrete Gaussian free field on $\mathbb{Z}^2$ pinned at the origin, we equip the edge (u, v) with conductance $\mathrm{e}^{\gamma(\eta_u+\eta_v)}$. The metric structure of effective resistance plays a crucial role in our proof of the main result in Chapter 4. The primary motivation behind this metric is to understand the random walk on $\mathbb{Z}^2$ where the edge (u, v) has weight $\mathrm{e}^{\gamma(\eta_u+\eta_v)}$. Using the estimates from Chapter 4 we show in Chapter 5 that for almost every η, this random walk is recurrent and that, with probability tending to 1 as $T \to \infty$, the return probability at time $2T$ decays as $T^{-1+o(1)}$. In addition, we prove a version of subdiffusive behavior by showing that the expected exit time from a ball of radius N scales as $N^{\psi(\gamma)+o(1)}$ with $\psi(\gamma) > 2$ for all $\gamma > 0$. The contents of these chapters are based on a joint work with Marek Biskup and Jian Ding [13].

CHAPTER 1

INTRODUCTION

Gaussian free field (GFF) appears as a natural analogue of Brownian motion when one replaces the underlying parameter space with a multi-dimensional domain. As such, two-dimensional GFF is a canonical model for random surfaces. It is an extremely rich and intriguing mathematical object emerging in a wide range of contexts in probability theory and statistical physics. An important property peculiar to the two-dimensional GFF is conformal invariance (see, e.g., [74]) which relates it to Schramm-Loewner Evolution (SLE) in several ways (see [37], [70], [71]). Planar GFF has been found to be the scaling limit of height function of uniform random planar domino tilings [51]. In [67] it was shown that the fluctuations of the characteristic polynomial of a particular random matrix model tends to the planar Gaussian free field conditioned to be harmonic outside the unit disk.

Several important properties of GFF have been explored. Among them are its various metric properties which have attracted substantial amount of research in recent years. A random pseudo-metric was defined in [56] via the zero-set of GFF on the metric graph whose scaling limit (in the planar case) should describe the distance between CLE_4 loops (see [79]). [33] initiated the study on chemical distances of percolation clusters for level sets of planar (discrete) GFF. In this thesis, we will focus on three other metrics namely the Liouville FPP [36, 29, 32, 35], the Liouville graph distance [32] and an effective resistance metric [13]. In the following two sections we discuss the contexts in which they arise along with the relevant definitions. We will use these definitions in subsequent chapters where we discuss them in greater detail.

1.1 Liouville FPP and Liouville graph distance

1.1.1 Definitions

Let $D \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary. Denoting the euclidean distance between any two subsets S and S' of $\mathbb{R}^2$ by $d_{\ell_2}(S, S')$, let us define $D_\delta = \{v \in D : d_{\ell_2}(v, \partial D) > \delta\}$ where $\delta > 0$. For simplicity we will only consider domains D such that $V \equiv [0,1]^2 \subseteq D_\epsilon$ for some fixed ϵ. Let h be a (continuum) Gaussian free field (GFF) on D with Dirichlet boundary condition. We will desist from providing a detailed introduction to the GFF in this thesis for which there are several nice expositions (see e.g. [74, 9]). Although h is not a function on D (it is a *random distribution*), it is regular enough so that we can make sense of its Lebesgue integrals over sufficiently nice Borel sets in a rigorous way. In particular we can take its average along a circle of radius δ around v (where $d_{\ell_2}(v, \partial D) > \delta$) and define the *circle average process* $\{h_\delta(v) : v \in D, d_{\ell_2}(v, \partial D) > \delta\}$ which is a centered Gaussian field with covariance

$$\mathrm{Cov}(h_\delta(v), h_{\delta'}(v')) = \int_{\partial B_\delta(v) \times \partial B_{\delta'}(v')} G_D(z, z') \mu_\delta^v(dz) \mu_{\delta'}^{v'}(dz') \,.$$

Here $B_r(z)$ is the open ball with radius r centered at z, μ_r^z is the uniform probability measure on $\partial B_r(z)$ and $G_D(z, z')$ is the Green function for domain D, which we define by

$$G_D(z, z') = \int_{(0,\infty)} p_D(s; z, z') ds \,,$$

where $p_D(s; z, z')$ is the transition probability density of Brownian motion killed when exiting D. It was shown in [43] that there exists a version of the circle average process which is jointly Hölder continuous in v and δ of order $\vartheta < 1/2$ on all compact subsets of $\{(v, \delta) : v \in D, 0 < \delta < d_{\ell_2}(z, \partial D)\}$. Given such an instance of h_δ and a fixed inverse-temperature parameter $\gamma > 0$, the *Liouville first-passage percolation (Liouville FPP) metric* $D_{\gamma,\delta}(\cdot,\cdot)$ is defined by

$$D_{\gamma,\delta}(v,w) = \inf_P \int_P \mathrm{e}^{\gamma h_\delta(z)} |dz| \,, \tag{1.1.1}$$

where P ranges over all piecewise C^1 paths in V connecting v and w. The infimum is well-defined and measurable since we are dealing with a continuous field on a compact space. In fact $D_{\gamma,\delta}(\cdot,\cdot)$ does not change if we only restrict to C^1 paths.

The second notion of metric i.e. the Liouville graph distance comes from the so-called *Liouville quantum gravity* (LQG) measure M_γ^D. For any $\gamma < 2$, M_γ^D is defined as the almost sure weak limit of the sequence of measures $M_{\gamma,n}^D$ given by

$$M_{\gamma,n}^D = \mathrm{e}^{\gamma h_{2^{-n}}(z)} 2^{-\pi^{-1} n \gamma^2/2} \sigma(dz) \,, \tag{1.1.2}$$

where σ is the Lebesgue measure (the factor π^{-1} in the exponent is purely due to our particular definition of Green function). Much on the LQG measure has been understood (see e.g., [50, 43, 65, 66, 73] including the existence of the limit in (1.1.2), the uniqueness in law for the limiting measure via different approximation schemes, as well as a KPZ correspondence through a uniformization of the random lattice seen as a Riemann surface. Our focus in the current thesis is the *metric* aspect of LQG. Given $\delta \in (0,1)$, we say that a closed Euclidean ball $B \subseteq D$ is a (M_γ^D, δ)-ball if $M_\gamma^D(B) \leq \delta^2$ and the center of B is rational (to avoid unnecessary measurability considerations). The *Liouville graph distance* $\tilde{D}_{\gamma,\delta}(v,w)$ between $v, w \in V$ is the minimum number of (M_γ^D, δ) balls whose union contains a path between v and w. It is called Liouville graph distance since it corresponds to the shortest path distance in a graph indexed on $\mathbb{Q}^2$ where neighboring relation corresponds to the intersection of the (M_γ^D, δ) balls. A very related graph distance was mentioned in [60] which proposed to keep dividing each squares until the LQG measure is below δ.

One can similarly consider Liouville FPP for discrete planar GFF (which was explicitly mentioned in [7]). Given a two-dimensional box $V_N \subseteq \mathbb{Z}^2$ of side length N, let ∂V_N denote the set of vertices in $\mathbb{Z}^2 \setminus V_N$ that have a neighbor in V_N. The discrete GFF in V_N with

Dirichlet boundary condition is a mean-zero Gaussian process $\{\eta_{N,v} : v \in \mathbb{Z}^2\}$ such that

$$\eta_{N,v} = 0 \text{ for all } v \in \mathbb{Z}^2 \setminus V_N\,, \text{ and } \mathbb{E}\eta_{N,v}\eta_{N,w} = G_{V_N}(v,w) \text{ for all } v, w \in V_N\,,$$

where $G_{V_N}(v,w)$ is the Green's function for simple random walk on V_N i.e. the expected number of visits to v by the simple random walk on $\mathbb{Z}^2$ started at u and killed upon hitting ∂V_N. As before, for a fixed inverse-temperature parameter $\gamma > 0$, the Liouville FPP metric $D_{\gamma,N}(\cdot,\cdot)$ on V_N is defined by

$$D_{\gamma,N}(v_1, v_2) = \min_{\pi} \sum_{v\in\pi} e^{\gamma\eta_{N,v}}, \tag{1.1.3}$$

where π ranges over all paths in V_N connecting v_1 and v_2.

1.1.2 Motivation and related works

Much effort has been devoted to understanding classical first-passage percolation (FPP), with independent and identically distributed edge/vertex weights. We refer the reader to [4, 47] and their references for reviews of the literature on this subject. We argue that FPP with strongly-correlated weights is also a rich and interesting subject, involving questions both analogous to and divergent from those asked in the classical case. Since the Gaussian free field is in some sense the canonical strongly-correlated random medium, we see strong motivation to study Liouville FPP.

Our primary motivation behind Liouville FPP and Liouville graph distance, however, comes from the random metric associated with Liouville quantum gravity (LQG) [63, 43, 66]. Informally, LQG is a random surface whose "Riemannian metric tensor" can be described as $e^{\gamma X(x)}dx^2$, where X is a Gaussian free field on some planar domain D. Therefore the metrics $D_{\gamma,\delta}$ and $D_{\gamma,N}$ appear as natural approximations for the LQG metric. On the other hand, one can interpret the LQG measure M_γ^D (1.1.2) as the volume measure for LQG metric. By analogy with the Euclidean scenario (i.e. when $\gamma = 0$), we then see that $\tilde{D}_{\gamma,\delta}$ is yet another natural approximation for the LQG metric.

We remark that the random metric of LQG is a major open problem in contemporary probability theory, even just to make rigorous sense of it (we refer to [64] for a rather up-to-date review). In a recent series of works of Miller and Sheffield, much understanding has been obtained for the LQG metric (in the special case when $\gamma = \sqrt{8/3}$), and we note that an essentially equivalent metric to Liouville graph distance was mentioned in [60] as a natural approximation. While no mathematical result was obtained (perhaps not attempted either) on such approximations, the main achievement of this series of works by Miller and Sheffield (see [60, 61] and references therein) is to produce candidate scaling limits and to establish a deep connection to the Brownian map. Our approach is different, in the sense that we aim to understand the random metric of LQG via approximations by natural discrete metrics.

Furthermore, we expect that the Liouville FPP and Liouville graph distance are related to the heat kernel estimate for Liouville Brownian motion (LBM), which is essentially a time change of the standard Brownian motion by an exponential of GFF. In fact, we expect a direct and strong connection between Liouville graph distance and the LBM heat kernel. The mathematical construction (of the diffusion) for LBM was provided in [44, 8] and the heat kernel was constructed in [45]. The LBM is closely related to the geometry of LQG; in [26, 10] the Knizhnik–Polyakov–Zamolodchikov (KPZ) formula was derived from Liouville heat kernel. In [59] some nontrivial bounds for LBM heat kernel were established. The non-universality of the Liouville heat kernel over a class of log-correlated fields was shown in [34]. A very interesting direction is to compute the heat kernel of LBM with high precision. It is plausible that understanding the Liouville graph distance is of crucial importance in computing the LBM heat kernel.

1.2 Effective resistance metric

Let $\eta = \{\eta_v\}_{v\in\mathbb{Z}^2}$ denote a sample of the discrete GFF on $\mathbb{Z}^2$ pinned to 0 at the origin. Thus, $\{\eta_v : v \in \mathbb{Z}^2\}$ is a centered Gaussian process such that

$$\eta_0 = 0 \quad \text{and} \quad \mathbb{E}(\eta_u \eta_v) = G_{\mathbb{Z}^2\setminus\{0\}}(u,v) \text{ for all } u, v \in \mathbb{Z}^2\,,$$

where $G_{\mathbb{Z}^2\setminus\{0\}}(u,v)$ is the Green's function for simple random walk on $\mathbb{Z}^2 \setminus \{0\}$. For $\gamma > 0$ and conditional on the sample η of the GFF, let $\{X_t\}_{t\geq 0}$ be a discrete-time Markov chain with transition probabilities given by

$$p_\eta(u,v) := \frac{\mathrm{e}^{\gamma(\eta_v - \eta_u)}}{\sum_{w:|w-u|_1=1} \mathrm{e}^{\gamma(\eta_w-\eta_u)}} \mathbf{1}_{|v-u|_1=1}\,, \tag{1.2.1}$$

where $|\cdot|_1$ denotes the ℓ^1-norm on $\mathbb{Z}^2$.

The transition probabilities p_η are such that the walk prefers to move along the edges where η increases; the walk is thus driven towards larger values of the field. This has been predicted (e.g., in [21, 22]) to result in a subdiffusive behavior. Furthermore the diffusive exponent was predicted to undergo a continuous phase transition around a critical value of γ.

Another way to look at this problem is to rewrite the transition kernel as,

$$p_\eta(u,v) = \frac{\mathrm{e}^{\gamma(\eta_v + \eta_u)}}{\sum_{w:|w-u|_1} \mathrm{e}^{\gamma(\eta_w+\eta_u)}} \mathbf{1}_{|v-u|_1=1}. \tag{1.2.2}$$

This represents $\{X_t\}_{t\geq 0}$ as a random walk among random conductances where conductance of the edge (u,v) is given by $\mathrm{e}^{\gamma(\eta_u+\eta_v)}$. A large body of literature has been dedicated to Random Conductance Models in recent years (see [12, 52] for reviews). Unfortunately the law of the conductance is not translation invariant in this case which makes most of the existing theory in random conductance model inapplicable. Nevertheless, one can still hope to be able to explain the subdiffusivity using the connection between random walks and effective resistance of the underlying network (see [57]). Indeed, it turns out that all we need is a delicate control on effective resistances which is a fundamental metric for a graph. Properties of this metric like scaling limit etc. are of independent interest.

CHAPTER 2

LIOUVILLE FIRST-PASSAGE PERCOLATION

2.1 Upper bound on the expected distance

The order of the expected Liouville FPP distance between any two points in $V \equiv [0,1]^2$ or $V_N \equiv [0, N-1]^2$ is of enormous importance. In particular the expected distance between two facing boundaries of V (or V_N) seems to be the appropriate scaling factor for obtaining a scaling limit of LFPP [29]. In the following result we obtain an upper bound on this quantity (see (1.1.1), Chapter 1 for definition).

Theorem 2.1.1. *There exists $C_{\gamma,D,\epsilon} > 0$ (depending on (γ, ϵ, D)) and positive (small) absolute constants c^*, γ_0 such that for all $\gamma \leq \gamma_0$, we have*

$$\max_{v,w \in V} \mathbb{E} D_{\gamma,\delta}(v,w) \leq C_{\gamma,D,\epsilon} \delta^{c^* \frac{\gamma^{4/3}}{\log \gamma^{-1}}} .$$

As we explain in Section 2.6, the proof of Theorem 2.1.1 can be adapted to derive a similar result for the discrete GFF (see (1.1.3), Chapter 1).

Theorem 2.1.2. *Given any fixed $0 < \epsilon < 1/2$, there exists $C_{\gamma,\epsilon} > 0$ (depending on (γ, ϵ)) and positive (small) absolute constant c^*, γ_0 such that for all $\gamma \leq \gamma_0$, we have*

$$\max_{v,w \in V_{N,\epsilon}} \mathbb{E} D_{\gamma,N}(v,w) \leq C_\gamma N^{1 - c^* \frac{\gamma^{4/3}}{\log \gamma^{-1}}} ,$$

where $V_{N,\epsilon}$ is the square $\{v \in V_N : d_\infty(v, \partial V_N) \geq \epsilon N\}$.

Remark 2.1.3. Theorem 2.1.2 still holds if we restrict π to be a path within $V_{N,\epsilon}$ in (1.1.3).

2.1.1 Discussion on Watabiki's prediction

As already mentioned in the introductory chapter, the Liouville FPP and the Liouville graph distance are two (related) natural discrete approximations for the random metric associated with the Liouville quantum gravity (LQG) [63, 43, 66]. Precise predictions on various exponents regarding to LQG metric have been made by Watabiki [78] (see also, [3, 2]). In particular, the Hausdorff dimension for the LQG metric is predicted to be

$$d_H(\gamma) = 1 + \frac{\gamma^2}{4} + \sqrt{(1+\frac{\gamma^2}{4})^2 + \gamma^2}\,. \tag{2.1.1}$$

The prediction in (2.1.1) was widely believed. In a recent work [60], Miller and Sheffield introduced and studied a process called *quantum Loewner evolution*. As a *byproduct* of their work they gave a non-rigorous analysis on exponents of the LQG metric which matched Watabiki's prediction — we also note that in [60] the authors did express some reservations on their non-rigorous analysis. For other discussions on Watabiki's prediction in mathematical literature, see e.g., [59, 48].

The precise mathematical interpretation of Watabiki's prediction is not completely clear to us. However, there are a number of reasonable "folklore" interpretations for Liouville FPP that seem to be widely accepted. [1] For instance, see [2, Equation (17), (18)]. We would like to point out that in [2, Equation (17)] the term ρ_δ was not defined — some reasonable interpretations include $\rho_\delta = \mathrm{e}^{\gamma h_\delta(z)}$ and $\rho_\delta = \mathrm{e}^{\gamma h_\delta(z)}\delta^{\frac{\gamma^2}{2}}$ as well as possibly replacing γ by $\frac{\gamma}{d_H(\gamma)}$ as suggested in the footnote. For all these interpretations, [2, Equation (18)] would then imply that there exist constants $c, C > 0$ such that for sufficiently small but fixed $\gamma > 0$ the Liouville FPP distance between two generic points is between $\delta^{C\gamma^2}$ and $\delta^{c\gamma^2}$ as $\delta \to 0$. However, Theorem 2.1.1 contradicts with all aforementioned interpretations of (2.1.1) for Liouville FPP at high temperatures.

Currently, we do not have any reasonable conjecture on the precise value of the exponent for Liouville FPP — we regard a precise computation of the exponent as a major challenge.

1. For instance, we learned from Rémi Rhodes and Vincent Vargas that, according to [78], the physically appropriate approximation for the γ-LQG metric should involve $\inf_P \int_P \mathrm{e}^{\frac{\gamma}{d_H(\gamma)} h_\delta(z)} |dz|$, i.e., the parameter in the exponential of GFF is $\gamma/d_H(\gamma)$ instead of γ.

2.1.2 Discussion on non-universality

Combined with [36], Theorem 2.1.2 shows that the weight exponent for first passage percolation on the exponential of log-correlated Gaussian fields is non-universal, i.e., the exponents may differ for different families of log-correlated Gaussian fields. In contrast, we note that the behavior for the maximum is universal among log-correlated Gaussian fields (see e.g., [20, 19, 58, 27]) in a sense that their expectations are the same up to additive $O(1)$ term and that the laws of the centered maxima for all these fields are in the same universal family known as Gumbel distribution with random shifts (but the random shifts may not have the same law for different fields).

While non-universality suggests subtlety for the weight exponent of Liouville FPP, the proof in the current chapter does not see complication due to such subtlety. In fact, our proof should be adaptable to general log-correlated Gaussian fields with $\star$-scale invariant kernels as in [42]. The following question remains an interesting challenge, especially (in light of the non-universality) for log-correlated Gaussian fields for which a kernel representation is not known to exist.

Question 2.1.4. Let $\{\varphi_{N,v} : v \in V_N\}$ be an arbitrary mean-zero Gaussian field satisfying $|\mathbb{E}\varphi_{N,v}\varphi_{N,u} - \log \frac{N}{1+\|u-v\|} \leq K$. Does an analogue of Theorem 2.1.2 hold for C_γ, c^* depending on K?

2.1.3 Further related works

In a recent work [48] some upper and lower bounds have been obtained for a type of distance related to LQG and that their bounds are consistent with Watabiki's prediction. We further remark that currently we see no connection between our work and [60, 61, 48].

There has been a number of other recent works on Liouville FPP (while they focus on the case for the discrete GFF, these results are expected to extend to the case of continuum GFF). In a recent work [29], it was shown that at high temperatures the appropriately normalized Liouville FPP converges subsequentially in the Gromov-Hausdorff sense to a random metric on the unit square, where all the (conjecturally unique) limiting metrics are equivalent to the Euclidean metric. We remark that the proof method in the current chapter bears little similarity to that in [29]. In a very recent work [35], it was shown that the dimension of the geodesic for Liouville FPP is strictly larger than 1. In fact, in [35] it proved that all paths with dimension close to 1 has weight exponent close to 1, which combined with Theorem 2.1.2 yields that the lower bound on the dimension of the geodesic. While both the proofs in [35] and this chapter use multi-scale analysis method, the details are drastically different.

2.1.4 A historical remark and the proof strategy

Our proof strategy naturally inherits that of [31] which proved a weak version of Theorem 2.1.2 in the context of Branching random walk (BRW), and we encourage the reader to flip through [31] (in particular Section 1.2) which contains a prototype of the multi-scale analysis carried out in the current chapter. In fact, prior to the work presented in this chapter, we posted an article [30] on arXiv which proved that the weight exponent is less than $1-\gamma^2/10^3$. Our current results are stronger than [30]. In addition, the proof simplifies that of [30] and is self-contained. As a result, the work in the current chapter supersedes [30].

However, some historical remarks might be interesting and helpful. During the work of [30], we had in mind that the second leading term for the weight exponent is of order γ^2 in light of (2.1.1). As a result, we followed [31] and designed a strategy of constructing light crossings inductively to prove an upper bound of $1-\gamma^2/10^3$. In the multi-scale construction, the order of γ^2 is exactly the order of both the gain and the loss for our strategy, and thus a much delicate analysis was carried out in [30] since we fought between two constants for the loss and the gain. A curious reader may quickly flip through [30] for an impression on the level of technicality.

A key component in both [31, 30] is an inductive construction where one constructs light crossings in a bigger scale from crossings in smaller scale, where one switches between two layers of candidate crossings in smaller scale based on the value of Gaussian variables in the bigger scale (it is to be noted here that there is a hierarchical structure for both BRW and GFF). In those papers, vertical crossings were used as switching gadgets to connect horizontal crossings in top and bottom layers. A crucial improvement in this chapter arises from a simple observation that a *sloped* switching gadget is much more efficient (see Figure 2.5). In order to give a flavor of how it works we discuss the following toy problem.

Let $\Gamma = \Gamma(\gamma)$ be a large positive number and $\{\zeta(v) : v \in V^\Gamma\}$ be a continuous, centered Gaussian field on the rectangle $V^\Gamma = [0, \Gamma] \times [0, 1]$. Suppose that ζ satisfies the following properties:

(a) $\mathrm{Var}(\zeta(v)) = 1$ for all $v \in V^\Gamma$.

(b) For any straight line segment $\mathcal{L}$, $\mathrm{Var}(\int_{\mathcal{L}} \zeta(z)|dz|) = O(|\mathcal{L}|)$, where $|\mathcal{L}|$ is the (euclidean) length of $\mathcal{L}$. Furthermore if $v \in \mathbb{R}^2$ is orthogonal to $\mathcal{L}$ such that $||v|| = \Omega(1)$, then

$$\mathrm{Var}\Big(\int_{\mathcal{L}} \zeta(z)|dz| - \int_{\mathcal{L}+v} \zeta(z)|dz|\Big) = \Theta(|\mathcal{L}|)\,.$$

We want to construct a piecewise smooth path P connecting the shorter boundaries of V^Γ that has a small "random length" given by $\int_P \mathrm{e}^{\gamma\zeta(z)}|dz|$. Due to condition (a), we can approximate $\mathrm{e}^{\gamma\zeta(z)}$ with $1+\gamma\zeta(z)+\frac{\gamma^2}{2}$ when γ is sufficiently small. Thus the random length of P is approximately

$$(1+\tfrac{\gamma^2}{2})|P| + \gamma\int_P \zeta(z)|dz|\,. \tag{2.1.2}$$

Henceforth we will treat the above expression as the "true" random length of P. Now consider the segments $\mathcal{L}_1 = [0, \Gamma] \times \{0.75\}$ and $\mathcal{L}_2 = [0, \Gamma] \times \{0.25\}$. Choose β such that $\Gamma \gg \beta \gg 1$ and divide $\mathcal{L}_i$ (here $i \in [2]$) into segments $\mathcal{L}_{i,1}, \mathcal{L}_{i,2}, \cdots, \mathcal{L}_{i,\Gamma/\beta}$ of length β from left to right. Given $i_j \in \{1, 2\}$ for each $j \in [\Gamma/\beta]$ (called a *strategy*), we can construct a crossing i.e. a path connecting the shorter boundaries of V^Γ as follows. If $i_j = i_{j+1}$, let $\mathcal{L}'_j$ be the segment $\mathcal{L}_{i_j,j}$. Otherwise set $\mathcal{L}'_j$ as the segment joining the left endpoints of $\mathcal{L}_{i_j,j}$

and $\mathcal{L}_{i_{j+1},j+1}$ (the sloped gadget). It is clear that the segments $\mathcal{L}'_1, \mathcal{L}'_2, \cdots, \mathcal{L}'_{\Gamma/\beta}$ define a crossing. The random length (see (2.1.2)) of this crossing is given by

$$\big(1+\frac{\gamma^2}{2}\big)\Gamma + \big(1+\frac{\gamma^2}{2}\big)\sum_{j\in[\Gamma/\beta-1]} \mathbf{1}_{\{i_j\neq i_{j+1}\}}(|\mathcal{L}'_j|-\beta) + \gamma\sum_{j\in[\Gamma/\beta]}\int_{\mathcal{L}'_j}\zeta(z)|dz|\,.$$

Since $\beta \gg 1$, $|\mathcal{L}'_j| - \beta = \sqrt{O(1)+\beta^2} - \beta = O(\beta^{-1})$ whenever $i_j \neq i_{j+1}$. On the other hand by condition (b), $\big(\int_{\mathcal{L}'_j}\zeta(z)|dz| - \int_{\mathcal{L}_{i_j,j}}\zeta(z)|dz|\big)$ is a centered Gaussian variable with variance

$$\mathrm{Var}\big(\int_{\mathcal{L}'_j}\zeta(z)|dz| - \int_{\mathcal{L}_{i_j,j}}\zeta(z)|dz|\big) = O(|\mathcal{L}_j|) = O(\beta)\,,$$

whenever $i_j \neq i_{j+1}$ and thus

$$\mathbb{E}\big(\gamma\int_{\mathcal{L}'_j}\zeta(z)|dz| - \gamma\int_{\mathcal{L}_{i_j,j}}\zeta(z)|dz|\big)^+ = O(\sqrt{\beta})\,.$$

Therefore if we choose our strategy so that $i_j \neq i_{j+1}$ only on a *fixed* set $J = \{j_1, j_2, \cdots, j_{|J|}\}$, then we can bound (from above) the expected random length of the crossing by

$$\big(1+\frac{\gamma^2}{2}\big)\Gamma + \big(1+\frac{\gamma^2}{2}\big)|J|C\beta^{-1} + \gamma|J|C'\sqrt{\beta} + \gamma\sum_{k\in|J|}\mathbb{E}\big(\int_{\overline{\mathcal{L}}_{i_{j_k},J}}\zeta(z)|dz|\big)\,. \tag{2.1.3}$$

Here C, C' are positive constants and $\overline{\mathcal{L}}_{i_{j_k},J}$ is the union of segments $\mathcal{L}_{i_{j_{k-1}},j_{k-1}+1}, \cdots, \mathcal{L}_{i_{j_{k-1}},j_k}$ with $j_0 = 0$. Now notice that

$$\mathbb{E}\big(\int_{\overline{\mathcal{L}}_{i_{j_k},J}}\zeta(z)|dz|\big) = \frac{1}{2}\mathbb{E}(-1)^{i_{j_k}+1}\big(\int_{\overline{\mathcal{L}}_{1,J}}\zeta(z)|dz| - \int_{\overline{\mathcal{L}}_{2,J}}\zeta(z)|dz|\big)\,,$$

as $\int_{\overline{\mathcal{L}}_{1,J}}\zeta(z)|dz|$ and $\int_{\overline{\mathcal{L}}_{2,J}}\zeta(z)|dz|$ are centered. But by condition (b), $\int_{\overline{\mathcal{L}}_{1,j}}\zeta(z)|dz| - \int_{\overline{\mathcal{L}}_{2,j}}\zeta(z)|dz|$ is a centered Gaussian variable with variance $\geq c(j_k - j_{k-1})\beta$ for some positive constant c. Thus if we allow $j_k - j_{k-1}$ to be only large enough so that

$$\frac{\gamma}{2}\mathbb{E}\Big|\int_{\overline{\mathcal{L}}_{1,j}} \zeta(z)|dz| - \int_{\overline{\mathcal{L}}_{2,j}} \zeta(z)|dz|\Big| \geq 2(1+\frac{\gamma^2}{2})C\beta^{-1} + 2\gamma C'\sqrt{\beta}\,,$$

and set $i_j = 1$ or 2 accordingly as $\int_{\overline{\mathcal{L}}_{1,j}} \zeta(z)|dz| - \int_{\overline{\mathcal{L}}_{2,j}} \zeta(z)|dz| <$ or > 0, then

$$\gamma\mathbb{E}\big(\int_{\overline{\mathcal{L}}_{i_{j_k},J}} \zeta(z)|dz|\big) \leq -2\Big((1+\frac{\gamma^2}{2})C\beta^{-1} + 2\gamma C'\sqrt{\beta}\Big)\,,$$

and

$$|J| = \frac{\Omega(\frac{\Gamma}{\beta})}{\frac{\Big((1+\frac{\gamma^2}{2})C\beta^{-1}+2\gamma C'\sqrt{\beta}\Big)^2}{\beta\gamma^2}} = \frac{\Omega(\Gamma\gamma^2)}{\Big((1+\frac{\gamma^2}{2})C\beta^{-1}+2\gamma C'\sqrt{\beta}\Big)^2}\,.$$

Let us call the path given by this strategy as $P^\star$. Plugging the previous two expressions into (2.1.3), we find that the expected random length of $P^\star$ can be at most

$$\begin{aligned}
&(1+\frac{\gamma^2}{2})\Gamma + |J|\Big((1+\frac{\gamma^2}{2})C\beta^{-1} + 2\gamma C'\sqrt{\beta}\Big) - 2|J|\Big((1+\frac{\gamma^2}{2})C\beta^{-1} + 2\gamma C'\sqrt{\beta}\Big)\\
= \;& (1+\frac{\gamma^2}{2})\Gamma - \frac{\Omega(\Gamma\gamma^2)}{\Big((1+\frac{\gamma^2}{2})C\beta^{-1}+2\gamma C'\sqrt{\beta}\Big)^2}\Big((1+\frac{\gamma^2}{2})C\beta^{-1} + 2\gamma C'\sqrt{\beta}\Big)\\
= \;& (1+\frac{\gamma^2}{2})\Gamma - \frac{\Omega(\Gamma\gamma^2)}{\gamma\sqrt{\beta}+\beta^{-1}}\,.
\end{aligned}$$

The above expression is minimized for $\beta = \Theta(\gamma^{-2/3})$ and the optimal value is $\Gamma(1-\Omega(\gamma^{4/3}))$ when γ is small. This shows, on a high level, why we get a contraction as in Theorem 2.1.1.

We remark that the simple observation on the slopped switching strategy is more natural when considering continuous path in the plane — this is why our main proof focuses on the case of continuous GFF. In the case for discrete GFF, we first bound the distance minimizing the lengths over all continuous path and then argue that for each continuous path there is a lattice path whose weight grows by a factor that is negligible.

We now give a brief guide on the organization. In Section 2.2, we introduce a new

Gaussian field which has a simpler hierarchical structure than the circle average process — our main proof will be carried out for this new field. In Section 2.3 we describe our inductive construction on light crossings as scales increase and in particular we introduce the aforementioned sloped switching gadget. In Section 2.4 we analyze the construction in Section 2.3 and derive an upper bound on the weight exponent for the Gaussian field from Section 2.2. In Section 2.5, we show that the circle average process of GFF is well-approximated by the field from Section 2.2, thereby proving Theorem 2.1.1. Finally, in Section 2.6 we explain how to adapt the proof to deduce Theorem 2.1.2.

2.1.5 Conventions, notations and some useful definitions

We assume that γ is small enough (less than some small, positive absolute constant) for our bounds or inequalities to hold although we keep this implicit in our discussions. Γ is the smallest (integral) power of 2 that is $\geq \gamma^{-2}$. Thus $1 \leq \Gamma\gamma^2 < 2$. (It will be clear from our analysis that any exponent $< -4/3$ should work.) For any $w \in \mathbb{R}^2$, $\ell \in \mathbb{N}$ and $r > 0$, $V_\ell^{r;w}$ denotes the rectangle $w + [0, r2^{-\ell}] \times [0, 2^{-\ell}]$. We will suppress ℓ or w from this notation whenever they are 0. We will also omit r when it is 1. We call two rectangles R and R' to be *copies* of each other if R can be obtained from R' via translation and / or rotation by an angle. The rectangles R and R' are called *non-overlapping* is their interiors are disjoint. If R and R' have same dimensions then we say that they are *adjacent* if they share one of their shorter boundary segments. A *smooth path* is a C^1 map $P : [0,1] \to \mathbb{R}^2$. We also use P to denote the image set of P which is a subset of $\mathbb{R}^2$. This distinction should be clear from the context. For any rectangle $R = [a,b] \times [c,d]$ with sides parallel to the coordinate axes, we define its left, right, upper and lower boundary segments in the obvious way and denote them as $\partial_{\text{left}} R, \partial_{\text{right}} R, \partial_{\text{up}} R$ and $\partial_{\text{down}} R$ respectively. Thus $\partial_{\text{left}} R$ is the path described by $(a, c + t(d-c)); t \in [0,1]$ etc. For convenience, we will identify (and denote) the points in $\mathbb{R}^2$ as complex numbers.

For (nonnegative) functions $F(.)$ and $G(.)$ we write $F = O(G)$ (or $\Omega(G)$) if there exists an absolute constant $C > 0$ such that $F \leq CG$ (respectively $\geq CG$) everywhere in the domain. If the constant depends on variables $x_1, x_2, \ldots, x_n$, we modify these notations as $O_{x_1,x_2,\ldots,x_n}(G)$ and $\Omega_{x_1,x_2,\ldots,x_n}(G)$ respectively. $F = \Theta(G)$ if F is both $O(G)$ and $\Omega(G)$. For any positive integer i, the notations C_i and c_i indicate positive, absolute constants whose values are assumed to be same throughout this chapter. Similarly we use $C_{x_1,x_2,\cdots,x_k}, C'_{x_1,x_2,\cdots,x_k}, C''_{x_1,x_2,\cdots,x_k}, \cdots$ to denote fixed positive functions $C, C', C'', \cdots$ of $x_1, x_2, \cdots, x_k$. However we keep these qualifications i.e. "positive", "absolute constant", "depends on $x_1, x_2, \cdots, x_k$" etc. implicit in our discussion.

2.2 Preliminaries

2.2.1 White noise decomposition of some Gaussian processes

A white noise W distributed on $\mathbb{R}^2 \times \mathbb{R}^+$ refers to a centered Gaussian process $\{(W, f) : f \in L^2(\mathbb{R}^2 \times \mathbb{R}^+)\}$ whose covariance kernel is given by $\mathbb{E}(W, f)(W, g) = \int_{\mathbb{R}^2 \times \mathbb{R}^+} fg dw ds$. An alternative notation for (W, f) is $\int_{\mathbb{R}^2 \times \mathbb{R}^+} fW(dw, ds)$, which we will use in this chapter. For any $D \in \mathcal{B}(R^2)$ and $I \in \mathcal{B}(R^+)$, we let $\int_{D \times I} fW(dw, ds)$ denote the variable $\int_{\mathbb{R}^2 \times \mathbb{R}^+} f_{D \times I} W(dw, ds)$, where $f_{D \times I}$ is the restriction of f to $D \times I$. Now define a Gaussian process $\{h'_\delta(v) : v \in D_\delta\}$ as follows:

$$h'_\delta(v) = \int_{D \times (0,\infty)} \Big(\int_{\partial B_\delta(v)} p_D(s/2; v', w) \mu^v_\delta(dv') \Big) W(dw, ds) \tag{2.2.1}$$

Since $G_D(v, w) = \int_{(0,\infty)} p_D(s; v, w) ds$, it is easy to check that the processes h_δ and $h_{\delta'}$ are identically distributed for all $\delta \in (0, \text{diam}(D))$. This provides an automatic coupling between h_δ and a "convenient" field (to be defined shortly) which will be useful in our proof of Theorem 2.1.1 in Section 2.5. Henceforth we will work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a white noise is defined.

It turns out that the field $\{h_\delta\}_{\delta > 0}$ can be reasonably approximated (see Section 2.5) by a new family of fields which enjoy certain nice properties. To this end, we define a Gaussian process $\eta \equiv \{\eta_\delta^{\delta'}(v) : v \in \mathbb{R}^2, 0 < \delta < \delta' \le 1\}$ as:

$$\eta_\delta^{\delta'}(v) = \int_{\mathbb{R}^2 \times [\delta^2, \delta'^2]} p(s/2; v, w) W(dw, ds) \,. \tag{2.2.2}$$

where $p(s; v, w)$ is the transition probability density function of standard two-dimensional Brownian motion. We can immediately deduce the following properties of η from this representation:

(a) *Invariance with respect to symmetries of the plane.* Law of η remains same under any distance preserving transformation (i.e. translation, rotation, reflection etc.) of $\mathbb{R}^2$.

(b) *Scaling property.* The fields $\{\eta_{\delta^*\delta}^{\delta^*\delta'}(\delta^* v) : v \in \mathbb{R}^2\}$ and $\{\eta_\delta^{\delta'}(v) : v \in \mathbb{R}^2\}$ are identically distributed for all $0 < \delta < \delta' \le 1$ and $\delta^* \in (0, 1]$.

(c) *Independent increment.* The fields $\{\eta_\delta^{\delta'}(v) : v \in \mathbb{R}^2\}$ and $\{\eta_{\delta''}^{\delta'''}(v) : v \in \mathbb{R}^2\}$ are independent for all $0 < \delta < \delta' < \delta'' < \delta'' \le 1$.

We will suppress the superscript δ' in $\eta_\delta^{\delta'}$ whenever $\delta' = 1$. Notice that

$$\mathrm{Var}(\eta_\delta(v)) = \int_{[\delta^2, 1]} p(s; v, v) ds = \pi^{-1} \log \delta^{-1} \,. \tag{2.2.3}$$

In Lemma 2.2.1 we show that $\mathrm{Var}(\eta_\delta(v) - \eta_\delta(w)) = O(\frac{|v-w|^2}{\delta^2})$. As η_δ is a Gaussian process, this property implies by Kolmogorov-Centsov theorem that there is a version of η_δ with continuous sample paths. Thus we can work with a continuous version η_δ for any given δ and hence for any fixed, finite collection of δ's that we consider at any given instant.

2.2.2 Some variance and covariance estimates

Lemma 2.2.1. *For all $v, w \in \mathbb{R}^2$, we have* $\mathrm{Var}(\eta_\delta(v) - \eta_\delta(w)) \le \frac{|v-w|^2}{\delta^2}$.

Proof. These follow from (2.2.2) by straightforward computations:

$$\mathrm{Var}(\eta_\delta(v)-\eta_\delta(w)) = \pi^{-1}\int_{[\delta^2,1]} \frac{1-\mathrm{e}^{-\frac{|v-w|^2}{2s}}}{s} ds \le \pi^{-1}\int_{s\in[\delta^2,1]} \frac{|v-w|^2}{2s^2} ds \le \frac{|v-w|^2}{\delta^2}\,. \quad \square$$

We need similar results for a different class of random variables as well. To this end let us define some new objects. Let $\mathcal{P}$ be a finite, non-empty collection of smooth paths in $\mathbb{R}^2$. A *random polypath* (or simply a polypath) ξ from $\mathcal{P}$ is a collection of $\{0,1\}$-valued random variables $\{e_{\xi,P}\}_{P\in\mathcal{P}}$ such that $\cup_{P\in\mathcal{P}:e_{\xi,P}=1} P$ is a connected subset of $\mathbb{R}^2$. Thus one can view ξ as a random sub-collection of $\mathcal{P}$ forming a connected set. We treat any smooth path P as a polypath from $\mathcal{P}=\{P\}$ with $e_{P,P}=1$. We will often omit the reference to $\mathcal{P}$ when it is clear from the context and simply say that ξ is a polypath. If $X=\{X(v): v\in D\}$ is a continuous field and ξ is a polypath from $\mathcal{P}$, then we define its *weight computed with respect to* X or alternatively *weight computed with* X *as the underlying field* as the quantity $\sum_{P\in\mathcal{P}} e_{\xi,P}\int_P \mathrm{e}^{\gamma X(z)}|dz|$. For continuous random fields $X=\{X(v): v\in\mathbb{R}^2\}$ and $Y=\{Y(v): v\in\mathbb{R}^2\}$, and a polypath ξ such that (ξ, X) is independent with Y, consider the random variable

$$Z(\xi,X,Y;\gamma) = \sum_{P\in\mathcal{P}} e_{\xi,P}\int_P \mathrm{e}^{\gamma X(z)} Y(z)|dz|\,. \tag{2.2.4}$$

It is a simple consequence of Fubini's theorem that $\mathbb{E}Z(\xi,X,Y;\gamma)$ is finite if $\sup_{w\in\mathbb{R}^2}\mathbb{E}|Y(w)|$ and $\sup_{P\in\mathcal{P}}\mathbb{E}\int_P \mathrm{e}^{\gamma X(z)}|dz|$ are both finite. In this case we can express $\overline{Z}(\xi,X,Y;\gamma) = \mathbb{E}(Z(\xi,X,Y;\gamma)|Y)$ as

$$\overline{Z}(\xi,X,Y;\gamma) = \sum_{P\in\mathcal{P}}\int_P \mathbb{E}(e_{\xi,P}\mathrm{e}^{\gamma X(z)})Y(z)|dz|\,. \tag{2.2.5}$$

Furthermore, $\overline{Z}(\xi,X,Y;\gamma)$ is a centered Gaussian variable if Y is a centered Gaussian field. If $X\equiv 0$, we drop X and γ from the notation and write is simply as $\overline{Z}(\xi,Y)$. Another quantity of interest is the expected weight of ξ computed with respect to X i.e.

$$L(\xi, X; \gamma) \stackrel{\text{def}}{=} \sum_{P \in \mathcal{P}} \int_P \mathbb{E}(e_{\xi,P} \mathrm{e}^{\gamma X(z)})|dz| \,. \tag{2.2.6}$$

Now fix some $N \in \mathbb{N}$, $w \in \mathbb{R}^2$ and $\nu \in \mathbb{R}$. For $v \in \{w, w + \iota\nu\}$ (here $\iota = \sqrt{-1}$) subdivide the rectangle $V_{2m_\Gamma}^{N\Gamma;v}$ into N non-overlapping translates of $V_{2m_\Gamma}^{\Gamma}$ namely $R_{1,v}, R_{2,v}, \cdots, R_{N,v}$ ordered from left to right. Suppose that $\xi_{j,v}$ is a polypath contained in $R_{j,v}$ such that $(\xi_{j,v}, X)$ is independent with $\eta_{0.5}$ and $L(\xi_{j,v}, X; \gamma) = 1$. Our next lemma deals with the random variables $\sum_{j \in [N]} \overline{Z}(\xi_{j,v}, X, \eta_{0.5}; \gamma)$.

Lemma 2.2.2. *If $N\Gamma^{-1} \geq 1$ and $\nu \geq 0.1$, then there exist c_1 and C_1 such that*

$$\sqrt{\mathrm{Var}\Big(\sum_{j \in [N]} \overline{Z}(\xi_{j,w+\iota\nu}, X, \eta_{0.5}; \gamma) - \sum_{j \in [N]} \overline{Z}(\xi_{j,w}, X, \eta_{0.5}; \gamma)\Big)} \geq c_1 \sqrt{N\Gamma} - C_1 N\Gamma^{-1} \,.$$

On the other hand $\mathrm{Var}\big(\sum_{j \in [N]} \overline{Z}(\xi_{j,w}, X, \eta_{0.5}; \gamma)\big) = O(N\Gamma + N^2\Gamma^{-2})$ *for all N.*

Proof. First we will show that $\mathrm{Var}\big(\overline{Z}(\xi_{j,v}, X, \eta_{0.5}; \gamma) - \overline{Z}(\Gamma \partial_{\mathrm{up}} R_{j,v}, \eta_{0.5})\big)$ is small. To this end let $u \in R_{j,v}$. By Fubini, we can write $\mathrm{Var}\big(\overline{Z}(\Gamma \partial_{\mathrm{up}} R_{j,v}, \eta_{0.5}) - \eta_{0.5}(u)\big)$ as

$$\int_{[0,1]^2} \mathrm{Cov}\Big(\eta_{0.5}(v_j + \Gamma^{-1}s) - \eta_{0.5}(u), \eta_{0.5}(v_j + \Gamma^{-1}t) - \eta_{0.5}(u)\Big) ds dt \,,$$

where v_j is the upper-left vertex of the rectangle $R_{j,v}$. Since the diameter of $R_{j,v}$ is $O(\Gamma^{-1})$, applying Lemma 2.2.1 to the last expression we get

$$\mathrm{Var}\big(\overline{Z}(\Gamma \partial_{\mathrm{up}} R_{j,v}, \eta_{0.5}) - \eta_{0.5}(u)\big) = O(\Gamma^{-2}) \tag{2.2.7}$$

for any $u \in R_{j,v}$. Now suppose $\mathcal{P}_{j,v}$ is the collection of paths corresponding to $\xi_{j,v}$. Denote, for any path P in $\mathcal{P}_{j,v}$, the quantity $\int_P \mathbb{E}(e_{\xi_{j,v},P} \mathrm{e}^{\gamma X(z)})|dz|$ as $q_{P,j,v}$. Using Fubini and

(2.2.7) in a similar way as we used Fubini and Lemma 2.2.1 for (2.2.7), one gets

$$\operatorname{Var}\Big(\int_P \mathbb{E}(e_{\xi_{j,v},P} \mathrm{e}^{\gamma X(z)}) \eta(z)|dz| - q_{P,j,v} \overline{Z}(\Gamma \partial_{\mathrm{up}} R_{j,v}, \eta_{0.5}) \Big) = O(q_{P,j,v}^2 \Gamma^{-2}) \,, \tag{2.2.8}$$

for all $P \in \mathcal{P}_{j,v}$. Since

$$\overline{Z}(\xi_{j,v}, X, \eta_{0.5}; \gamma) = \sum_{P \in \mathcal{P}_{j,v}} \int_P \mathbb{E}(e_{\xi_{j,v},P} \mathrm{e}^{\gamma X(z)}) \eta(z)|dz| \,,$$

and $\sum_{P \in \mathcal{P}_{j,v}} q_{P,j,v} = L(\xi_{j,v}, X; \gamma) = 1$, (2.2.8) gives us

$$\operatorname{Var}\big(\overline{Z}(\xi_{j,v}, X, \eta_{0.5}; \gamma) - \overline{Z}(\Gamma \partial_{\mathrm{up}} R_{j,v}, \eta_{0.5}) \big) = O(\Gamma^{-2}) \,.$$

Denoting $\sum_{j \in [N]} \overline{Z}(\Gamma \partial_{\mathrm{up}} R_{j,v}, \eta_{0.5})$ as $\overline{Z}_{v,N}$, we then have

$$\begin{aligned}
&\operatorname{Var}\Big(\big(\sum_{j \in [N]} \overline{Z}(\xi_{j,w+\iota\nu}, X, \eta_{0.5}; \gamma) - \overline{Z}_{w+i\nu,N} \big) - \big(\sum_{j \in [N]} \overline{Z}(\xi_{j,w}, X, \eta_{0.5}; \gamma) - \overline{Z}_{w,N} \big) \Big) \\
&= O(N^2 \Gamma^{-2}) \,.
\end{aligned}$$

In order to estimate $\operatorname{Var}(\overline{Z}_{w+\iota\nu,N} - \overline{Z}_{w,N})$, on the other hand, we can use the definition of $\eta_{0.5}(v)$ in (2.2.2) and Fubini to obtain:

$$\begin{aligned}
\operatorname{Var}(\overline{Z}_{w+\iota\nu,N} - \overline{Z}_{w,N}) &= \frac{\Gamma^2}{\pi} \int_{[0,N/\Gamma]^2 \times [0.25,1]} s^{-1} \mathrm{e}^{-\frac{(x-z)^2}{2s}} (1 - \mathrm{e}^{-\frac{\nu^2}{2s}}) dx dz ds \\
&= \Omega(\Gamma^2) \int_{[0.25,1]} \int_{[0,N/\Gamma]} \int_{[0,N/\Gamma]} s^{-1} \mathrm{e}^{-\frac{(x-z)^2}{2s}} dx dz ds \\
&= \Omega(N\Gamma) \,,
\end{aligned}$$

where in the second step we used $\nu \geq 0.1$ and in the final step the fact $\int_{[0,b]} \mathrm{e}^{-ax^2} dx = \Omega_{a,b}(1)$. The last two displays yield the required bound on the standard deviation of $\sum_{j \in [N]} \overline{Z}(\xi_{j,w+i\nu}, X, \eta_{0.5}; \gamma) - \sum_{j \in [N]} \overline{Z}(\xi_{j,w}, X, \eta_{0.5}; \gamma)$. $\operatorname{Var}\big(\sum_{j \in [N]} \overline{Z}(\xi_{j,w}, X, \eta_{0.5}; \gamma) \big)$

can be bounded by similar computations. □

2.3 Inductive constructions for light paths

In this section we will discuss algorithms to construct light paths between the shorter boundaries of V^Γ when the underlying field is $\eta_{2^{-n}}$. Belowe we introduce some terms that will be used repeatedly.

A polypath ξ is said to *connect* two polypaths ξ' and ξ'' if ξ always intersects ξ' and ξ'' considered as subsets of $\mathbb{R}^2$. More generally we say that the polypaths $\xi_1, \xi_2, \cdots, \xi_k$ *form* or *define* a polypath if their union is always a connected subset of $\mathbb{R}^2$. A *crossing* for a rectangle $\mathcal{R}$ is any polypath ξ that stays entirely within $\mathcal{R}$ and connects two shorter boundaries of $\mathcal{R}$.

Depending on the value of current scale n, we will use one of two different strategies for constructing a crossing cross_n for V^Γ. To be more precise, let $2a_n m_\Gamma \leq n < 2(a_n+1)m_\Gamma$ where $2^{m_\Gamma} = \Gamma$ and $a_n \in \mathbb{N} \cup \{0\}$. We will use a simple strategy called *Strategy I* when $2a_n m_\Gamma \leq n < 2a_n m_\Gamma + 2m_\Gamma - 1$ and a different strategy called *Strategy II* otherwise. We detail these two strategies in separate subsections.

2.3.1 Strategy I

We will adopt an inductive approach. Consider the rectangle $0.5\iota + [0, \Gamma] \times [0, 2^{2a_n m_\Gamma - 1 - n}]$. Notice that this is same as $V_{n-2a_n m_\Gamma+1}^{2^{n-2a_n m_\Gamma+1}\Gamma; 0.5\iota}$. Subdivide it into non-overlapping translates of $V^\Gamma_{n-2a_n m_\Gamma+1}$ and denote them by $R_{1;n-2a_n m_\Gamma+1}, R_{2;n-2a_n m_\Gamma+1}, \cdots$ from left to right (see Figure 2.1).

Now suppose that for all $\ell \leq 2a_n m_\Gamma - 1$, we already have an algorithm $\mathcal{A}_{2a_n m_\Gamma - 1}$ that constructs a crossing through V^Γ and takes only the fields $\{\eta_{2^{-k}}\}_{k\in[\ell]}$ as input. Due to the scaling and translation invariance property of η we can then use $\mathcal{A}_{2a_n m_\Gamma-1}$ to construct a crossing $\mathrm{cross}_{j;n}$ through $R_{j;n-2a_n m_\Gamma+1}$ using only the fields $\{\eta_{2^{-k}}^{2^{2a_n m_\Gamma - 1 - n}}\}_{n-2a_n m_\Gamma+1<k\leq n}$ as its input. Henceforth whenever we talk about applying $\mathcal{A}_{\ell'}$ to construct a crossing at scale

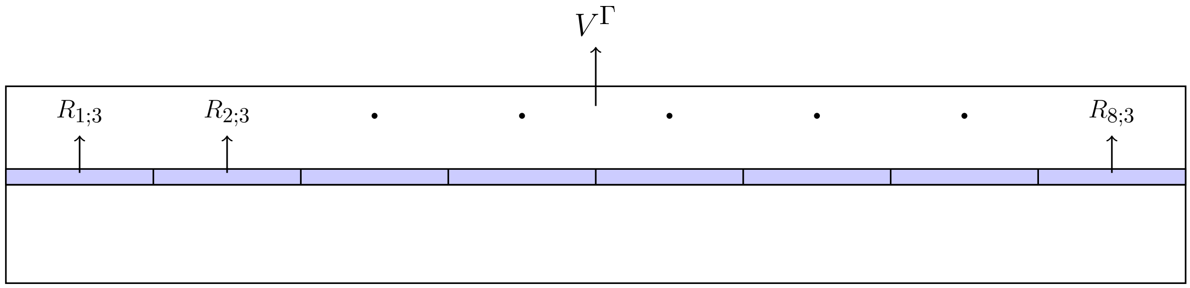

Figure 2.1: **The rectangles** $R_{1;n-2a_nm_\Gamma+1}, R_{2;n-2a_nm_\Gamma+1}, \cdots$**.** Here $n = 2a_nm_\Gamma + 2$.

n, we will suppress the statement that the fields used to construct it are $\{\eta_{2^{-\ell}}^{2^{-(n-\ell')}}\}_{n-\ell'<\ell\leq n}$. The remaining job is to link the pair of crossings $\text{cross}_{j;n}$ and $\text{cross}_{j+1;n}$. This can be done in a simple way which we call *tying* for convenience. We describe this technique in a general setting as it will be used several times in the future. The reader is referred to Figure 2.2 for an illustration.

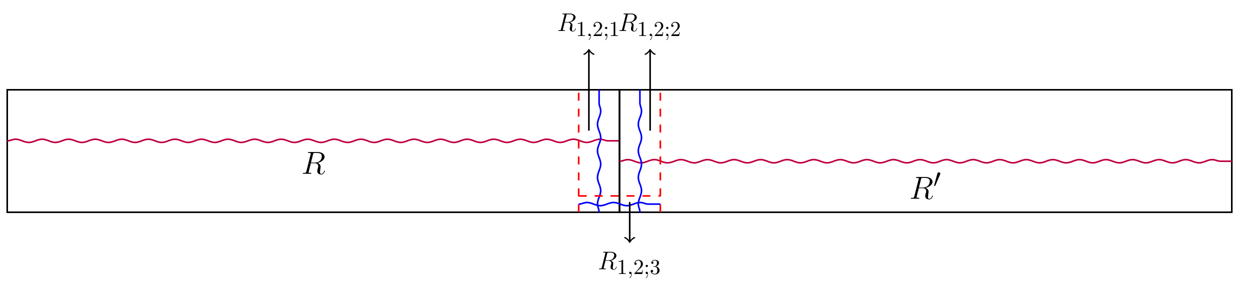

Figure 2.2: **Tying** cross_R **and** $\text{cross}_{R'}$**.** The crossings cross_R and $\text{cross}_{R'}$ are indicated by purple lines. The two vertical blue lines indicate the crossings $\text{cross}^{*,R_{1,2;1}}$ (left) and $\text{cross}^{*,R_{1,2;2}}$ (right). The horizontal blue line indicates the crossing $\text{cross}^{*,R_{1,2;3}}$.

Let $k \in [n-1]$. Consider two adjacent copies of V_k^Γ. Without any loss of generality (because of the rotational invariance property of $\eta_\delta^{\delta'}$), assume that their longer boundary segments are aligned with the horizontal axis. Call the left one as $R = I \times J$ and the right one as $R' = I' \times J'$. We want to link two crossings cross_R and $\text{cross}_{R'}$ through R and R' respectively to build a crossing for $R \cup R'$. To this end define three additional rectangles $R_{1,2;1} = [r_I - 2^{-k-m_\Gamma}, r_I] \times J$, $R_{1,2;2} = [r_I, r_I + 2^{-k-m_\Gamma}] \times J$ and $R_{1,2;3} = [r_I - 2^{-k-m_\Gamma}, r_I + 2^{-k-m_\Gamma}] \times [\ell_J, \ell_J + 2^{-k-2m_\Gamma+1}]$, where ℓ_J and r_I are the left and right endpoints of J and I respectively. We use $\mathcal{A}_{n-k-m_\Gamma}$ to construct up-down crossings

$\text{cross}_{R_{1,2;1}}$ and $\text{cross}_{R_{1,2;2}}$ for $R_{1,2;1}$ and $R_{1,2;2}$ respectively. Similarly we apply $\mathcal{A}_{n-k-2m_\Gamma+1}$ to construct a left-right crossing $\text{cross}_{R_{1,2;3}}$ through $R_{1,2;3}$. Let us also make it clear that $\mathcal{A}_\ell$ constructs a straight line connecting midpoints of the shorter boundary segments of V^Γ when $\ell \leq 0$. Finally notice that the union of crossings $\text{cross}_R, \text{cross}_{R'}, \text{cross}_{R_{1,2;1}}, \text{cross}_{R_{1,2;2}}$ and $\text{cross}_{R_{1,2;3}}$ is a crossing for the rectangle $R \cup R'$. We refer to this as the crossing obtained from tying cross_R and $\text{cross}_{R'}$.

Thus we tie together the sequence of crossings $\text{cross}_{1;n}, \text{cross}_{2;n}, \cdots, \text{cross}_{2^{n-2a_n m_\Gamma+1};n}$ (i.e. every pair of successive crossings) to form cross_n. Figure 2.3 provides an illustration of this construction.

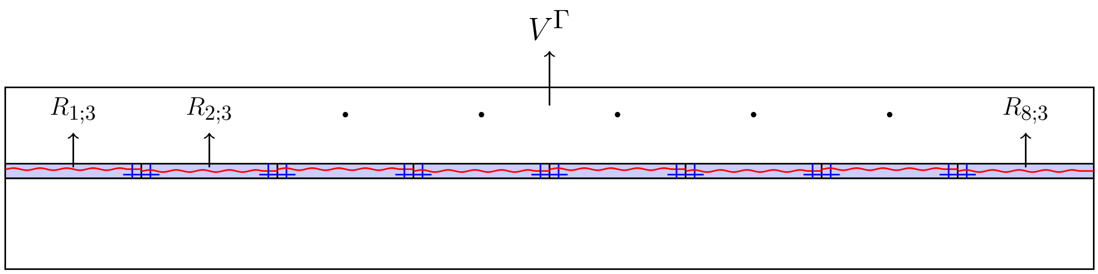

Figure 2.3: **Construction of cross_n using Strategy I.** Here $n = 2a_n m_\Gamma + 2$. The red lines indicate the crossings $\text{cross}_{j;n}$'s while the blue lines indicate the crossings used for tying the pairs $(\text{cross}_{j;n}, \text{cross}_{j+1;n})$'s.

2.3.2 Strategy II

This is our main strategy which employs switching using sloped gadgets in order to build efficient crossings. Recall that $n = 2(a_n+1)m_\Gamma - 1$ in this case. Unlike in Strategy I here we start with two strips $V_{2m_\Gamma}^{\Gamma^2;0.25\iota}$ and $V_{2m_\Gamma}^{\Gamma^2;0.75\iota}$. We subdivide $V_{2m_\Gamma}^{\Gamma^2;0.75\iota}$ and $V_{2m_\Gamma}^{\Gamma^2;0.25\iota}$ into non-overlapping translates of $V_{2m_\Gamma}^{\Gamma\beta}$ where β is the smallest power of 2 that is $\geq \gamma^{-2/3}$. Let us denote them as $R_{1,1}, R_{1,2}, \cdots, R_{1,\Gamma/\beta}$ and $R_{2,1}, R_{2,2}, \cdots, R_{2,\Gamma/\beta}$ respectively from left to right. Similarly one can subdivide each $R_{i,j}$ into non-overlapping translates of $V_{2m_\Gamma}^{\Gamma^2}$ which we call as its *blocks*. See Figure 2.4 below for an illustration of this set-up. We can use $\mathcal{A}_{2a_n m_\Gamma - 1}$ to construct a crossing cross_R through each block R. Let $R_{i,j,\text{left}}$ and

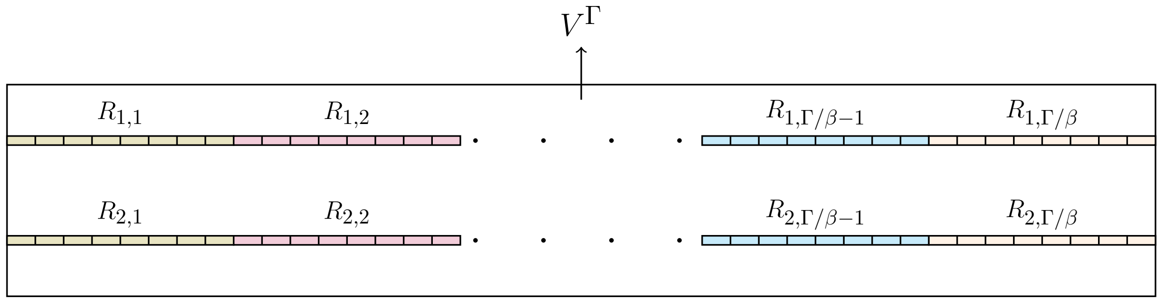

Figure 2.4: **The rectangles $R_{i,j}$'s and its blocks.** In this (hypothetical) example each $R_{i,j}$ consists of 8 blocks and thus $\Gamma = 8$.

$R_{i,j,\text{right}}$ respectively denote the leftmost and rightmost blocks of $R_{i,j}$. We will construct a new sequence of crossings which, when tied, gives a polypath connecting $\text{cross}_{R_{1,j,\text{left}}}$ and $\text{cross}_{R_{2,j,\text{right}}}$. Observe that, due to the choice of Γ and the fact $d_{\ell_2}(V_{2m_\Gamma}^{\Gamma^2;0.25\iota}, V_{2m_\Gamma}^{\Gamma^2;0.75\iota}) = \Omega(1)$, there exists an integer L_γ and a copy $S_{1,j}$ of $V_{2m_\Gamma}^{L_\gamma\Gamma}$ such that:

(I) The length of $S_{1,j}$ is at most $d_{\ell_2}(c_{R_{1,j,\text{left}}}, c_{R_{2,j,\text{right}}}) + 2/\Gamma$ where c_R denotes the center of a rectangle R.

(II) $S_{1,j}, R_{1,j,\text{left}}$ and $R_{2,j,\text{right}}$ are arranged as in Figure 2.5.

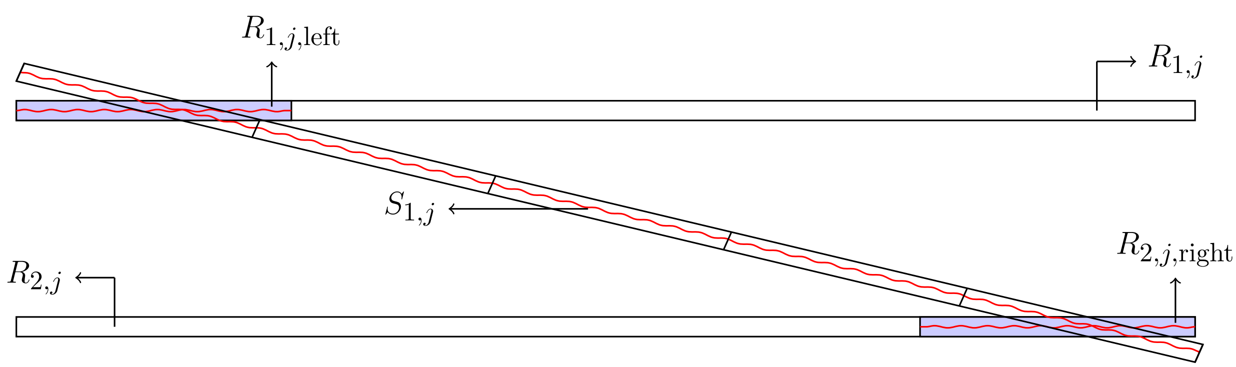

Figure 2.5: **The rectangles $R_{1,j,\text{left}}$, $R_{2,j,\text{right}}$ and $S_{1,j}$.** Each of the five rectangles comprising $S_{1,j}$ is a copy of $V_{2m_\Gamma}^{\Gamma}$. Hence $L_\gamma = 5$ in this example. The red lines inside each rectangle indicate the corresponding crossings.

It is clear from the arrangement depicted in (II) (or in Figure 2.5 for that matter) that any crossing through $S_{1,j}$ intersects both $\text{cross}_{R_{1,j,\text{left}}}$ and $\text{cross}_{R_{2,j,\text{right}}}$. Now subdivide $S_{1,j}$ into L_γ non-overlapping copies of $V_{2m_\Gamma}^{\Gamma}$ and construct a crossing through each one of them using $\mathcal{A}_{2a_n m_\Gamma - 1}$. Tying these crossings would then give a crossing of $S_{1,j}$ which connects $\text{cross}_{R_{1,j,\text{left}}}$ and $\text{cross}_{R_{2,j,\text{right}}}$. Similarly we can construct a copy $S_{2,j}$ of $V_{2m_\Gamma}^{L_\gamma \Gamma}$ and a corresponding sequence of crossings which connect $\text{cross}_{R_{2,j,\text{left}}}$ and $\text{cross}_{R_{1,j,\text{right}}}$ after they are tied. The L_γ non-overlapping copies of $V_{2m_\Gamma}^{\Gamma}$ comprising $S_{i,j}$ are also called its *blocks*. Henceforth we will refer to the collection of blocks of $S_{i,j}$'s and $R_{i,j}$'s as Block_γ. We now have all the ingredients for defining our strategy which is essentially encoded by the numbers $i_j \in [2]$. Given these numbers, we define a collection $\mathscr{C}_{a_n}$ of crossings as follows. If $i_j = i_{j+1}$, we include cross_R in $\mathscr{C}_{a_n}$ for all the blocks R of $R_{i_j,j}$. Otherwise we include cross_R for all the blocks R of $S_{i_j,j}$ as well as $\text{cross}_{R_{i_j,j,\text{left}}}$ and $\text{cross}_{R_{3-i_j,j,\text{right}}}$ (notice that $3 - i_j$ switches 1 and 2). We refer to the collection of blocks included in $\mathscr{C}_{a_n}$ from a "location" j as the *bridge* at that location. Unless there is a switch at location 1 (as $S_{i,1}$ can potentially intersect $\mathbb{R}^2 \setminus V^\Gamma$), the crossings in $\mathscr{C}_{a_n}$ define a crossing for V^Γ after we tie every pair of crossings $(\text{cross}_R, \text{cross}_{R'})$ in $\mathscr{C}_{a_n}$ for adjacent R, R'. See Figure 2.6 below for an illustration. The particular choice of i_j's will be determined by the field $\eta_{0.5}$ which we discuss in the next section.

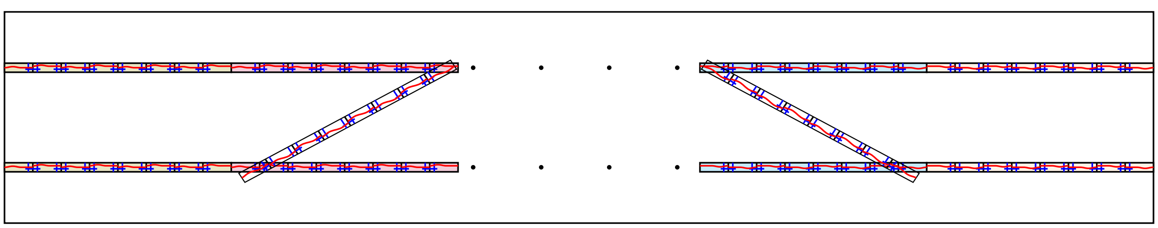

Figure 2.6: **Construction of cross_n using Strategy II.** In this example $i_1 = i_2 = 2$ and $i_3 = 1$; $i_{\Gamma/\beta - 1} = 1$ and $i_{\Gamma/\beta} = 2$. The red lines indicate the crossings in $\mathscr{C}_{a_n}$ and the blue lines indicate the crossings used to tie them.

2.4 Multi-scale analysis on expected weight of crossings

Let $D_{\gamma,n}$ denote the total weight of cross_n computed with $\eta_{2^{-n}}$ as the underlying field and $d_{\gamma,n}$ denote its expectation. In Sections 2.4.1 through 2.4.3 we will derive recurrence relations involving $d_{\gamma,n}$'s for $n \in \mathbb{N}$. It is useful to recall at this point that $d_{\gamma,n} = \Gamma$ whenever $n \leq 0$.

In Section 2.4.4 we show how these relations lead to a bound on $d_{\gamma,n}$.

2.4.1 Strategy I: a recurrence relation involving $d_{\gamma,n}$

We will assume $n > 0$. For convenience we use $[n]_\gamma$ to denote $n-2a_n m_\Gamma+1$. Let $D_{\gamma,n,\text{main}}$ denote the total weight of $\text{cross}_{1;n}, \text{cross}_{2;n}, \cdots, \text{cross}_{2^{[n]_\gamma};n}$ (see Section 2.3.1) and $D_{\gamma,n,\text{gadget}}$ denote the total weight of crossings used to tie them. These weights are all computed with respect to $\eta_{2^{-n}}$ and thus $D_{\gamma,n} = D_{\gamma,n,\text{main}} + D_{\gamma,n,\text{gadget}}$. Notice that the weight of $\text{cross}_{j;n}$ is $Z(\text{cross}_{j;n}, \eta_{2^{-n}}^{2^{-[n]_\gamma}}, \mathrm{e}^{\gamma\eta_{2^{-[n]_\gamma}}}; \gamma)$ (see (2.2.4) for the definition of $Z(\cdot,\cdot,\cdot;\gamma)$). Hence from Fubini and the translation invariance property of η_δ we get

$$\mathbb{E}D_{\gamma,n,\text{main}} \leq \mathbb{E}\mathrm{e}^{\gamma\eta_{2^{-[n]_\gamma}}(0)} 2^{[n]_\gamma} \frac{d_{\gamma,2a_n m_\Gamma-1}}{2^{[n]_\gamma}}\,, \tag{2.4.1}$$

where the divisor $2^{[n]_\gamma}$ comes from scaling property (compare to the situation when $\gamma = 0$). From this point onwards any expression of the form "$\frac{d_{\gamma,k}}{2^{n-k}}$" would implicitly mean that the divisor 2^{n-k} originates from a similar consideration. Now since $\mathrm{Var}(\eta_\delta(0)) = O(\log\delta^{-1})$ and $m_\Gamma = O(\log\gamma^{-1})$, the last display gives us

$$\mathbb{E}D_{\gamma,n,\text{main}} = (1 + O(\gamma^2\log\gamma^{-1}))d_{\gamma,2a_n m_\Gamma-1}\,. \tag{2.4.2}$$

As to the estimation of $\mathbb{E}D_{\gamma,n,\text{gadget}}$, recall from Section 2.3.1 that we spend three crossings for tying the pair $(\text{cross}_{j;n}, \text{cross}_{j+1;n})$. Two of these are constructed using $\mathcal{A}_{(2a_n-1)m_\Gamma-1}$ and the other one using $\mathcal{A}_{2(a_n-1)m_\Gamma}$. Hence by a similar reasoning as used for (2.4.1), the expected weight of these crossings is given by

$$2\mathbb{E}\mathrm{e}^{\gamma\eta_{2^{-[n]_\gamma-m_\Gamma}}(0)}\frac{d_{\gamma,(2a_n-1)m_\Gamma-1}}{2^{[n]_\gamma+m_\Gamma}} + \mathbb{E}\mathrm{e}^{\gamma\eta_{2^{-[n]_\gamma-2m_\Gamma}}(0)}\frac{d_{\gamma,2(a_n-1)m_\Gamma}}{2^{[n]_\gamma+2m_\Gamma-1}}\,.$$

Since there are $2^{[n]_\gamma}-1$ many tyings, this implies (along with the variance bounds given by

(2.2.3))

$$\mathbb{E}D_{\gamma,n,\text{gadget}} \leq (1+O(\gamma^2\log\gamma^{-1}))(2\Gamma^{-1}d_{\gamma,(2a_n-1)m_\Gamma-1}+\Gamma^{-2}d_{\gamma,2(a_n-1)m_\Gamma})\,.$$

Combined with (2.4.2), the last inequality gives us

$$d_{\gamma,n} \leq (1+O(\gamma^2\log\gamma^{-1}))(d_{\gamma,2a_nm_\Gamma-1}+2\Gamma^{-1}d_{\gamma,(2a_n-1)m_\Gamma-1}+\Gamma^{-2}d_{\gamma,2(a_n-1)m_\Gamma})\,. \quad (2.4.3)$$

2.4.2 Strategy II: choosing the particular strategy

As in Section 2.4.1, we begin with a decomposition of $D_{\gamma,n}$ into two components. To this end denote by $D_{\gamma,n,\text{main}}$ the total weight of crossings in $\mathscr{C}_{a_n}$ where $n = 2a_nm_\Gamma + 2m_\Gamma - 1$. The other component $D_{\gamma,n,\text{gadget}}$ is the total weight of gadgets that we use to tie pairs of crossings $(\text{cross}_R, \text{cross}_{R'})$ in $\mathscr{C}_{a_n}$ for adjacent R, R' (see Section 2.3.2). All the weights are computed with respect to the field $\eta_{2^{-n}}$. $D_{\gamma,n,\text{main}}$ is the major component and will inform our choice of strategy.

We, in fact, devise our strategy based on an approximate expression of $\mathbb{E}(D_{\gamma,n,\text{main}}|\eta_{0.5})$. For this we need to analyze $D_{\gamma,n,\text{main};j}$ which is the combined weight of crossings through all the blocks in the bridge at location j. In our analysis we rely heavily on the fact that our strategy is determined by $\eta_{0.5}$. Also along the way we make several approximations that will be justified in a later subsection. Let us begin with the case $i_j = i_{j+1}$. In this case

$$\mathbb{E}(D_{\gamma,n,\text{main};j}|\eta_{0.5}) = \sum_{R\in\text{Block}_\gamma, R\subseteq R_{i_j,j}} \overline{Z}(\text{cross}_R, \eta^{0.5}_{2^{-n}}, \mathrm{e}^{\gamma\eta_{0.5}};\gamma)\,.$$

Now we replace $\eta^{0.5}_{2^{-n}}$ in the above expression with $\eta^{\Gamma^{-2}}_{2^{-n}}$ which results in

$$\text{approx}_{j,1} = \sum_{R\in\text{Block}_\gamma, R\subseteq R_{i_j,j}} \overline{Z}(\text{cross}_R, \eta^{\Gamma^{-2}}_{2^{-n}}, \mathrm{e}^{\gamma\eta_{0.5}};\gamma)\,.$$

We further approximate $\mathrm{e}^{\gamma\eta_{0.5}(z)}$ with $1+\gamma\eta_{0.5}(z)$ and obtain a new expression as follows (recall the definitions (2.2.5) and (2.2.6)):

$$\begin{aligned}\mathrm{approx}_{j,2} &= \sum_{R\in\mathrm{Block}_\gamma, R\subseteq R_{i_j,j}} \overline{Z}(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, 1+\gamma\eta_{0.5};\gamma)\\ &= \sum_{R\in\mathrm{Block}_\gamma, R\subseteq R_{i_j,j}} \Big(L(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}};\gamma) + \overline{Z}(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \gamma\eta_{0.5};\gamma)\Big)\\ &= \beta\Gamma\frac{d_{\gamma,2a_n m_\Gamma-1}}{\Gamma^2} + \gamma\sum_{R\in\mathrm{Block}_\gamma, R\subseteq R_{i_j,j}} \overline{Z}(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \eta_{0.5};\gamma)\\ &= \beta\Gamma^{-1}d_{\gamma,2a_n m_\Gamma-1} + \overline{Z}_{\gamma,n,i_j,j}\end{aligned}$$

where $\overline{Z}_{\gamma,n,i_j,j} = \gamma\sum_{R\in\mathrm{Block}_\gamma, R\subseteq R_{i_j,j}} \overline{Z}(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \eta_{0.5};\gamma)$. Thus there is a "deterministic" part and a "random" part in $\mathrm{approx}_{j,2}$. The small magnitude of γ is crucial for these approximations. When $i_j \neq i_{j+1}$, i.e. there is a switch at the location j, deriving $\mathrm{approx}_{j,2}$ requires slightly more work. In this case

$$\mathbb{E}(D_{\gamma,n,\mathrm{main};j}|\eta_{0.5}) = \sum_{R\in\mathrm{Block}_\gamma, R\subseteq S_{i_j,j}\cup R_{i_j,j,\mathrm{left}}\cup R_{3-i_j,j,\mathrm{right}}} \overline{Z}(\mathrm{cross}_R, \eta_{2^{-n}}^{0.5}, \mathrm{e}^{\gamma\eta_{0.5}};\gamma)\,.$$

Similarly as before we ignore the contribution from $\eta_{\Gamma^{-2}}^{0.5}$ and the higher order terms in $\mathrm{e}^{\gamma\eta_{0.5}(z)}$ to obtain

$$\mathrm{approx}_{j,1} = \sum_{R\in\mathrm{Block}_\gamma, R\subseteq S_{i_j,j}\cup R_{i_j,j,\mathrm{left}}\cup R_{3-i_j,j,\mathrm{right}}} \overline{Z}(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \mathrm{e}^{\gamma\eta_{0.5}};\gamma)\,,$$

and

$$\mathrm{approx}_{j,2} = \sum_{R\in\mathrm{Block}_\gamma, R\subseteq S_{i_j,j}\cup R_{i_j,j,\mathrm{left}}\cup R_{3-i_j,j,\mathrm{right}}} L(\mathrm{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}};\gamma) + \overline{Z}'_{\gamma,n,i_j,j}\,,$$

where

$$\overline{Z}'_{\gamma,n,i_j,j} = \gamma \sum_{R\in\text{Block}_\gamma, R\subseteq S_{i_j,j}\cup R_{i_j,j,\text{left}}\cup R_{3-i_j,j,\text{right}}} \overline{Z}(\text{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \eta_{0.5}; \gamma)\,.$$

Recall from Section 2.3.2 that the total number of blocks in the bridge in this case is $L_\gamma + 2$. From property (I) of $S_{i_j,j}$ and the elementary fact $\sqrt{1+x^2} = 1 + \frac{x^2}{2} + o(x^2)$ as $x \to 0$, we get

$$\frac{L_\gamma}{\Gamma} = \beta + \frac{C'_{\gamma,n}}{\beta}\,,$$

for some $C'_{\gamma,n} = \Theta(1)$. Hence the deterministic part in $\text{approx}_{j,2}$ is

$$\beta\Gamma\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma^2} + C''_{\gamma,n}\frac{\Gamma}{\beta}\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma^2}\,,$$

where again $C''_{\gamma,n} = \Theta(1)$. Writing the random part $\overline{Z}'_{\gamma,n,i_j,j}$ as

$$\overline{Z}'_{\gamma,n,i_j,j} = \overline{Z}_{\gamma,n,i_j,j} + \text{Loss}_{\gamma,n,i_j,j}\,, \tag{2.4.4}$$

we obtain in this case

$$\text{approx}_{j,2} = \beta\Gamma\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma^2} + C''_{\gamma,n}\frac{\Gamma}{\beta}\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma^2} + \overline{Z}_{\gamma,n,i_j,j} + \text{Loss}_{\gamma,n,i_j,j}\,. \tag{2.4.5}$$

Now from Lemma 2.2.2 we have

$$\begin{aligned}\text{Var}\Big(\gamma \sum_{R\in\text{Block}_\gamma, R\subseteq S_{i_j,j}} \overline{Z}(\text{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \eta_{0.5}; \gamma)\Big) &= \gamma^2\Gamma^2\frac{d^2_{\gamma,2a_nm_\Gamma-1}}{\Gamma^4}O(\beta+\beta^2\Gamma^{-2})\\ = \quad O(\Gamma^2\gamma^2\beta)\frac{d^2_{\gamma,2a_nm_\Gamma-1}}{\Gamma^4}\,.\end{aligned}$$

The same bound holds for $\text{Var}(\overline{Z}_{\gamma,n,i_j,j})$ and (obviously) for $\text{Var}(\gamma\overline{Z}(\text{cross}_R, \eta_{2^{-n}}^{\Gamma^{-2}}, \eta_{0.5}; \gamma))$

when $R = R_{i_j,j,\text{left}}$ or $R_{i_j,j,\text{right}}$. Thus from (2.4.4) we get

$$\text{Var}(\text{Loss}_{\gamma,n,i,j}) = O(\Gamma^2\gamma^2\beta)\frac{d^2_{\gamma,2a_nm_\Gamma-1}}{\Gamma^4}\,.$$

As $\text{Loss}_{\gamma,n,i,j}$'s are centered Gaussian variables, the previous bound implies

$$\sum_{i\in[2]}\mathbb{E}(\text{Loss}^+_{\gamma,n,i,j}) = O(\Gamma\gamma\sqrt{\beta})\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma^2}\,.$$

Incorporating this bound into (2.4.5), we get the following upper bound on the expectation of $\text{approx}_2 = \sum_{j\in[\Gamma/\beta]}\text{approx}_{j,2}$.

$$\mathbb{E}(\text{approx}_2) \le d_{\gamma,2a_nm_\Gamma-1} + \mathbb{E}\sum_{j\in[\Gamma/\beta]}\overline{Z}_{\gamma,n,i_j,j} + C_{\gamma,n}\frac{(\beta^{-1}+\gamma\sqrt{\beta})d_{\gamma,2a_nm_\Gamma-1}}{\Gamma}N_{\text{switch}}\,, \quad (2.4.6)$$

where $C_{\gamma,n} = \Theta(1)$ and N_{switch} is total number of "potential" switching locations (deterministic). Since $\overline{Z}_{\gamma,n,i_j,j}$'s are centered, we can write

$$\mathbb{E}\sum_{j\in[\Gamma/\beta]}\overline{Z}_{\gamma,n,i_j,j} = \frac{1}{2}\mathbb{E}\sum_{j\in[\Gamma/\beta]}(-1)^{i_j+1}(\overline{Z}_{\gamma,n,1,j} - \overline{Z}_{\gamma,n,2,j})\,.$$

Hence we choose our strategy so that it gives a small value of the following expectation:

$$E_{\gamma,n} = \mathbb{E}\Big(\frac{1}{2}\sum_{j\in[\Gamma/\beta]}(-1)^{i_j+1}\Delta\overline{Z}_{\gamma,n,j} + C_{\gamma,n}(\beta^{-1}+\gamma\sqrt{\beta})\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma}N_{\text{switch}}\Big)\,, \quad (2.4.7)$$

where $\Delta\overline{Z}_{\gamma,n,j} = \overline{Z}_{\gamma,n,1,j} - \overline{Z}_{\gamma,n,2,j}$. From Lemma 2.2.2 we can deduce that for any $1 \le j_1 \le j_2 \le \lceil\Gamma/\beta\rceil$,

$$\begin{aligned}\text{Var}\Big(\sum_{j_1\le j\le j_2}\frac{1}{2}\Delta\overline{Z}_{\gamma,n,j}\Big) &= \Omega(\gamma^2)\frac{d^2_{\gamma,2a_nm_\Gamma-1}}{\Gamma^4}(j_2-j_1+1)\beta\Gamma^2\Big(1-O\big(\frac{\sqrt{(j_2-j_1+1)\beta}}{\Gamma}\big)\Big)\\ &\ge c_2\gamma^2(j_2-j_1+1)\beta\frac{d^2_{\gamma,2a_nm_\Gamma-1}}{\Gamma^2}\,.\end{aligned}$$

As a consequence we have

$$\mathbb{E}\Big|\sum_{j_1\le j\le j_2}\frac{1}{2}\Delta\overline{Z}_{\gamma,n,j}\Big| \ge 2C_{\gamma,n}(\beta^{-1}+\gamma\sqrt{\beta})\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma} \tag{2.4.8}$$

whenever

$$j_2-j_1+1\ge \frac{4C^2_{\gamma,n}(\beta^{-1}+\gamma\sqrt{\beta})^2}{\frac{2}{\pi}\gamma^2\beta}=N'_{\gamma,n}\,. \tag{2.4.9}$$

Here we use the simple fact that $\mathbb{E}|Z|=\sqrt{\frac{2}{\pi}}$ for a standard Gaussian Z. Let $N_{\gamma,n}$ be the smallest power of 2 that is $\ge N'_{\gamma,n}$. We are now ready to define our strategy. Set

$$i_j=\begin{cases}0 & \text{if } \sum_{(k_j-1)N_{\gamma,n}+1\le j'\le k_jN_{\gamma,n}}\Delta\overline{Z}_{\gamma,n,j'}>0\,,\\ 1 & \text{otherwise}\,,\end{cases}$$

where $k_j\in\mathbb{N}$ is such that $(k_j-1)N_{\gamma,n}+1\le j\le k_jN_{\gamma,n}$. It then follows from (2.4.7), (2.4.8) and (2.4.9), and the choice of β as $\Theta(\gamma^{-2/3})$ that

$$\begin{aligned} E_{\gamma,n} &\le \frac{\Gamma}{N_{\gamma,n}\beta}\times -C_{\gamma,n}(\beta^{-1}+\gamma\sqrt{\beta})\frac{d_{\gamma,2a_nm_\Gamma-1}}{\Gamma} = -\Omega\Big(\frac{\gamma^2}{\beta^{-1}+\gamma\sqrt{\beta}}\Big)d_{\gamma,2a_nm_\Gamma-1}\\ &= -\Omega(\gamma^{4/3})d_{\gamma,2a_nm_\Gamma-1}\,. \end{aligned} \tag{2.4.10}$$

Notice that this strategy ensures $i_1=i_2$ i.e. there is no switch at location 1 which implies we get a "legitimate" crossing (see the discussions at the end of Section 2.3.2).

2.4.3 Strategy II: a recurrence relation involving $d_{\gamma,n}$

Let us first estimate the expected errors that we made in every stage of approximation in the previous subsection. Denote the sum $\sum_{j\in[\Gamma/\beta]}\text{approx}_{j,1}$ as approx_1. Since the choice of crossings in $\mathscr{C}_{a_n}$ is independent with $\eta^{0.5}_{\Gamma^{-2}}$, from Fubini and translation invariance of η we get

$$\mathbb{E}D_{\gamma,n,\text{main}}=\mathbb{E}\mathrm{e}^{\gamma\eta^{0.5}_{\Gamma^{-2}}(0)}\mathbb{E}(\text{approx}_1)=(1+O(\gamma^2\log\gamma^{-1}))\mathbb{E}(\text{approx}_1)\,.$$

Next we take care of the approximation of approx_1 with approx_2. Since $e^x \geq 1+x$, it follows that $\text{approx}_1 \geq \text{approx}_2$. On the other hand, a reasoning similar to the one used for last display gives us

$$\mathbb{E}(\text{approx}_1 - \text{approx}_2) \leq \mathbb{E}(e^{\gamma\eta_{0.5}(0)} - 1 - \gamma\eta_{0.5}(0))\frac{d_{\gamma,2a_n m_\Gamma - 1}}{\Gamma^2}|\text{Block}_\gamma| \,.$$

It is straightforward that $|\text{Block}_\gamma| = O(\Gamma^2)$ and hence

$$\mathbb{E}(\text{approx}_1 - \text{approx}_2) \leq O(\gamma^2) d_{\gamma,2a_n m_\Gamma - 1} \,.$$

Since $\mathbb{E}(\text{approx}_2) \leq d_{\gamma,2a_n m_\Gamma - 1} + E_{\gamma,n}$ (see (2.4.6), (2.4.7)), the bounds from the previous displays and (2.4.10) together imply

$$\mathbb{E}D_{\gamma,n,\text{main}} \leq d_{\gamma,2a_n m_\Gamma - 1}(1 - \Omega(\gamma^{4/3})) \,. \tag{2.4.11}$$

It only remains to deal with $\mathbb{E}D_{\gamma,n,\text{gadget}}$. In fact the argument that we used to bound $\mathbb{E}D_{\gamma,n,\text{gadget}}$ for Strategy I can be applied directly in this case to obtain

$$\mathbb{E}D_{\gamma,n,\text{gadget}} \leq (2\frac{d_{\gamma,(2a_n-1)m_\Gamma - 1}}{\Gamma^3}\mathbb{E}e^{\gamma\eta_{\Gamma-3}(0)} + \frac{d_{\gamma,2(a_n-1)m_\Gamma}}{\Gamma^4}\mathbb{E}e^{\gamma\eta_{2\Gamma-4}(0)})|\text{Block}_\gamma| \,.$$

which implies

$$\mathbb{E}D_{\gamma,n,\text{gadget}} = O(1)(\Gamma^{-1}d_{\gamma,(2a_n-1)m_\Gamma - 1} + \Gamma^{-2}d_{\gamma,2(a_n-1)m_\Gamma}) \,. \tag{2.4.12}$$

Finally (2.4.11) and (2.4.12) give us

$$d_{\gamma,n} \leq d_{\gamma,2a_n m_\Gamma - 1}(1 - \Omega(\gamma^{4/3})) + O(1)(\Gamma^{-1}d_{\gamma,(2a_n-1)m_\Gamma - 1} + \Gamma^{-2}d_{\gamma,2(a_n-1)m_\Gamma}) \,. \tag{2.4.13}$$

2.4.4 Upper bound on $d_{\gamma,n}$

We will use the recursion relations (2.4.3), (2.4.13) and an induction argument to derive an upper bound on $d_{\gamma,n}$. To this end let C_2 be a positive, absolute constant (from (2.4.13)) such that

$$d_{\gamma,2(a+1)m_\Gamma-1} \leq d_{\gamma,2am_\Gamma-1}(1-C_2\gamma^{4/3}) + O(1)\Big(\frac{d_{\gamma,(2a-1)m_\Gamma-1}}{\Gamma} + \frac{d_{\gamma,2(a-1)m_\Gamma}}{\Gamma^{-2}}\Big), \quad (2.4.14)$$

for all $a \geq 0$. Fixing an $a \in \mathbb{N} \cup \{0\}$, we formulate our induction hypotheses as:

(a) $d_{\gamma,2am_\Gamma-1} \leq \Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^a$.

(b) $d_{\gamma,n} \leq 2\Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a_n+1}$ for all $n < 2am_\Gamma$.

Hypotheses (a) and (b) obviously hold for $a=0$ since $d_{\gamma,n} = \Gamma$ for $n \leq 0$. Now combined with (2.4.14) and the fact $\Gamma \geq \gamma^{-2}$, these two hypotheses imply

$$\begin{aligned} d_{\gamma,2(a+1)m_\Gamma} &\leq \Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^a(1-C_2\gamma^{4/3}) + O(\gamma^2)\Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a-1} \\ &= \Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a+1}(1-\Omega(\gamma^{4/3})+O(\gamma^2)) \leq \Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a+1}. \end{aligned}$$

On the other hand for $2am_\Gamma \leq n < 2(a+1)m_\Gamma$, we can apply (2.4.3) and hypotheses (a), (b) to obtain

$$\begin{aligned} d_{\gamma,2(a+1)m_\Gamma} &\leq (1+O(\gamma^2\log\gamma^{-1}))\Gamma\Big(\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^a + O(\gamma^2)\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a-1}\Big) \\ &= \Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a+1}(1+O(\gamma^{4/3})) \leq 2\Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a+1}. \end{aligned}$$

Thus by induction it follows that

$$d_{\gamma,n} \leq 2\Gamma\big(1-\frac{C_2\gamma^{4/3}}{2}\big)^{a_n+1}, \quad (2.4.15)$$

for all $n \geq 0$.

2.5 Proof of Theorem 2.1.1

For the purpose of proving Theorem 2.1.1, we can identify h_δ with its *white noise decomposition* given in (2.2.1). Since this representation involves some special functions, it would be helpful to have convenient notations for them. To this end we denote $p(s;v,w) - p_D(s;v,w)$ as $\overline{p}_D(s;v,w)$. Also for any function f defined on $\mathbb{R}^+ \times D_\epsilon \times D_\epsilon$ and $\delta \le \epsilon$, the function $f^\delta(.;v,.)$ denotes the average $\int_{\partial B_\delta(v)} f(.;v',.)\mu_\delta^v(dv')$. Now notice that we can decompose the difference $h_\delta(v) - \eta_\delta(v)$ into four components as follows:

$$h_\delta(v) - \eta_\delta(v) = G_{v;1} + G_{v;2} + G_{v;3} + G_{v;4}\,,$$

where

$$G_{v;1} = \int_{D\times[1,\infty)} p_D^\delta(s/2;v,w)W(dw,ds)\,, G_{v;2} = \int_{D\times(0,\delta^2]} p_D^\delta(s/2;v,w)W(dw,ds)\,,$$
$$G_{v;3} = -\int_{\mathbb{R}^2\times[\delta^2,1]} \overline{p}_D^\delta(s/2;v,w)W(dw,ds) \text{ and}$$
$$G_{v;4} = \int_{\mathbb{R}^2\times[\delta^2,1]} \big(p^\delta(s/2;v,w) - p(s/2;v,w)\big)W(dw,ds)\,.$$

We will show that the variance of each component is $O_{D,\epsilon}(1)$. Let us begin with $\mathrm{Var}(G_{v;1})$. Observe that

$$\begin{aligned}\mathrm{Var}(G_{v;1}) &= \int_{[1,\infty)}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} p_D(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds \\ &= \int_{[1,\infty)}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} p(s;v',v'')\mathscr{P}^D(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds\,, \qquad (2.5.1)\end{aligned}$$

where $\mathscr{P}^D(s;v',v'')$ is the probability that a (two dimensional) Brownian bridge of duration s remains in D. Since squared absolute norm of a standard Brownian motion at time t is distributed as an exponential variable with mean $2t$, a simple computation gives us

$$\mathscr{P}^D(s;v',v'') = O(1)(1-\mathrm{e}^{-O\left(\frac{(d_{\ell_2}(v',\partial D)+|v'-v''|)^2}{s}\right)})\,.$$

Plugging this into (2.5.1) we get

$$\begin{aligned}\mathrm{Var}(G_{v;1}) &= O(1)\int_{[1,\infty)} s^{-1}(1-\mathrm{e}^{-O\left(\frac{(d_{\ell_2}(v',\partial D)+|v'-v''|)^2}{s}\right)})ds \\ &= O(1)\int_{[1,\infty)} s^{-2}(d_{\ell_2}(v',\partial D)+|v'-v''|)^2 ds = O(\mathrm{diam}(D)^2)\,.\end{aligned}$$

Next is $\mathrm{Var}(G_{v;2})$ which can be evaluated as

$$\begin{aligned}\mathrm{Var}(G_{v;2}) &= \int_{(0,\delta^2]}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} p_D(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds \\ &\le \int_{(0,\delta^2]}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} p(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds \\ &= (2\pi)^{-2}\int_{(0,\delta^2]} s^{-1}\int_{[0,2\pi]} \mathrm{e}^{-\frac{\delta^2(1-\cos\theta)}{s}}d\theta ds \\ &= O(1)\int_{(0,\delta^2]} \mathrm{e}^{-\frac{\Omega(\delta^2)}{s}}s^{-1}ds = O(1)\,.\end{aligned}$$

For $\mathrm{Var}(G_{v;3})$ we start with an upper bound:

$$\begin{aligned}\mathrm{Var}(G_{v;3}) &\le \int_{[\delta^2,1]}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} \overline{p}_D(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds \\ &= O(1)\int_{[\delta^2,1]}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} p(s;v',v'')\overline{\mathscr{P}}^D(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds\,, \end{aligned} \tag{2.5.2}$$

where $\overline{\mathscr{P}}^D(s;v',v'')$ is the probability that a Brownian bridge of duration s hits ∂D. Like $\mathscr{P}^D(s;v',v'')$, we can use tail probabilities of appropriate exponentials to bound this as

$$\overline{\mathscr{P}}^D(s;v',v'') = O(1)\mathrm{e}^{-\Omega\left(\frac{(d_{\ell_2}(v',\partial D)-|v'-v''|)^{+2}}{s}\right)}\,.$$

(2.5.2) and the previous bound together imply

$$\mathrm{Var}(G_{v;3}) \;\leq\; O(1)\int_{[\delta^2,1]} \mathrm{e}^{-\Omega\left(\frac{(d_{\ell_2}(v',\partial D)-|v'-v''|)^{+2}}{s}\right)} ds = O_\epsilon(1)\,,$$

We are only left with $\mathrm{Var}(G_{v;4})$ now. Notice that

$$\begin{aligned}
\mathrm{Var}(G_{v;4}) \;&=\; \int_{[\delta^2,1]} p(s;v,v)ds + \int_{[\delta^2,1]}\int_{\partial B_\delta(v)\times\partial B_\delta(v)} p(s;v',v'')\mu_\delta^v(dv')\mu_\delta^v(dv'')ds \\
&\quad -2\int_{[\delta^2,1]}\int_{\partial B_\delta(v)} p(s;v',v)\mu_\delta^v(dv')ds \\
&=\; I_1 + I_2 - 2I_3\,.
\end{aligned}$$

Since $p(s;v',v'') \leq (2\pi)^{-1}s^{-1}$, it follows that I_1 and I_2 are bounded above by $(2\pi)^{-1}\int_{[\delta^2,1]}\frac{ds}{s}$. On the other hand.

$$I_3 = (2\pi)^{-1}\int_{[\delta^2,1]} \mathrm{e}^{-\frac{\delta^2}{s}} s^{-1}ds \geq (2\pi)^{-1}\int_{[\delta^2,1]}(1-\delta^2 s^{-1})s^{-1}ds\,.$$

Putting all these estimates together we get $\mathrm{Var}(G_{v;4}) = O(1)$. Thus $\mathrm{Var}(h_\delta(v)-\eta_\delta(v)) = O_{D,\epsilon}(1)$ for all $v \in D_\epsilon$. In addition we claim that

$$\mathrm{Var}\Big((h_\delta(v)-\eta_\delta(v)) - (h_\delta(w)-\eta_\delta(w))\Big) = O\Big(\frac{|v-w|}{\delta}\Big) \tag{2.5.3}$$

for all $v, w \in D_\epsilon$ such that $|v-w| \leq \delta$. Thus, by Dudley's entropy bound on the supremum of a Gaussian process (see, e.g., [1, Theorem 4.1]) and Gaussian concentration inequality (see e.g., [54, Equation (7.4), Theorem 7.1]) we deduce that

$$\mathbb{P}\Big(\max_{v\in V}(h_\delta(v)-\eta_\delta(v)) > C_3\sqrt{\log\delta^{-1}} + x\Big) = \mathrm{e}^{-\Omega_{D,\epsilon}(x^2)}\,, \tag{2.5.4}$$

for all $x \geq 0$. We will verify (2.5.3) shortly, but before that let us show how (2.5.4) leads to a proof of Theorem 2.1.1. To this end define, for $v, w \in V$, $D_{\eta,\gamma,\delta}(v,w) = \inf_P \int_P \mathrm{e}^{\gamma\eta_\delta(z)}|dz|$

where P ranges over all piecewise smooth paths in V connecting v and w. Also denote by $D^{\text{straight}}_{h,\gamma,\delta}(v,w)$ the weight of the straight line joining v and w when the underlying field is h_δ. The following is straightforward:

$$D_{\gamma,\delta}(v,w) \le \mathrm{e}^{O(\gamma\sqrt{\log\delta^{-1}})} D_{V,\eta,\gamma,\delta}(v,w)\mathbf{1}_{E_V} + D^{\text{straight}}_{h,\gamma,\delta}(v,w)1_{E_V^c}\,, \tag{2.5.5}$$

where $M_V = \max_{v\in V}(h_\delta(v) - \eta_\delta(v))$ and E_V is the event $\{M_V \le (C_3+1)\sqrt{\log\delta^{-1}}\}$. Let n be the unique positive integer satisfying $2^{-n-1} < \delta \le 2^{-n}$. Notice that for any point $v \in V$ and any boundary segment V^∂ of V, there exists a sequence of rectangles $R_{1,v}, R_{2,v}, \cdots R_{K,v}$ with sides parallel to the coordinate axes such that:

(a) The shorter boundary of $R_{1,v}$ has length at most 2^{-n} and has v as one of its endpoints.

(b) $R_{K,v}$ intersects V^∂.

(c) The ratio of longer to shorter dimension of each $R_{i,v}$ is Γ.

(d) $R_{i,v} \subseteq V$ for all $i \le K-2$.

(e) $R_{i,v} \subseteq R_{i+1,v}$ for all $i \le K-2$. Furthermore one of the shorter boundary segments of $R_{i+1,v}$ is same as one of the longer boundary segments of $R_{i,v}$ for all such i.

(f) $R_{K-1,v} \cap V$ (also $R_{K,v} \cap V$) is a non-degenerate rectangle whose one boundary segment is same as one of the shorter boundary segments of $R_{K-1,v}$ (respectively $R_{K,v}$).

(g) $R_{K-1,v} \cap V \subseteq R_{K,v} \cap V$ and one of the shorter boundary segments of $R_{K,v}$ is contained in one of the longer boundary segments of $R_{K-1,v}$.

(h) $K = O(n)$.

Properties (a), (b), (d), (e), (f) and (g) imply that given any choice of a crossing $P_{i,v}$ through $R_{i,v}$, the union of $P^\star_v, P_{1,v}, \cdots, P_{K,v}$ contains a path between v and V^∂ that is contained in V (see Figure 2.7). Here $P^\star_v$ is the shorter boundary segment of $R_{1,v}$ containing v. Also notice that if we connect each of v and w to each of the four boundary segments of V by some paths in V, then there must exist a path from v and another path from w that intersect each other and hence contains a path between v and w (see Figure 2.8). Therefore we can build

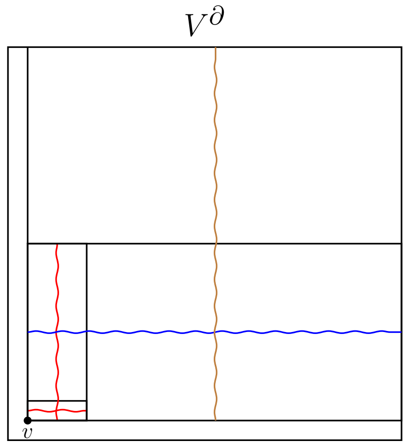

Figure 2.7: **The rectangles $R_{1,v}, R_{2,v}, \cdots, R_{K,v}$.** In this case $K = 4$. The portions of $R_{3,v}$ and $R_{4,v}$ lying outside V have been omitted. The red curved lines indicate $P_{1,v}$ and $P_{2,v}$. The blue and brown curved lines respectively indicate the portions of $P_{3,v}$ and $P_{4,v}$ that lie within V.

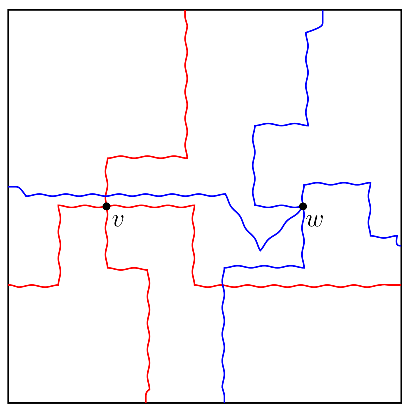

Figure 2.8: **Paths from v and w to each of the four boundary segments of V.**

an efficient path between v and w by choosing $P_{i,v}$ (or $P_{i,w}$) to be the crossing constructed by $\mathcal{A}_{n-\lceil r \rceil}$ where 2^{-r} is the shorter dimension of $R_{i,v}$ (respectively $R_{i,w}$). Recall again from Subsection 2.3.1 that we only use the fields $\{\eta_{2^{-\ell}}^{2^{-\lceil r \rceil}}\}_{\lceil r \rceil < \ell \leq n}$ to construct a crossing at scale n using $\mathcal{A}_{n-\lceil r \rceil}$. Thus by (2.4.15), independent increment and the scaling property of η and (2.2.3), we can bound the expected weight of $P_{i,v}$ (or $P_{i,w}$) computed with respect to η_δ by the following:

$$\frac{O(\Gamma) 2^{-(n-\lceil r \rceil)\Omega(\frac{\gamma^{4/3}}{\log \gamma^{-1}})}}{2^{\lceil r \rceil}} 2^{O(\gamma^2)\lceil r \rceil} = O(\Gamma) \delta^{\Omega(\frac{\gamma^{4/3}}{\log \gamma^{-1}})} 2^{-\Omega(\lceil r \rceil)}.$$

Consequently

$$\mathbb{E}D_{\eta,\gamma,\delta}(v,w) = O(\Gamma)\delta^{\Omega(\frac{\gamma^{4/3}}{\log\gamma^{-1}})}\,. \tag{2.5.6}$$

As to $D^{\text{straight}}_{h,\gamma,\delta}(v,w)1_{E_v^c}$, we can use (2.5.4) and Cauchy-Schwarz inequality to obtain

$$\mathbb{E}D_{\eta,\gamma,\delta}(v,w) \le \mathrm{e}^{-\Omega_{D,\epsilon}(\log\delta^{-1})}\sqrt{\mathbb{E}\big(D^{\text{straight}}_{h,\gamma,\delta}(v,w)\big)^2}\,. \tag{2.5.7}$$

Since $\mathrm{Var}(h_\delta(v)-\eta_\delta(v)) = O_{D,\epsilon}(1)$ and $\mathrm{Var}(\eta_\delta(v)) = O(\log\delta^{-1})$, we get from Fubini

$$\mathbb{E}\big(D^{\text{straight}}_{h,\gamma,\delta}(v,w)\big)^2 = \int_{[0,1]^2}\mathrm{e}^{\gamma(h_\delta(x+0.5\iota)+h_\delta(y+0.5\iota))}dxdy \le O_{D,\epsilon}(1)\delta^{-O(\gamma^2)}\,. \tag{2.5.8}$$

(2.5.5), (2.5.6), (2.5.7) and (2.5.8) together imply

$$\mathbb{E}D_{\gamma,\delta}(v,w) = O_{\gamma,D,\epsilon}(1)\delta^{\Omega(\frac{\gamma^{4/3}}{\log\gamma^{-1}})}\,,$$

which proves Theorem 2.1.1.

It only remains to verify (2.5.3). Since

$$(h_\delta(v)-\eta_\delta(v)) - (h_\delta(w)-\eta_\delta(w)) = (h_\delta(v)-h_\delta(w)) - (\eta_\delta(v)-\eta_\delta(w))\,,$$

it suffices to prove similar bounds for each of $\mathrm{Var}(h_\delta(v)-h_\delta(w))$ and $\mathrm{Var}(\eta_\delta(v)-\eta_\delta(w))$. The latter can be obtained from Lemma 2.2.1. The bound on $\mathrm{Var}(h_\delta(v)-h_\delta(w))$ was derived (in a more general set-up) in the proof of [49, Proposition 2.1].

2.6 Adapting to discrete GFF

Let $N = 2^n$, $V_N^\Gamma \equiv ([0,\Gamma N-1]\times[0,N-1])\cap\mathbb{Z}^2$ and $V_N^{\Gamma,\epsilon} = ([-\lfloor\epsilon\Gamma N\rfloor, \Gamma N+\lfloor\epsilon\Gamma N\rfloor-1]\times[-\lfloor\epsilon N\rfloor, N+\lfloor\epsilon N\rfloor+1])\cap\mathbb{Z}^2$. Consider a discrete Gaussian free field $\{\eta_{\gamma,N}(v) : v\in V_N^{\Gamma,\epsilon}\}$ on $V_N^{\Gamma,\epsilon}$ with Dirichlet boundary condition. By interpolation we can extend $\eta_{\gamma,N}$ to a continuous field on the rectangle $[-\epsilon\Gamma N,(1+\epsilon)\Gamma N]\times[-\epsilon N,(1+\epsilon)N]$. After appropriate scaling we then get a continuous Gaussian field $\tilde{\eta}_{\gamma,N}$ on the domain $V^{\Gamma,\epsilon}=(-\epsilon\Gamma,(1+\epsilon)\Gamma)\times(-\epsilon,(1+\epsilon))$. It is clear that we need to find a suitable decomposition for the covariance kernel of $\eta_{\gamma,N}$ in order to get a decomposition of $\tilde{\eta}_{\gamma,N}$ similar to the white noise decomposition of η_δ. The covariance between $\eta_{\gamma,N}(v)$ and $\eta_{\gamma,N}(w)$ is given by the simple random walk Green function $G_{V_N^{\Gamma,\epsilon}}(v,w)$. There is a simple representation of $G_{V_N^{\Gamma,\epsilon}}(\cdot,\cdot)$ as a sum of simple random walk probabilities. However here we represent it in terms of *lazy* simple random walk probabilities for reasons that would become clear shortly. To this end we write

$$G_{V_N^{\Gamma,\epsilon}}(v,w)=\frac{1}{2}\sum_{k=0}^{\infty}\mathbb{P}^v(S_k=w,\tau_{\gamma,\epsilon}>k)\,,$$

where $\{S_k\}_{k\geq 0}$ is a lazy simple random walk on $\mathbb{Z}^2$ i.e. it stays put for each step with probability $\frac{1}{2}$ and jumps to each of its four neighbors with probability $\frac{1}{8}$, $\mathbb{P}^v$ is the measure corresponding to the random walk starting from v and $\tau_{\gamma,\epsilon}$ is the first time the random walk hits $\partial V_N^{\Gamma,\epsilon}$. Emulating our approach to the approximation of circle average process with η_δ, we replace $\tau_{\gamma,\epsilon}$ in the above representation with the order of it expectation i.e. N^2 (on V_N^{Γ}, of course) and obtain a new kernel:

$$K_N(v,w)=\frac{1}{2}\sum_{k=1}^{N^2-1}\mathbb{P}^v(S_k=w)\,.$$

Notice that, thanks to the laziness of S_k, each matrix $(\mathbb{P}^v(S_t=w))_{v,w\in V_N^{\Gamma,\epsilon}}$ is non-negative definite. The similarity of this expression with the integral representation of the covariance between $\eta_\delta(v)$ and $\eta_\delta(w)$ prompts the following decomposition of $K_N(\cdot,\cdot)$:

$$K_N(v,w)=\sum_{k'\in[n]}\frac{1}{2}\sum_{4^{k'-1}\leq k<4^{k'}}\mathbb{P}^v(S_k=w)=\sum_{k'\in[n]}K_{N,k'}(v,w)\,.$$

Hence we can "approximate" $\tilde{\eta}_{\gamma,N}$ with a sum of independent, stationary fields $\Delta\tilde{\eta}_{N,k'}$ on V_N where the covariance kernel of $\Delta\tilde{\eta}_{N,k'}$ is "given" by $K_{N,k'}$. Denote $\tilde{\eta}_{N,k'} = \sum_{k''\in[k']} \Delta\tilde{\eta}_{N,k''}$. It is immediate that the sequence of fields $\tilde{\eta}_{N,k'}$'s are stationary and have independent increments. Using standard results on discrete planar random walk and local central limit theorem estimates (see, e.g., Chapters 2 and 4 in [53]) one can also prove the following properties:

(a) $\mathrm{Var}(\Delta\tilde{\eta}_{N,k'}(v)) = O(1)$ and $\mathrm{Var}(\Delta\tilde{\eta}_{N,k'}(v) - \Delta\tilde{\eta}_{N,k'}(w)) = 4^{n-k'}O(|v-w|^2)$ for all v, w. Compare this to Lemma 2.2.1.

(b) For any straight line segment $\mathcal{L}$ of length at most $\Gamma 2^{k'-n}$, $\mathrm{Var}(\int_{\mathcal{L}} \Delta\tilde{\eta}_{N,k'}(z)|dz|) = 4^{k'-n}|\mathcal{L}|$. Here $|\mathcal{L}|$ is the length of $\mathcal{L}$. Furthermore if $v \in \mathbb{R}^2$ is orthogonal to $\mathcal{L}$, then

$$\mathrm{Var}\Big(\int_{\mathcal{L}} \Delta\tilde{\eta}_{N,k'}(z)|dz| - \int_{\mathcal{L}+v} \Delta\tilde{\eta}_{N,k'}(z)|dz|\Big) = 4^{k'-n}\Theta(|\mathcal{L}|)\,,$$

whenever $|\mathcal{L}| \geq 2^{k'-n}$ and $|v| = \Theta(1)$. Compare this to a similar estimate derived in the proof of Lemma 2.2.2 and also to the property (b) of the field ζ discussed in the introduction.

We can now use strategies similar to those used for constructing cross_n. Since the fields $\tilde{\eta}_{N,k'}$'s do not have rotational invariance, we will actually construct crossings in all possible directions at any given scale (through appropriately scaled rectangles) and consider the *maximum expected weight* of these crossings. In view of properties (a) and (b), we can then obtain recursion relations like (2.4.3) and (2.4.13) on the maximum expected weight without any significant change in the analysis. Next we build a (lattice) crossing $P_n^\star$ of $\frac{1}{N}V_N^\Gamma$ from the crossing P_n which we constructed for V^Γ so that

$$\mathbb{E}\Big(\sum_{v\in P_n^\star} \mathrm{e}^{\gamma\tilde{\eta}_{N,n}(v)}\Big) = O_{\gamma,\epsilon}(N^{1-\Omega(\gamma^{4/3}/\log\gamma^{-1})})\,.$$

We can do this by following a procedure detailed in the proof of Lemma 3.2.4 in Chapter 3. Indeed we have an analogous upper bound on $\mathrm{Var}(\tilde{\eta}_{N,n}(v) - \tilde{\eta}_{N,n}(w))$ for adjacent v, w:

$$\max_{v,w \in V_N^\Gamma, |v-w|=1} \mathrm{Var}(\tilde{\eta}_{N,n}(v) - \tilde{\eta}_{N,n}(w)) = 1\,,$$

which makes all the arguments employed in the proof of Lemma 3.2.4 work smoothly. The approximation of $\tilde{\eta}_{\gamma,N}$ with $\tilde{\eta}_{N,n}$ can be tackled in a similar way as we tackled the approximation of h_δ with $\eta_{2^{-\lfloor \log_2 \delta \rfloor}}$ in Section 2.5. Once we have bounds on expected weights of crossings between shorter boundaries of rectangles at all scales, we can use such crossings to build an efficient path connecting any two given points in V_N (we discussed this idea in Section 2.5 in greater detail). This leads to a proof of Theorem 2.1.2.

CHAPTER 3

LIOUVILLE GRAPH DISTANCE

3.1 Upper bound on the expected distance

There is also a similar upper bound on the expected Louville graph distance.

Theorem 3.1.1. *There exists* $C_{\gamma,D,\epsilon} > 0$ *(depending on* (γ, ϵ, D)*) and positive (small) absolute constants* c^*, γ_0 *such that for all* $\gamma \leq \gamma_0$*, we have*

$$\max_{v,w \in V} \mathbb{E}\tilde{D}_{\gamma,\delta}(v,w) \leq C_{\gamma,D,\epsilon}\delta^{-1+c^*\frac{\gamma^{4/3}}{\log \gamma^{-1}}}\,.$$

3.1.1 Liouville graph distance and Watabiki's prediction

There are also reasonable interpretations of Watabiki's prediction in terms of Liouville graph distance. The scaling exponent $\chi = -\lim_{\delta\to 0}\frac{\mathbb{E}\tilde{D}_{\gamma,\delta}(v,w)}{\log\delta}$ is expected to exist and is expected to be given by (here we take v, w as two fixed generic points in the domain)

$$\chi = \frac{2}{d_H(\gamma)} = 1 - O_{\gamma\to 0}(\gamma^2)\,, \tag{3.1.1}$$

where in the last step we plugged in (2.1.1). A similar interpretation to (3.1.1) appeared in [48, Conjecture 1.14] though the graph structure considered in [48] is based on the peanosphere construction of LQG and so far we see no mathematical connection to Liouville graph distance considered in this thesis. Note that there is a difference of factor of 2, which is due to the fact that for the graph defined in [48] on average each ball contains LQG measure about ϵ (in their notation) while in our construction each ball contains LQG measure δ^2. We see that Theorem 3.1.1 contradicts (3.1.1).

3.2 Proof of Theorem 3.1.1

We will follow the notation convention laid down in the previous chapter as well as need some new ones. To this end let $\mathbb{D}$ is the open, unit disk centered at the origin. For any closed ball $B \equiv c_B + r\overline{\mathbb{D}}$, we let $\widetilde{B}$ denote the open ball $c_B + 4r\mathbb{D}$. If $h^\star$ is a GFF with Dirichlet boundary condition on some bounded domain $\mathcal{D}$ with smooth boundary, then for any $\delta > 0$ and $v \in \mathcal{D}_\delta$, we denote the average of $h^\star$ along the circle $v + \delta\partial\mathbb{D}$ as $h^\star_\delta(v)$.

Now consider a GFF $h^{\mathfrak{D}}$ on $\mathfrak{D}$ with Dirichlet boundary condition. If $\tilde{B} \subseteq \mathfrak{D}$, then *Markov property* (see [74, Section 2.6] or [9, Theorem 1.17]) of GFF states that $h^{\mathfrak{D}} = h^{\mathfrak{D},\widetilde{B}} + \varphi^{\mathfrak{D},B}$, where

- $h^{\mathfrak{D},\widetilde{B}}$ is a GFF on $\widetilde{B}$ with Dirichlet boundary condition ($= 0$ outside $\widetilde{B}$).
- $\varphi^{\mathfrak{D},B}$ is harmonic on $\tilde{B}$.
- $h^{\mathfrak{D},\widetilde{B}}, \varphi^{\mathfrak{D},B}$ are independent.

This decomposition has a useful consequence for us as follows. Since $\varphi^{\mathfrak{D},B}$ is harmonic on $\widetilde{B}$, we get

$$h^{\mathfrak{D}}_\delta(v) = h^{\mathfrak{D},\widetilde{B}}_\delta(v) + \varphi^{\mathfrak{D},B}(v) \tag{3.2.1}$$

for all $v \in \overline{B}^{2*} = c_B + 2r\overline{\mathbb{D}}$ and $\delta \in (0, r]$. The process $\{h^{\mathfrak{D},\widetilde{B}}_\delta(v) : v \in \overline{B}^{2*}, 0 < \delta \leq r\}$ is independent with $\{\varphi^{\mathfrak{D},\widetilde{B}}(v) : v \in \overline{B}^{2*}\}$ and also with $h^{\mathfrak{D}}_{\delta'}(w)$ for $w \in \mathfrak{D} \setminus \widetilde{B}, \delta' < d_{\ell_2}(w, \widetilde{B})$. The following lemma shows that the field $\varphi^{\mathfrak{D},B}$ is smooth on $\overline{B}^{2*}$.

Lemma 3.2.1. *Let $B \equiv c_B + r\mathbb{D} \subseteq \mathfrak{D}$ be a closed ball such that $\widetilde{B} \subseteq \mathfrak{D}$. Then we have for all $v, w \in \overline{B}^{2*}$*

$$\mathrm{Var}(\varphi^{\mathfrak{D},B}(v) - \varphi^{\mathfrak{D},B}(w)) = O\Big(\frac{|v-w|}{r}\Big)\,.$$

Also,

$$\sup_{v \in \overline{B}^{2*}} \mathrm{Var}(h^{\mathfrak{D}}_r(c_B) - \varphi^{\mathfrak{D},B}(v)) = O(1)\,.$$

Proof. Since $h_r^{\widetilde{B}}$ and $\varphi^{\mathfrak{D},B}$ are independent, we get from (3.2.1)

$$\mathrm{Var}(\varphi^{\mathfrak{D},B}(v) - \varphi^{\mathfrak{D},B}(w)) \leq \mathrm{Var}(h_r^{\mathfrak{D}}(v) - h_r^{\mathfrak{D}}(w))\,,$$

for all $v, w \in \overline{B}^{2*}$.But we know (see the proof of [49, Proposition 2.1])

$$\mathrm{Var}(h_r^{\mathfrak{D}}(v) - h_r^{\mathfrak{D}}(w)) = O\Big(\frac{|v-w|}{r}\Big)\,,$$

which gives the required bound on $\mathrm{Var}(\varphi^{\mathfrak{D},B}(v) - \varphi^{\mathfrak{D},B}(w))$. For the second part, notice that

$$\mathrm{Var}(h_r^{\mathfrak{D}}(c_B) - \varphi^{\mathfrak{D},B}(c_B)) = \mathrm{Var}(h_r^{\mathfrak{D},\widetilde{B}}(c_B))\,.$$

Thus it suffices to prove $\mathrm{Var}(h_r^{\mathfrak{D},\widetilde{B}}(c_B)) = O(1)$ in view of the bound on $\mathrm{Var}(\varphi^{\mathfrak{D},B}(v) - \varphi^{\mathfrak{D},B}(w))$. But $h_r^{\mathfrak{D},\widetilde{B}}(c_B)$ is identically distributed as $h_{0.25}^{\mathbb{D}}(0)$ by the scale and translation invariance of GFF and hence $\mathrm{Var}(h_r^{\mathfrak{D},\widetilde{B}}(c_B))$ is a finite constant (see the discussions in [74, Section 2.1] and also [9, Theorem 1.9]). $\square$

Now consider a Radon measure μ on $\mathfrak{D}$ and some $\delta > 0$. We call a closed Euclidean ball $B \subseteq \mathfrak{D}$ with a rational center as a (μ,δ)-*ball* if $\mu(B) \leq \delta^2$. For any compact $A \subseteq \mathfrak{D}$, let $N(\mu,\delta,A)$ denote the minimum number of (μ,δ)-balls required to cover A. Our next proposition provides a crude upper bound on the *second moment* of $N(M_\gamma^{\mathbb{D}},\delta,A)$ (see (1.1.2) for the definition of $M_\gamma^{\mathbb{D}}$) when A is a segment inside $\mathbb{D}$. We remark that the KPZ relation proved in [43] gives the sharp exponent on the first moment of $N(M_\gamma^{\mathbb{D}},\delta,A)$.

Proposition 3.2.2. *Let $\mathcal{H}$ denote the straight line segment joining -0.25 and 0.25.For any $\delta \in (0,1)$, we can find a collection of $(M_\gamma^{\mathbb{D}},\delta)$-balls $\mathscr{S}(M_\gamma^{\mathbb{D}},\delta,\mathcal{H})$ such that*

(a) Balls in $\mathscr{S}(M_\gamma^{\mathbb{D}},\delta,\mathcal{H})$ cover $\mathcal{H}$.

(b) All the balls in $\mathscr{S}(M_\gamma^{\mathbb{D}},\delta,\mathcal{H})$ are contained in $0.25\overline{\mathbb{D}}$.

(c) For some C_4,

$$\mathbb{E}(|\mathscr{S}(M_\gamma^{\mathbb{D}},\delta,\mathcal{H})|^2) = O_\gamma(\delta^{-2-C_4\gamma}),$$

Proof. For each $k \in \mathbb{N}$, let $\mathscr{B}_k$ denote the collection of all (closed) balls of radius 2^{-k-1} whose centers lie in the set $\{-\frac{1}{4}+2^{-k-1}, -\frac{1}{4}+32^{-k-1}, \cdots, \frac{1}{4}-2^{-k-1}\}$. The balls in $\mathscr{B}_k$ are nested in a natural way. In particular any ball B in $\mathscr{B}_k$ has a unique *parent* $B^{k'}$ in $\mathscr{B}_{k'}$ (where $k' \leq k$) such that $B \subseteq B^{k'}$. Let $\mathscr{B}(M_\gamma^{\mathbb{D}}, k, \delta)$ denote the collection of balls in $\mathscr{B}_k$ with $M_\gamma^{\mathbb{D}}$ volume $> \delta^2$. We include a $(M_\gamma^{\mathbb{D}}, \delta)$-ball $B \in \mathscr{B}_k$ in $\mathscr{S}(M_\gamma^{\mathbb{D}}, \delta, \mathcal{H})$ if the $M_\gamma^{\mathbb{D}}$ volume of the *most recent* parent of B is bigger than δ^2. Since the measure $M_\gamma^{\mathbb{D}}$ is a.s. is finite and has no atoms (see [43] and [9, Theorem 2.1]), it follows that $\mathscr{S}(M_\gamma^{\mathbb{D}}, \delta, \mathcal{H})$ satisfies condition (a) (and obviously (b)). It also follows from the construction that

$$|\mathscr{S}(M_\gamma^{\mathbb{D}},\delta,\mathcal{H})| \leq 2\delta^{-1-C_4'\gamma} + \sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} |\mathscr{B}(M_\gamma^{\mathbb{D}},k,\delta)|, \tag{3.2.2}$$

where $C_4' > 1$ is some fixed constant to be specified later. Observing that $|\mathscr{B}(M_\gamma^{\mathbb{D}}, k, \delta)|$ is the total number of balls in $\mathscr{B}_k$ with $M_\gamma^{\mathbb{D}}$ volume $> \delta^2$ a naive bound can be obtained as

$$\begin{aligned}
&\Big(\sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} |\mathscr{B}(M_\gamma^{\mathbb{D}},k,\delta)|\Big)^2 \\
&\leq \sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} \sum_{B\in\mathscr{B}_k} 2\sum_{k'\leq k}\sum_{B'\in\mathscr{B}_{k'}} \mathbf{1}_{\{M_\gamma^{\mathbb{D}}(B)>\delta^2, M_\gamma^{\mathbb{D}}(B')>\delta^2\}} \\
&\leq \sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} \sum_{B\in\mathscr{B}_k} 2\sum_{k'\leq k}\sum_{B'\in\mathscr{B}_{k'}} \mathbf{1}_{\{M_\gamma^{\mathbb{D}}(B)>\delta^2\}} \\
&\leq \sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} 2^{k+1}\sum_{B\in\mathscr{B}_k} \mathbf{1}_{\{M_\gamma^{\mathbb{D}}(B)>\delta^2\}}.
\end{aligned} \tag{3.2.3}$$

Next we compute the probability that any given ball $B \equiv c_B + 2^{-k}\mathbb{D}$ in $\mathscr{B}_k$ has $M_\gamma^{\mathbb{D}}$ volume at least δ^2. Since $M_\gamma^{\mathbb{D}}$ (or $M_{B,k}^{\mathbb{D}}$) is the weak limit of measures $M_{\gamma,n}^{D}$'s (respectively $M_{B,k}^{\mathbb{D}}$'s) defined in (1.1.2), we have

$$M_\gamma^{\mathbb{D}}(B) \leq 4^{-k} 2^{-\frac{k\pi^{-1}\gamma^2}{2}} \mathrm{e}^{\gamma h^{\mathbb{D}}_{2^{-k}}(c_B)} \times \mathrm{e}^{\gamma M^{\mathbb{D}}_{B,k}} \times 4^k M^{\mathbb{D}}_{\gamma,B}(B)\,, \tag{3.2.4}$$

where $M_{B,k} = \max_{v \in \overline{B}^{2*}} (\varphi^{\mathbb{D},B}(v) - h^{\mathbb{D}}_{2^{-k}}(c_B))$ and $M^{\mathbb{D}}_{\gamma,B}$ is the LQG measure on $\widetilde{B}$ obtained from $h^{\mathbb{D},B}$. From the scale and translation invariance property of GFF it follows that $4^k M^{\mathbb{D}}_{\gamma,B}(B)$ is identically distributed as $M^{\mathbb{D}}_\gamma(\frac{1}{4}\mathbb{D})$. Using this observation and (3.2.4) we can write,

$$\begin{aligned}\mathbb{P}(M_\gamma^{\mathbb{D}}(B) > \delta^2) \ \ \leq \ \ & \mathbb{P}\big(h^{\mathbb{D}}_{2^{-k}}(c_B) \geq \frac{2}{3\gamma}\log(\delta 2^k)\big) + \mathbb{P}\big(M^{\mathbb{D}}_{B,k} \geq \frac{2}{3\gamma}\log(\delta 2^k)\big) \\ & + \mathbb{P}(M^{\mathbb{D}}_\gamma(\tfrac{1}{4}\overline{\mathbb{D}}) \geq 4^{k/3}\delta^{2/3})\,.\end{aligned}$$

Since $\mathrm{Var}(h^{\mathbb{D}}_{2^{-k}}(c_B)) = k\log 2 + C_B$ for $|C_B| = \Theta(1)$ and $\delta^{1+C_4'\gamma} > 2^{-k}$, the first term on the right hand side of the previous display can be bounded as

$$\mathbb{P}\big(\eta_{2^{-k}}(c_B) \geq \frac{2}{3\gamma}\log(\delta 2^k)\big) \leq \mathbb{P}\Big(Z \geq \frac{C_4' k \log 2}{3\sqrt{k\log 2 + C_{B,\gamma}}}\Big) \leq \mathrm{e}^{-C_4'^2 \Omega(k)\log 2} = 2^{-C_4'^2\Omega(k)}\,.$$

Here Z is a standard Gaussian variable. Thus we can choose C_4' big enough so that the bound above becomes $< 2^{-10k}$. From Lemma 3.2.1 we know that

$$\max_{v \in B} \mathrm{Var}(\varphi^{\mathbb{D},B}(v) - h^{\mathbb{D}}_{2^{-k}}(c_B)) = O(1) \text{ and } \mathrm{Var}(\varphi^{\mathbb{D},B}(v) - \varphi^{\mathbb{D},B}(w)) \leq O\big(\tfrac{|v-w|}{2^{-k}}\big)$$

for all $v, w \in B$. Hence, similar to the derivation of (2.5.4), by Dudley's entropy bound and Gaussian concentration inequality we get for all sufficiently large k

$$\mathbb{P}\big(M^{\mathbb{D}}_{B,k} \geq \frac{2}{3\gamma}\log(\delta 2^k)\big) \leq 2^{-10k}\,.$$

The only remaining term is $\mathbb{P}(M^{\mathbb{D}}_\gamma(\frac{1}{4}\overline{\mathbb{D}}) \geq 4^{k/3}\delta^{2/3})$. In order to bound this probability we will use the fact that $\mathbb{E}\big(M^{\mathbb{D}}_\gamma(\frac{1}{4}\overline{\mathbb{D}})\big)^4 < \infty$ (see [50] and also [66, Theorem 2.11 and

Theorem 5.5]). Hence by Chebychev's inequality

$$\mathbb{P}(M_\gamma^{\mathbb{D}}(\tfrac{1}{4}\mathbb{D}) = O_\gamma(\delta^{-8/3})2^{-8k/3}\,.$$

Plugging the last three estimates into the expression for the upper bound on $\mathbb{P}(M_\gamma(B) \geq \delta^2)$ we get

$$\mathbb{P}(M_\gamma^{\mathbb{D}}(B) > \delta^2) \leq O_\gamma(\delta^{-8/3})2^{-8k/3}\,.$$

Taking expectation on both sides in (3.2.3) and using the bound above one gets:

$$\begin{aligned}\mathbb{E}\Big(\sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} |\mathscr{B}(M_\gamma,k,\delta)|\Big)^2 &\leq \sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} 2^{2k+1}O_\gamma(\delta^{-8/3})2^{-8k/3}\\ = \;\; O_\gamma(\delta^{-8/3})\sum_{k>(1+C_4'\gamma)\log_2\delta^{-1}} 2^{-2k/3} &= O_\gamma(\delta^{-8/3})\delta^{2/3+8\gamma} = O_\gamma(\delta^{-2+8\gamma})\,.\end{aligned}$$

The lemma follows from this bound and (3.2.2) for $C_4 = \max(C_4', 8)$. □

The proof of Proposition 3.2.2 can be easily adapted to accommodate the following set-ups.

Corollary 3.2.3. *Let $S \subseteq V$ be a closed square of length 2^{-k} whose vertices lie in $2^{-k}\mathbb{Z}^2$. Then for any $\delta \in (0, 2^{-k})$ we have*

$$\mathbb{E}N(M_\gamma^D, \delta, S)^2 = O_{\gamma,D,\epsilon}((2^k\delta)^{-4-O(\gamma)}2^{kO(\gamma)})\,.$$

Now given a $\delta \in (0,1)$ and $v, w \in V$, we will construct a collection of (M_γ^D, δ) balls $\mathcal{S}(\delta, v, w)$ such that the union of these balls contains a path between v and w. Thus it would suffice to show

$$\mathbb{E}|\mathcal{S}(\delta, v, w)| = O_{\gamma,D,\epsilon}(1)\delta^{-1+\Omega(\frac{\gamma^{4/3}}{\log\gamma^{-1}})}$$

for proving Theorem 3.1.1. Before we describe the construction of $\mathcal{S}(\delta, v, w)$, we need to discuss a related construction which will be very useful. To this end define, for any fixed

$k > 4$, the set $\mathscr{D}_k$ as $2^{-(k-3)}\mathbb{Z}^2 \cap V$. We can treat $\mathscr{D}_k$ as a subgraph of the lattice $2^{-(k-3)}\mathbb{Z}^2$. The centers of the squares in $\mathscr{D}_k$ (i.e. the squares of side length $2^{-(k-3)}$ with vertices in $\mathscr{D}_k$) form another set $\mathscr{D}_k^\star \subseteq V$ which will be treated as the dual graph of $\mathscr{D}_k$. We can define a LFPP metric $D^\star_{\gamma,k}(\cdot,\cdot)$ on $\mathscr{D}_k^\star$ in a similar way as in (1.1.3) with $h_{2^{-k}}$ as the underlying field and $\gamma(1+C_4\gamma)/2$ as the inverse temperature parameter (see Proposition 3.2.2 for the definition of C_4). The next lemma is a consequence of our proof of Theorem 2.1.1

Lemma 3.2.4. *For all $\gamma > 0$ sufficiently small,*

$$\max_{u,u' \in \mathscr{D}_k^\star} \mathbb{E} D^\star_{\gamma,k}(u,u') = O_{\gamma,D,\epsilon}(1) 2^{k(1-\Omega(\gamma^{4/3}/\log \gamma^{-1}))}\,.$$

Proof. Let $V_k^\star$ denote the square $[2^{-(k-3)}, 1-2^{-(k-3)}]^2$ so that $\mathscr{D}_k^\star \subseteq V_k^\star$. Following the proof of Theorem 2.1.1 in the last chapter, we get a *fixed*, finite collection $\mathcal{P}_k(u,u')$ of piecewise smooth paths in $V_k^\star$ between $u, u' \in V_k^\star$ and a (randomly chosen) simple, piecewise smooth path $P_{k,\gamma}(u,u') \in \mathcal{P}_k$ such that

$$\mathbb{E}\Big(\int_{P_{k,\gamma}(u,u')} \mathrm{e}^{\gamma(1+C_4\gamma)h_{2^{-k}}(z)/2} |dz| \Big) = O_{\gamma,D,\epsilon}(2^{-k\Omega(\gamma^{4/3}/\log\gamma^{-1})})\,. \tag{3.2.5}$$

In order to create a lattice path (i.e. in $\mathscr{D}_k^\star$) between u and u' from $P_{k,\gamma}(u,u')$ we follow a simple procedure. Starting from the initial point $p_{k,\gamma;0}$ of $P_{k,\gamma}(u,u')$, wait until it exits the smallest square S_0 satisfying (a) $p_{k,\gamma;0} \in S_0$, (b) $d_{\ell_2}(p_{n,0}, \partial S_0) \geq 2^{-(k-3)}$ and (c) the vertices of S_0 are in $\mathscr{D}_k^\star$. Repeat the same procedure with the exit point of $P_{k,\gamma}(u,u')$ and continue until it reaches u'. At the end of this procedure we will get a sequence of squares $S_0, S_1, \cdots$, where each S_i has diameter at most $32^{-(k-3)}$ and the vertices of S_i's contain a lattice path $P^\star_{k,\gamma}(u,u')$ between u and u'. Now let us recall from Section 2.5 in Chapter 2 that

$$\max_{z,z'\in V_k^\star, |z-z'|\le 2^{-(k-3)}} \mathrm{Var}(h_{2^{-k}}(z) - h_{2^{-k}}(z')) = O(1)\,, \text{ and}$$

$$\max_{z\in V_k^\star} \mathrm{Var}(h_{2^{-k}}(z)) = O(k) + O_{D,\epsilon}(1)\,.$$

Then from the arguments involving the extreme values of Gaussian processes as used for (2.5.4), we can find C_5 such that

$$\mathbb{P}\Big(\max_{z,z'\in V_k^\star, |z-z'|\le 2^{-(k-3)}} (h_{2^{-k}}(z) - h_{2^{-k}}(z')) \ge C_5\sqrt{k} + x\Big) = \mathrm{e}^{-\Omega_{D,\epsilon}(x^2)}\,, \tag{3.2.6}$$

for all $x \ge 0$. Now define an event E_k as

$$E_k = \{\max_{z,z'\in V_k^\star, |z-z'|\le 2^{-(k-3)}} (h_{2^{-k}}(z) - h_{2^{-k}}(z')) \le (C_5+1)\sqrt{k}\}\,.$$

As the euclidean length of $P_{k,\gamma}(u,u')$ inside each S_i is $\Omega(2^{-k})$, from (3.2.5) it follows that

$$\begin{aligned}\mathbb{E}\Big(\big(\sum_{z\in P_{k,\gamma}^\star(u,u')} \mathrm{e}^{\gamma(1+C_4\gamma)h_{2^{-k}}(z)}\big)\mathbf{1}_{E_k}\Big) &= O(2^k)\mathrm{e}^{(C_5+1)\sqrt{k}} O_{\gamma,D,\epsilon}(2^{-k\Omega(\gamma^{4/3}/\log\gamma^{-1})}) \\ &= O_{\gamma,D,\epsilon}(2^{k(1-\Omega(\gamma^{4/3}/\log\gamma^{-1}))})\,.\end{aligned}$$

On the other hand, from (3.2.6) and Cauchy-Schwarz inequality (similar to (2.5.7) and (2.5.8)) we obtain

$$\mathbb{E}\Big(\big(\sum_{z\in P_k(u,u')} \mathrm{e}^{\gamma(1+C_4\gamma)h_{2^{-k}}(z)}\big)\mathbf{1}_{E_k^c}\Big) = O_{D,\epsilon}(2^k)2^{-k(\Omega_{D,\epsilon}(1)-O(\gamma^2))} = O_{D,\epsilon}\big(2^{k(1-\Omega_{D,\epsilon}(1))}\big)\,,$$

where $P_k(u,u')$ is the shortest path between u and u' in the graph $\mathscr{D}_k^\star$. Choosing $P_{k,\gamma}^\star(u,u')$ and $P_k(u,u')$ on E_k and E_k^c respectively as a lattice path between u and u', we get the desired bound on $\mathbb{E}D_{\gamma,k}^\star(u,u')$ from the previous two displays. □

We will call the path minimizing $D^\star_{\gamma,k}(u,u')$ as the (γ,k)-*geodesic* between u and u'. Now given v,w in V, we pick squares v_k and w_k in $\mathscr{D}_k$ that contain v and w respectively. There are several ways to do this and we follow an arbitrary but fixed convention. Define $\mathcal{S}^\star(k,v,w)$ as the collection of squares in $\mathscr{D}_k$ which correspond to the points in the (γ,k)-geodesic between $c([v]_k)$ and $c([w]_k)$ in $\mathbb{D}^\star_k$. Here $c([v]_k)$ and $c([w]_k)$ are the centers of squares $[v]_k$ and $[w]_k$ respectively. Thus $\mathcal{S}^\star(k,v,w)$ is actually a *chain of squares* connecting v and w (see Figure 3.1). An important observation is the following.

Observation 3.2.5. *The euclidean distance between the boundary of any square in $\mathcal{S}^\star(k,v,w)$ and $\mathscr{D}^\star_k$ is at least $2^{-(k-2)}$.*

Given $S \in \mathcal{S}^\star(k,v,w)$ that is not $[v]_k$ or $[w]_k$, divide each boundary segment of S into 16 segments (with disjoint interiors) of length $2^{-(k+1)}$. For any such segment T, let B_T denote the closed ball of radius $2^{-(k+2)}$ centered at the midpoint of T. Thus T is a diameter segment of B_T. Cover T with the minimum possible number of $(M^D_{\gamma,B_T}, \delta \mathrm{e}^{-\gamma h^D_{2^{-k}}(c(S))/2}\mathrm{e}^{-C_6\gamma\sqrt{k\log 2}})$-balls contained in B_T where M^D_{γ,B_T} is the LQG measure on $\widetilde{B}_T$ constructed from $h^{D,\widetilde{B}_T}$, $c(S)$ is the center of S and C_6 is an absolute constant to be specified later. Denote the collection of all such balls from all the segments of ∂S as $\mathcal{S}(S,\delta)$. If $S = [v]_k$ or $[w]_k$, we simply cover S with minimum possible number of (M^D_γ, δ)-balls and include them in $\mathcal{S}(S,\delta)$. Finally define

$$\mathcal{S}^{\star\star}(k,\delta,v,w) = \bigcup_{S\in\mathcal{S}^\star(k,v,w)} \mathcal{S}(S,\delta)\,.$$

It is clear that the union of balls in $\mathcal{S}^{\star\star}(k,v,w)$ contains a path between v and w. Figure 3.1 gives an illustration of this construction.

We will now describe the construction of $\mathcal{S}(\delta,v,w)$. By Lemma 3.2.1, the bounds on $\mathrm{Var}(h_{\delta^\star}(u) - h_{\delta^\star}(u'))$ and $\mathrm{Var}(h_{\delta^\star}(v))$, and tail estimates as used in (2.5.4) and (3.2.6), we get C_6 such that for all k sufficiently large (depending on D,ϵ)

(a) $\mathbb{P}(\min_{u\in V} h_{2^{-k}}(u) < -2C_6 k\log 2) \le 2^{-3k}$ and

(b) $\mathbb{P}(\max_S \max_B \max_{v\in\overline{B}^{2*}}(\varphi^{D,B}(v) - h^D_{2^{-k}}(c(S))) > 2C_6\sqrt{k\log 2}) \le 2^{-3k}$,

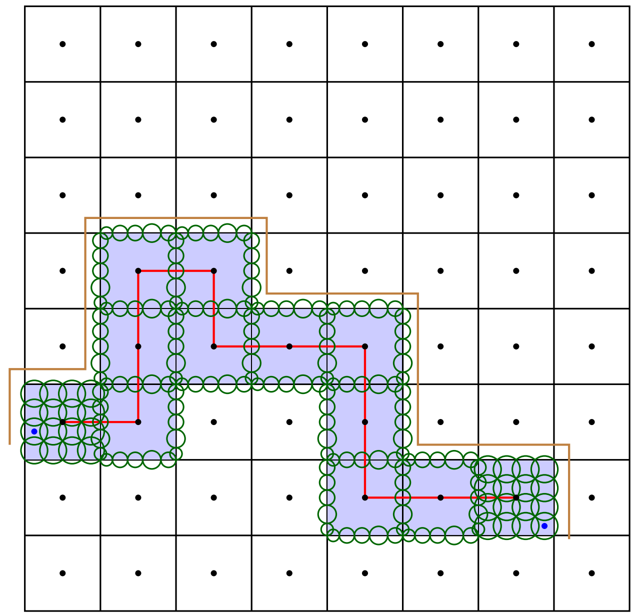

Figure 3.1: **An instance of $S^{\star\star}(k,\delta,v,w)$.** Squares in $S^{\star}(k,\delta,v,w)$ are filled with light blue color. The black dotted points lie in $\mathscr{D}_k^{\star}$. v (left) and w (right) are indicated as blue dotted points. The red (lattice) path is the LFPP path between $c([v]_k)$ and $c([w]_k)$. The green circles indicate the balls in $S^{\star\star}(k,\delta,v,w)$. Balls that lie parallel to the brown segments define a chain of ball connecting v and w.

where in (b), S ranges over all squares in $\mathscr{D}_k$ and B ranges over all balls of radius $2^{-(k+2)}$ around S that we described in the last paragraph. Choose δ' as the smallest number of the form 2^{-k} (where $k \in \mathbb{N}$) such that $\delta' \geq \delta^{1-2C_6\gamma}$. Now if

$$\min_{u\in V} h_{\delta'}(u) < -2C_6 \log {\delta'}^{-1}$$

or if

$$\max_S \max_B \max_{v\in \overline{B}^{2*}} (\varphi^{D,B}(v) - h^D_{2^{-k}}(c(S))) > 2C_6\sqrt{k\log 2}\,,$$

(we call the union of these two events as E_δ) simply cover the straight line segment joining v and w with the minimum possible number of (M_γ^D,δ)-balls. Otherwise (i.e. on E_δ^c) set $\mathcal{S}(\delta,v,w) = \mathcal{S}^{\star\star}(k',\delta,v,w)$ where $\delta' = 2^{-k'}$. Notice that $\mathcal{S}^{\star\star}(k,\delta,v,w)$ is a valid choice for

$\mathcal{S}(\delta, v, w)$ on E_δ^c as

$$M_\gamma^D(A) \le \mathrm{e}^{\gamma \max_S \max_B \max_{v \in \overline{B}^{2*}} (\varphi^{D,B}(v) - h^D_{2^{-k}}(c(S)))} \mathrm{e}^{\gamma h^D_{2^{-k}}(c(S))} M^D_{\gamma, B_T}(A)$$

for all B_T and all compact $A \subseteq B_T$ (this again follows from the definition of LQG measure as a weak limit).

Upper bound on $\mathbb{E}(|\mathcal{S}(\delta, v, w)|)$*:* Let us denote the σ-field generated by $\{h_{\delta'}(v) : v \in \mathscr{D}^\star_{k'}\}$ as $\mathfrak{F}_\delta$ and the event $\{\min_{v \in \mathscr{D}^\star_{k'}} h_{\delta'}(v) \ge -2C_6 \log \delta'^{-1}\}$ as F_δ. We then have

$$\begin{aligned}
\mathbb{E}(|\mathcal{S}(\delta, v, w)| \big| \mathfrak{F}_\delta) \le & \sum_{S \in \mathcal{S}^\star(k', v, w)} \sum_T \mathbb{E}\big(N^\star(M^D_{\gamma, B_T}, \delta \mathrm{e}^{-\gamma h^D_{\delta'}(c(S))/2} \mathrm{e}^{-C_6 \gamma \sqrt{k' \log 2}}, T) \big| \mathfrak{F}_\delta\big) \mathbf{1}_{F_\delta} \\
& + \mathbb{E}\big(N(M^D_\gamma, \delta, \overline{vw}) \mathbf{1}_{E_{\delta'}} | \mathfrak{F}_\delta\big) + \mathbb{E}\big(N(M^D_\gamma, \delta, [v]_{k'}) | \mathfrak{F}_\delta\big) \\
& + \mathbb{E}\big(N(M^D_\gamma, \delta, [w]_{k'}) | \mathfrak{F}_\delta\big),
\end{aligned}$$

where T ranges over all the 16×4 segments of ∂S and $N^\star(M^D_{\gamma, B_T}, r, T)$ is the minimum possible number of (M_{γ, B_T}, r)-balls contained in B_T that are required to cover T. By the Markov property of GFF (see the discussions around (3.2.1)) and Observation 3.2.5 it follows that M^D_{γ, B_T} is identically distributed as $M_\gamma^{\tilde{B}_T}$ and is independent with $\mathfrak{F}_\delta$. The latter is identically distributed as $\frac{\delta'^2}{16} M_\gamma^{\mathbb{D}}$ by scale and translation invariance property of GFF. Also on F_δ,

$$\delta \mathrm{e}^{-\gamma h^D_{\delta'}(c(S))/2} \mathrm{e}^{-C_6 \gamma \sqrt{k' \log 2}} < \delta \delta'^{-C_6 \gamma} \le \delta'^{(1 - 2C_6\gamma)^{-1} - C_6 \gamma} < \delta' .$$

We can then apply Proposition 3.2.2 to the first term in the right hand side of the previous display to get

$$\begin{aligned}
\mathbb{E}|\mathcal{S}(\delta, v, w)| \ \le \ & O_\gamma((\delta/\delta')^{-1 - C_4\gamma/2}) \mathrm{e}^{C_6 \gamma \sqrt{\log 2k'}(1 + C_4\gamma/2)} \mathbb{E}\Big(\sum_{S \in \mathcal{S}^\star(k', v, w)} \mathrm{e}^{\frac{\gamma(1 + C_4\gamma) h_{\delta'}(c(S))}{2}} \Big) \\
& + \mathbb{E} N(M^D_\gamma, \delta, \overline{vw}) \mathbf{1}_{E_{\delta'}} + \ \mathbb{E} N(M^D_\gamma, \delta, [v]_{k'}) + \mathbb{E} N(M^D_\gamma, \delta, [w]_{k'}) .
\end{aligned}$$

The first term on the right hand side equals, by Lemma 3.2.4

$$
\begin{aligned}
& O_\gamma(\delta^{-2C_6\gamma(1+C_4\gamma/2)}) O_{\gamma,D,\epsilon}(1) \delta'^{-1+\Omega(\gamma^{4/3}/\log\gamma^{-1})} \\
= \; & O_{\gamma,D,\epsilon}(\delta^{-2C_6\gamma(1+C_4\gamma/2)}) \delta^{2C_6\gamma} \delta^{-1+\Omega(\gamma^{4/3}/\log\gamma^{-1})} = O_{\gamma,D,\epsilon}(\delta^{-1+\Omega(\gamma^{4/3}/\log\gamma^{-1})}) \,.
\end{aligned}
$$

The second term is $O(1)$ as a consequence of bounds (a), (b), Corollary 3.2.3 and Cauchy-Schwarz inequality (similar to (2.5.7) and (2.5.8)). The last two terms are $O_{\gamma,D,\epsilon}(\delta^{-O(\gamma)})$ by Corollary 3.2.3. Adding up these four terms, we get the required bound on $\mathbb{E}|\mathcal{S}(\delta, v, w)|$.

CHAPTER 4

EFFECTIVE RESISTANCE METRIC

4.1 Upper and lower bounds on effective resistance

As hinted in Section 1.2 of Chapter 1, the key to analyzing the random walk defined by (1.2.2) (or equivalently (1.2.1)) is to estimate the effective resistances of the underlying network. The precise statement is the subject of:

Theorem 4.1.1. *Let us regard $B(N) := [-N,N]^2 \cap \mathbb{Z}^2$ as a conductance network where edge (u,v) has conductance $\mathrm{e}^{\gamma(\eta_u+\eta_v)}$. Let $R_{B(N)_\eta}(u,v)$ denote the effective resistance between u and v in network $B(N)$. For each $\gamma>0$ there are $C, C' \in (0,\infty)$ such that*

$$\max_{u,v\in B(N)} \mathbb{P}\Big(R_{B(N)_\eta}(u,v) \ge C\mathrm{e}^{Ct\sqrt{\log N}}\Big) \le C'\mathrm{e}^{-t^2}\,\log N \tag{4.1.1}$$

holds for each $N \ge 1$ and each $t \ge 0$. Moreover, for the corresponding network $\mathbb{Z}^2_\eta$ on all of $\mathbb{Z}^2$, there is a constant $\tilde{C} > 0$ such that

$$\limsup_{N\to\infty} \frac{\log R_{\mathbb{Z}^2_\eta}(0, B(N)^{\mathrm{c}})}{(\log N)^{1/2}(\log\log N)^{1/2}} \le \tilde{C}, \qquad \mathbb{P}\text{-a.s.} \tag{4.1.2}$$

and, for each $\gamma > 0$ and each $\delta > 0$, also

$$\liminf_{N\to\infty} \frac{\log R_{\mathbb{Z}^2_\eta}(0, B(N)^{\mathrm{c}})}{(\log N)^{1/2}/(\log\log N)^{1+\delta}} > 0, \qquad \mathbb{P}\text{-a.s.} \tag{4.1.3}$$

The effective resistance and further background on the theory of resistor networks are discussed in detail in Section 4.2. We note that, in light of monotonicity of $N \mapsto R_{\mathbb{Z}^2_\eta}(0, B(N)^{\mathrm{c}})$, the bounds in Theorem 4.1.1 readily imply recurrence of the random walk as well.

4.1.1 A word on proof strategy

Theorem 4.1.1 is proved by a novel combination of planar and electrostatic duality, Gaussian concentration inequality and the Russo-Seymour-Welsh theory, as we outline below.

Duality considerations for planar electric networks are quite classical. They invariably boil down to the simple fact that, in a planar network, every harmonic function comes hand-in-hand with its harmonic conjugate. An example of a duality statement, and a source of inspiration for us, is [57, Proposition 9.4], where it is shown that, for locally-finite planar networks with sufficient connectivity, the wired effective resistance across an edge (with the edge removed) is equal to the free effective conductance across the dual edge in the dual network (with the dual edge removed). However, the need to deal with more complex geometric settings steered us to develop a version of duality that is phrased in purely geometric terms. In particular, we use that, in planar networks with a bounded degree, cutsets can naturally be associated with paths and *vice versa*.

The starting point of our proofs is thus a representation of the effective resistance, resp., conductance as a variational minimum of the Dirichlet energy for *families* of paths, resp., cutsets. Although these generalize well-known upper bounds on these quantities (e.g., the Nash-Williams estimate), we prefer to think of them merely as extensions of the Parallel and Series Law. Indeed, the variational characterizations are obtained by replacing individual edges by equivalent collections of new edges, connected either in series or parallel depending on the context, and noting that the said upper bounds become sharp once we allow for optimization over all such replacements. We refer to Propositions 4.2.1 and 4.2.3 in Section 4.2 for more details.

Another useful fact that we rely on heavily is the symmetry $\eta \overset{\text{law}}{=} -\eta$ which implies that the joint laws of the conductances are those of the resistances. Using this we can *almost* argue that the law of the effective resistance between the left and right boundaries of a square centered at the origin is the same as the law of the effective conductance between the top and bottom boundaries. The rotation symmetry of η and the (electrostatic) duality between the effective conductance and resistance would then imply that the law of the effective resistance through a square is the same as that of its reciprocal value. Combined with a Gaussian concentration inequality (see [76, 18]), this would readily show that, for the square of side N, this effective resistance is typically $N^{o(1)}$.

However, some care is needed to make the "almost duality" argument work. In fact, we do not expect an exact duality of the kind valid for critical bond percolation on $\mathbb{Z}^2$ to hold in our case. Indeed, such a duality might for instance entail that the law of the conductances on a minimal cutset (separating, say, the opposite sides of a square) in the primal network is the same as the law of the resistances on the dual path "cutting through" this cutset. Although the GFFs on a graph and its dual are quite closely related (see, e.g., [14]), we do not see how this property can possibly be true. Notwithstanding, we are more than happy to work with just an approximate duality which, as it turns out, requires only a uniform bound on the *ratio* of resistances of neighboring edges. This ratio would be unmanageably too large if applied the duality argument to the network based on the GFF itself. For this reason, we invoke a decomposition of GFF (see Lemma 4.3.12) into a sum of two independent fields, one of which has small variance and the other is a highly smooth field. We then apply the approximate duality to the network derived from the smooth field, and we argue that the influence from the other field is small since it has small variance.

We have so far explained only how to estimate the effective resistances between the boundaries of a *square*. However, in order to prove our theorems, we need to estimate effective resistances between vertices, for which a crucial ingredient is an estimate of the effective resistances between the two short boundaries of a *rectangle*. Questions of this type fall into the framework of the Russo-Seymour-Welsh (RSW) theory. This is an important technique in planar statistical physics, initiated in [68, 72, 69] with the aim to prove uniform positivity of the probability of a crossing of a rectangle in critical Bernoulli percolation. Recently, the theory has been adapted to include FK percolation, see e.g. [38, 6, 41], and, in [77], also Voronoi percolation. In fact, the beautiful method in [77] is widely applicable to percolation problems satisfying the FKG inequality, mild symmetry assumptions, and weak correlation between well-separated regions. For example, in [40], this method was used to give a simpler proof of the result of [6], and in [39], a RSW theorem was proved for the crossing probability of level sets of the planar GFF.

Our RSW proof is *hugely* inspired by [77], with the novelty of incorporating the (resistance) metric rather than merely considering connectivity. We remark that in [29], a RSW result was established for the Liouville FPP metric, again inspired by [77]. It is fair to say that the RSW result in this chapter is less complicated than that in [29], for the reason that we have the approximate duality in our context which was not available in [29]. However, our RSW proof has its own subtlety since, for instance, we need to consider crossings by whole collections of paths simultaneously. The RSW proof is carried out in Section 4.4. Finally in Section 4.5, we use some of these estimates along with a decomposition of η from [15] to derive an asymptotic rate for the effective resistance between origin and $\partial B(N)$.

4.2 Generalized parallel and series law for effective resistances

As noted above, our estimates of effective resistance between various sets in $\mathbb{Z}^2$ rely crucially on a certain duality between the effective resistance and the effective conductance which will itself be based on the distributional equality of η with $-\eta$. The exposition of our proofs thus starts with general versions of these duality statements. These can be viewed as refinements of [57, Proposition 9.4] and are therefore of general interest as well.

4.2.1 Variational characterization of effective resistance

Let $\mathfrak{G}$ be a finite, unoriented, connected graph where each edge e is equipped with a resistance $r_e \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of positive reals. We will use $\mathfrak{G}$ to denote both the corresponding network as well as the underlying graph. Let $V(\mathfrak{G})$ and $E(\mathfrak{G})$ respectively denote the set of vertices and edges of $\mathfrak{G}$. We assume for simplicity that $\mathfrak{G}$ has no self-loops although we allow distinct vertices to be connected by multiple edges. For the purpose of counting we identify the two orientations of each edge; $E(\mathfrak{G})$ thus includes both orientations as one edge.

Two edges e and e' of $\mathfrak{G}$ are said to be *adjacent* to each other, denoted as $e \sim e'$, if they share at least one endpoint. Similarly a vertex v and an edge e are adjacent, denoted as $v \sim e$, if v is an endpoint of the edge e. A *path* P is a sequence of vertices of $\mathfrak{G}$ such that any two successive vertices are adjacent. We also use P to denote the subgraph of $\mathfrak{G}$ induced by the edge set of P.

For $u, v \in V(\mathfrak{G})$, a *flow* θ from u to v is an assignment of a number $\theta(x, y)$ to each *oriented* edge (x, y) such that $\theta(x, y) = -\theta(y, x)$ and $\sum_{y\colon y\sim x} \theta(x, y) = 0$ whenever $x \neq u, v$. The *value of the flow* θ is then the number $\sum_{y\colon y\sim u} \theta(u, y)$; a unit flow then has this value equal to one. With these notions in place, the effective resistance $R_{\mathfrak{G}}(u, v)$ between u and v is defined by

$$R_{\mathfrak{G}}(u, v) := \inf_{\theta} \sum_{e\in E(\mathfrak{G})} r_e\, \theta_{\mathrm{e}}^2\,, \tag{4.2.1}$$

where the infimum (which is achieved because $\mathfrak{G}$ is finite) is over all unit flows from u to v. Note that we sum over each edge $e \in E(\mathfrak{G})$ only once, taking advantage of the fact that θ_e appears in a square in this, and later expressions.

Recall that a multiset of elements of A is a set of pairs $\{(a, i)\colon i = 1, \dots, n_a\}$ for some $n_a \in \{0, 1, \dots\}$ for each $a \in A$. We have the following alternative characterization of $R_{\mathfrak{G}}(u, v)$:

Proposition 4.2.1. *Let $\mathfrak{P}_{u,v}$ denote the set of all multisets of simple paths from u to v. Then*

$$R_{\mathfrak{G}}(u, v) = \inf_{\mathcal{P}\in\mathfrak{P}_{u,v}} \; \inf_{\{r_{e,P}\colon e\in E(\mathfrak{G}),\, P\in\mathcal{P}\}\in\mathfrak{R}_{\mathcal{P}}} \biggl(\sum_{P\in\mathcal{P}} \frac{1}{\sum_{e\in P} r_{e,P}}\biggr)^{-1}, \tag{4.2.2}$$

where $\mathfrak{R}_{\mathcal{P}}$ is the set of all assignments $\{r_{e,P} : e \in E(\mathfrak{G}), P \in \mathcal{P}\} \in \mathbb{R}_+^{E(\mathfrak{G})\times\mathcal{P}}$ such that

$$\sum_{P\in\mathcal{P}} \frac{1}{r_{e,P}} \leq \frac{1}{r_e} \text{ for all } e \in E(\mathcal{G})\,. \tag{4.2.3}$$

The infima in (4.2.2) *are (jointly) achieved.*

Proof. Let $R^\star$ denote the right hand side of (4.2.2). We will first prove $R_{\text{eff}}(u, v) \leq R^\star$. Let thus $\mathcal{P} \in \mathfrak{P}_{u,v}$ and $\{r_{e,P} \colon e \in E, P \in \mathcal{P}\} \in \mathfrak{R}_{\mathcal{P}}$ subject to (4.2.3) be given. We will view each edge e in $\mathfrak{G}$ as a *parallel* of a collection of edges $\{e_P \colon P \in \mathcal{P}\}$ where the resistance on e_P is $r_{e,P}$ and, if the inequality in (4.2.3) for edge e is strict, a dummy edge $\tilde{e}$ with resistance $r_{\tilde{e}}$ such that $1/r_{\tilde{e}} = 1/r_e - \sum_{P\in\mathcal{P}} 1/r_{e,P}$. In this new network, $\mathcal{P}$ can be identified with a collection of *disjoint* paths where (by the series law) each path $P \in \mathcal{P}$ has total resistance $\sum_{e\in P} r_{e,P}$. The parallel law now guarantees

$$R_{\mathfrak{G}}(u, v) \leq \Big(\sum_{P\in\mathcal{P}} \frac{1}{\sum_{e\in P} r_{e,P}} \Big)^{-1}$$

which proves $R_{\mathfrak{G}}(u, v) \leq R^\star$ as desired.

Next, we turn to proving that $R_{\mathfrak{G}}(u, v) \geq R^\star$ and that the infima in (4.2.2) are achieved. To this end, let $\theta^\star$ be the flow that achieves the minimum in (4.2.1). In light of the inequality $R_{\mathfrak{G}}(u, v) \leq R^\star$ it suffices to construct a collection of paths $\mathcal{P}^\star \in \mathfrak{P}_{u,v}$ and an assignment of resistances $\{r^\star_{e,P} : e \in P, P \in \mathcal{P}^\star\}$ such that

$$\Big(\sum_{P\in\mathcal{P}^\star} \frac{1}{\sum_{e\in P} r^\star_{e,P}} \Big)^{-1} \leq \sum_{e\in E(\mathfrak{G})} r_e\, (\theta^\star_e)^2. \tag{4.2.4}$$

The argument proceeds by constructing inductively a sequence of flows $\theta^{(j)}$ from u to v (whose value decreases to zero) and a sequence of collections of paths $\mathcal{P}_j$ as follows. We initiate the induction by setting

$\theta^{(0)} := \theta^\star$ and $\mathcal{P}^{(0)} := \emptyset$ and employ the following iteration for $j \geq 1$:

- If $\theta_e^{(j-1)} = 0$ for all $e \in E(\mathfrak{G})$, then set $J := j-1$ and stop.

- Otherwise, there exists a path P_j from u to v such that $\theta_e^{(j-1)} > 0$ for all $e \in P_j$. Denote $\alpha_j := \min_{e\in P_j} \theta_e^{(j-1)}$.

- Set $\mathcal{P}_j := \mathcal{P}_{j-1} \cup \{P_j\}$ and let $r_{e,P_j} := \frac{\theta_e^\star}{\alpha_j} r_e$ for all $e \in P_j$.

- Set $\theta_e^{(j)} := \theta_e^{(j-1)} - \alpha_j$ for all $e \in P_j$ and $\theta_e^{(j)} := \theta_e^{(j-1)}$ for all $e \notin P_j$, and repeat.

Since the set $\{e \in E(\mathfrak{G})\colon \theta_e^{(j)} = 0\}$ is strictly increasing (and our graph is finite), the procedure will stop after a finite number of iterations; the quantity J then gives the number of iterations used. Note that the same also shows that the paths P_j are distinct.

We will now show the desired inequality (4.2.4) with $\mathcal{P}^\star := \mathcal{P}_J$ and $r_{e,P} := r_{e,P_j}$ for $P = P_j$. First, abbreviating $[J]\colon = \{1, \dots, J\}$, we have

$$\sum_{j\in[J]\colon e\in P_j} \alpha_j = \theta_e^\star$$

for each $e \in E(\mathfrak{G})$. Employing the definition of r_{e,P_j} we get

$$\sum_{j\in[J]\colon e\in P_j} \alpha_j^2 r_{e,P_j} = \sum_{j\in[J]\colon e\in P_j} \alpha_j \theta_e^\star r_e = r_e\,(\theta_e^\star)^2$$

and so

$$\sum_{e\in E(\mathfrak{G})} \sum_{j\in[J]\colon e\in P_j} \alpha_j^2 r_{e,P_j} = \sum_{e\in E(\mathfrak{G})} r_e\,(\theta_e^\star)^2 .$$

Rearranging the sums yields

$$\sum_{e\in E(\mathfrak{G})} \sum_{j\in[J]\colon e\in P_j} \alpha_j^2 r_{e,P_j} = \sum_{j\in[J]} \alpha_j^2 \Big(\sum_{e\in P_j} r_{e,P_j} \Big),$$

where $\sum_{j\in[J]} \alpha_j = 1$. Abbreviating $R_j := \sum_{e\in P_j} r_{e,P_j}$, the right hand side of the preceding equality is minimized (subject to the stated constraint) at $\alpha_j := \frac{1/R_j}{\sum_{j\in[J]} 1/R_j}$, and therefore

$$\sum_{j\in[J]} \alpha_j^2 \Big(\sum_{e\in P_j} r_{e,P_j} \Big) \geq \Big(\sum_{j\in[J]} \frac{1}{R_j} \Big)^{-1} .$$

This completes the desired inequality (4.2.4) including the construction of a minimizer in (4.2.2). □

A slightly augmented version of the above proof in fact yields:

Proposition 4.2.2. *Let $\mathfrak{T}_{u,v}$ be the set of all multisets of edges of $\mathfrak{G}$ that, if considered as a graph on $V(\mathfrak{G})$, contain a path between u and v. Then*

$$R_{\mathfrak{G}}(u,v) = \inf_{\mathcal{T}\in\mathfrak{T}_{u,v}} \inf_{\{r_{e,T}: e\in E(\mathfrak{G}), T\in\mathcal{T}\}\in\mathfrak{R}_{\mathcal{T}}} \Big(\sum_{T\in\mathcal{T}} \frac{1}{\sum_{e\in T} r_{e,T}} \Big)^{-1}, \tag{4.2.5}$$

where $\mathfrak{R}_{\mathcal{T}}$ is the set of all assignments $\{r_{e,T}\colon e\in E(\mathfrak{G}), T\in\mathcal{T}\}\in\mathbb{R}_+^{E(\mathfrak{G})\times\mathcal{T}}$ such that

$$\sum_{T\in\mathcal{T}} \frac{1}{r_{e,T}} \leq \frac{1}{r_e} \text{ for all } e\in E(\mathcal{G}) . \tag{4.2.6}$$

The infima are jointly achieved for $\mathcal{T}$ being a subset of $\mathfrak{P}_{u,v}$.

Proof. Let $R^\star$ denote the right-hand side of (4.2.5). Obviously, $\mathfrak{P}_{u,v}\subseteq\mathfrak{T}_{u,v}$ so restricting the first infimum to $\mathcal{T}\in\mathfrak{P}_{u,v}$, Proposition 4.2.1 shows $R_{\mathfrak{G}}(u,v)\geq R^\star$. (This will also ultimately give that the minimum is achieved over collections of paths.) To get $R_{\mathfrak{G}}(u,v)\leq R^\star$, let us consider an assignment $\{r_{e,T}\colon e\in E(\mathfrak{G}), T\in\mathcal{T}\}$ satisfying (4.2.6). For each $T\in\mathcal{T}$, let P_T denote an arbitrarily chosen simple path between u and v formed by edges in T. Then, defining $r_{e,P_T} := r_{e,T}$ for each $T\in\mathcal{T}$, we find that the assignment $\{r_{e,P_T}\colon e\in E(\mathfrak{G}), T\in\mathcal{T}\}$ satisfies (4.2.3). Now the claim follows from the simple observation that $\sum_{e\in P_T} r_{e,p_T} \leq \sum_{e\in P_T} r_{e,T}$. □

4.2.2 Variational characterization of effective conductance

An alternative way to approach an electric network is using conductances. We write $c_e := 1/r_e$ for the edge conductance on e, and define the effective conductance between u and v by

$$C_{\mathfrak{G}}(u,v) := \inf_F \sum_{e\in E(\mathfrak{G})} c_e \left[F(e_+) - F(e_-)\right]^2, \tag{4.2.7}$$

where $e_\pm$ are the two endpoints of the edge e (in some *a priori* orientation) and the infimum is over all functions $F\colon V \to \mathbb{R}$ satisfying $F(u) = 1$ and $F(v) = 0$. The infimum is again achieved by the fact that $\mathfrak{G}$ is finite. The fundamental electrostatic duality is then expressed as

$$C_{\mathfrak{G}}(u,v) = \frac{1}{R_{\mathfrak{G}}(u,v)} \tag{4.2.8}$$

and our aim is to capitalize on this relation further by exploiting the geometric duality between paths and cutsets. Here we say that a set of edges π is a cutset between u and v (or that π separates u from v) if each path from u to v uses an edge in π.

Proposition 4.2.3. *Let $\mathit{\Pi}_{u,v}$ denote the set of all finite collections of cutsets between u and v. Then*

$$C_{\mathfrak{G}}(u,v) = \inf_{\Pi\in\mathfrak{k}_{u,v}} \inf_{\{c_{e,\pi}\colon e\in E(\mathfrak{G}),\pi\in\Pi\}\in\mathfrak{C}_\Pi} \Biggl(\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c_{e,\pi}}\Biggr)^{-1}, \tag{4.2.9}$$

where $\mathfrak{C}_\Pi$ is the set of all assignments $\{c_{e,\pi}\colon e \in E(\mathfrak{G}), \pi \in \Pi\} \in \mathbb{R}_+^{E(\mathfrak{G})\times\Pi}$ such that

$$\sum_{\pi\in\Pi} \frac{1}{c_{e,\pi}} \le \frac{1}{c_e} \text{ for all } e \in E(\mathfrak{G}). \tag{4.2.10}$$

The infima in (4.2.9) *are (jointly) achieved.*

Proof. The proof is structurally similar to that of Proposition 4.2.1. Denote by $C^\star$ the quantity on the right hand side of (4.2.9). We will first prove $C_{\text{eff}}(u,v) \le C^\star$. Pick $\Pi \in \mathit{\Pi}$ and $\{c_{e,\pi}\colon e \in E(\mathfrak{G}), \pi \in \Pi\} \in \mathfrak{C}_\Pi$ subject to (4.2.10). Now view each edge e as a *series* of a collection of edges $\{e_\pi : e \in \pi, \pi \in \Pi\}$ where the conductance on e_π is $c_{e,\pi}$ and, if the inequality in (4.2.10) is strict, a dummy edge $\tilde{e}$ with conductance $c_{\tilde{e}}$ such that $1/c_{\tilde{e}} = 1/c_e - \sum_{\pi\in\Pi} 1/c_{e,\pi}$. In this new network, Π can be identified with a collection of *disjoint* cutsets, where the cutset $\pi \in \Pi$ has total conductance $\sum_{e\in\pi} c_{e,\pi}$. The Nash-Williams Criterion then shows

$$C_{\mathfrak{G}}(u,v) \leq \Big(\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c_{e,\pi}} \Big)^{-1}$$

thus proving $C_{\mathfrak{G}}(u,v) \leq C^\star$ as desired.

Next, we turn to proving $C_{\mathfrak{G}}(u,v) \geq C^\star$ and that the infima in (4.2.9) are attained. Let $F^\star$ be a function that achieves the infimum in (4.2.7). This function is discrete harmonic in the sense that $\mathcal{L}F^\star(x) = 0$ for $x \neq u, v$, where

$$\mathcal{L}f(x) := \sum_{y:\, y\sim x} c_{(x,y)} \big[f(y) - f(x)\big].$$

In light of the inequality $C_{\mathfrak{G}}(u,v) \leq C^\star$, it suffices to construct a collection of cutsets $\Pi^\star$ and conductances $\{c^\star_{e,\pi} \colon e \in \pi, \pi \in \Pi^\star\}$ such that

$$\Big(\sum_{\pi\in\Pi^\star} \frac{1}{\sum_{e\in\pi} c^\star_{e,\pi}} \Big)^{-1} \leq \sum_{e\in E(\mathfrak{G})} c_e \big[F^\star(e_+) - F^\star(e_-)\big]^2 . \qquad (4.2.11)$$

We will now define a sequence of functions $F^{(j)}$ satisfying

$$\mathcal{L}F^{(j)}(x) = 0, \qquad \text{for } x \neq u, v \qquad (4.2.12)$$

and a sequence of collections of cutsets Π_j as follows. Initially, we set $F^{(0)} := F^\star$ and $\Pi^{(0)} := \emptyset$. Abbreviating $dF(e) := |F(e_+) - F(e_-)|$, we employ the following iteration for $j \geq 1$:

- If $F^{(j-1)}$ is constant on $V(\mathfrak{G})$, then set $J := j-1$ and stop.
- Otherwise, by (4.2.12) (and positivity of all c_e's) we have $F^{(j-1)}(u) \neq F^{(j-1)}(v)$ and

hence there exists a cutset π_j separating u from v such that $|dF^{(j-1)}(e)| > 0$ for all $e \in P_j$. We take π_j to be the closest cutset to u — that is, one that is not separated from u by another such cutset — and define $\alpha_j := \min_{e\in\pi_j} dF^{(j-1)}(e)$.

- Set $\Pi_j := \Pi_{j-1} \cup \{\pi_j\}$ and let $c_{e,\pi_j} := \frac{dF^\star(e)}{\alpha_j} c_e$ for all $e \in P_j$.
- Set $F^{(j)}(e_+) := F^{(j-1)}(e_+) - \alpha_j$ for all $e \in \pi_j$, where e_+ denotes the endpoint of e with a larger value of $F^{(j-1)}$. For all other vertices x, set $F^{(j)}(x) := F^{(j-1)}(x)$, and repeat.

We see that the above procedure will stop after a finite number of iterations, since all the cutsets π_j are different by our construction. The number J is then the total number of iterations used. The validity of (4.2.12) for all $j = 1, \dots, J$ follows directly from the construction.

In order to prove (4.2.11), we now proceed as follows. First, we have

$$\sum_{j\in[J]\colon e\in\pi_j} \alpha_j = dF^\star(e)$$

and so, by the definition of α_j,

$$\sum_{j\in[J]\colon e\in\pi_j} \alpha_j^2 c_{e,\pi_j} = \sum_{j\in[J]\colon e\in P_j} \alpha_j\, dF^\star(e) c_e = (dF^\star(e))^2 c_e\,.$$

It follows that

$$\sum_{e\in E(\mathfrak{G})} \sum_{j\in[J]\colon e\in\pi_j} \alpha_j^2 c_{e,\pi_j} = \sum_{e\in E(\mathfrak{G})} c_e \left[F^\star(e_+) - F^\star(e_-)\right]^2.$$

Rearranging the sums yields

$$\sum_{e\in E(\mathfrak{G})} \sum_{j\in[J]\colon e\in\pi_j} \alpha_j^2 c_{e,\pi_j} = \sum_{j\in[J]} \alpha_j^2 \Big(\sum_{e\in\pi_j} c_{e,\pi_j} \Big),$$

where $\sum_{j\in[J]}\alpha_j = 1$. Abbreviating $C_j := \sum_{e\in\pi_j} c_{e,\pi_j}$, the right hand side of the preceding equality is minimized (subject to the stated constraint) at $\alpha_j := \frac{1/C_j}{\sum_{j\in[J]} 1/C_j}$. Therefore,

$$\sum_{j\in[J]} \alpha_j^2 \Big(\sum_{e\in\pi_j} c_{e,\pi_j} \Big) \geq \Big(\sum_{j\in[J]} \frac{1}{C_j} \Big)^{-1}$$

which completes the proof of (4.2.11) including the existence of minimizers in (4.2.9). □

Propositions 4.2.1 and 4.2.3 seem to be closely related to various variational characterizations of effective resistance/conductance by way of of optimizing over *random* paths and cutsets. These are rooted in the Nash-Williams criterion and Terry Lyons' random-path method for bounding effective resistance (which can be shown to be sharp). The ultimate statements of these characterizations can be found in Berman and Konsowa [11].

4.2.3 Restricted notion of effective resistance

Propositions 4.2.1 and 4.2.3 naturally lead to restricted notions of resistance and conductance obtained by limiting the optimization to only *subsets* of paths and cutsets, respectively. For the purpose of current chapter we will only be concerned with effective resistance. To this end, for each collection $\mathcal{A}$ of finite sets of elements from $E(\mathfrak{G})$, we define

$$R_{\mathfrak{G}}(\mathcal{A}) := \inf_{\{r_{e,A}: e\in E(\mathcal{A}),\, A\in\mathcal{A}\}\in\mathfrak{R}_{\mathcal{A}}} \Big(\sum_{A\in\mathcal{A}} \frac{1}{\sum_{e\in A} r_{e,A}} \Big)^{-1}, \tag{4.2.13}$$

where $E(\mathcal{A}) := \bigcup_{A\in\mathcal{A}} A$ and where $\mathfrak{R}_{\mathcal{A}}$ is the set of all $\{r_{e,A} : e \in E(\mathcal{A}),\ A \in \mathcal{A}\} \in \mathbb{R}_+^{E(\mathcal{A})\times\mathcal{A}}$ such that

$$\sum_{A\in\mathcal{A}} \frac{1}{r_{e,A}} \leq \frac{1}{r_e} \text{ for all } e \in E(\mathcal{A})\,. \tag{4.2.14}$$

We refer to $R_{\mathfrak{G}}(\mathcal{A})$ as the effective resistance *restricted to* $\mathcal{A}$. By taking suitable $r_{e,P}$, the map $\mathcal{A} \mapsto R_{\mathfrak{G}}(\mathcal{A})$ is shown to be non-decreasing with respect to the set inclusion. We will mostly be interested in $R_{\mathfrak{G}}(\mathcal{A})$ when $\mathcal{A}$ is a set of simple paths from u to v. The following result is analogous to metric property of effective resistance.

Lemma 4.2.4. *Let $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ be collections of paths such that for any choice of P_i from $\mathcal{P}_i$ for each $1 \le i \le k$, the graph union $\bigcup_{1\le i\le k} P_i$ contains a path between u and v. Then*

$$R_{\mathfrak{G}}(u,v) \le \sum_{i=1}^{k} R_{\mathfrak{G}}(\mathcal{P}_i)\,.$$

Proof. Define the edge sets $E_1, E_2, \cdots, E_k$ recursively by setting $E_1 := \bigcup_{P\in\mathcal{P}_1} E(P)$ and letting $E_j := \bigcup_{P\in\mathcal{P}_j} E(P) \setminus \bigcup_{i<j} E_i$ for $k \ge j > 1$. Let $\{r_{e,P} \colon e \in E(\mathfrak{G}), P \in \mathcal{P}_i\}$ be a vector in $\mathbb{R}_+^{E(\mathfrak{G})\times\mathcal{P}_i}$ satisfying (4.2.14) for all i. For each $i = 1, \ldots, k$ and each $P \in \mathcal{P}_i$, define $\rho_{i,P}$ by

$$\rho_{i,P} := \frac{\Big(\sum_{e\in E(P)} r_{e,P}\Big)^{-1}}{\sum_{P\in\mathcal{P}_i}\Big(\sum_{e\in E(P)} r_{e,P}\Big)^{-1}}\,.$$

Also for $e \in E_i$ and $P_1, P_2, \cdots, P_k$ in $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k$ respectively, define

$$r_{e;P_1,P_2,\cdots,P_k} := r_{e,P_i} \prod_{j\neq i} \frac{1}{\rho_{j,P_j}}\,.$$

Notice that for any $e \in E_i$,

$$\sum_{\substack{P_j\in\mathcal{P}_j,\\ 1\le j\le k}} \frac{1}{r_{e;P_1,P_2,\cdots,P_k}} = \sum_{P_i\in\mathcal{P}_i} \frac{1}{r_{e,P_i}} \le \frac{1}{r_e}\,, \tag{4.2.15}$$

where the first equality follows from the fact that $\sum_{P\in\mathcal{P}_j} \rho_{j,P} = 1$ for all j and the last inequality is a consequence of (4.2.14).

The above definitions also immediately give

$$\begin{aligned}\sum_{e\in\bigcup_{1\le i\le k}E(P_j)} r_{e;P_1,P_2,\cdots,P_k} &\le \sum_{1\le i\le k}\sum_{e\in E(P_i)}\frac{r_{e,P_i}}{\prod_{j\ne i}\rho_{j,P_j}}\\ &= \sum_{1\le i\le k}\frac{\big(\sum_{P\in\mathcal{P}_i}\frac{1}{\sum_{e\in E(P)}r_{e,P}}\big)^{-1}}{\prod_{1\le j\le k}\rho_{j,P_j}}\end{aligned} \tag{4.2.16}$$

As (4.2.15) holds, Proposition 4.2.2 with T being the set of edges in $P_1,\dots,P_k$ yields

$$\begin{aligned}R_{\mathfrak{G}}(u,v) &\le \Bigg(\sum_{\substack{P_j\in\mathcal{P}_j,\\1\le j\le k}}\frac{1}{\sum\limits_{e\in\bigcup_{1\le i\le k}E(P_j)} r_{e;P_1,P_2,\cdots,P_k}}\Bigg)^{-1}\\ &\le \Bigg(\Big[\sum_{1\le i\le k}\Big(\sum_{P\in\mathcal{P}_i}\frac{1}{\sum_{e\in E(P)}r_{e,P}}\Big)^{-1}\Big]^{-1}\sum_{\substack{P_j\in\mathcal{P}_j,\\1\le j\le k}}\prod_{1\le j\le k}\rho_{j,P_j}\Bigg)^{-1}\\ &= \sum_{1\le i\le k}\Big(\sum_{P\in\mathcal{P}_i}\frac{1}{\sum_{e\in E(P)}r_{e,P}}\Big)^{-1},\end{aligned} \tag{4.2.17}$$

where we again used that $\sum_{P\in\mathcal{P}_j}\rho_{j,P}=1$ in the last step. Since (4.2.17) holds for all choices of $\{r_{e,P}\colon e\in E(\mathfrak{G}), P\in\mathcal{P}_i\}$ satisfying (4.2.14), the claim follows from (4.2.13). □

A similar upper bound holds also for the effective conductance.

Lemma 4.2.5. *Let $\mathcal{P}_1,\dots,\mathcal{P}_k\in\mathfrak{P}_{u,v}$ be such that every path from u to v lies in $\bigcup_{1\le i\le k}\mathcal{P}_i$. Then*

$$C_{\mathfrak{G}}(u,v)\le\sum_{1\le i\le k}R_{\mathfrak{G}}(\mathcal{P}_i)^{-1}\,.$$

Proof. This is a straightforward consequence of Proposition 4.2.1. Indeed, write $R_{\mathfrak{G}}(u,v)^{-1}$ as suprema of $\sum_{P\in\mathcal{P}}(\sum_{e\in P}r_{e,P})^{-1}$ over $\mathcal{P}$ and $r_{e,P}$ satisfying (4.2.3). Next bound the sum over P by the sum over $i=1,\dots,k$ and the sum over $P\in\mathcal{P}\cap\mathcal{P}_i$ and observe, since

$\sum_{P\in\mathcal{P}\cap\mathcal{P}_i} 1/r_{e,P} \le \sum_{P\in\mathcal{P}} 1/r_{e,P} \le 1/r_e$, we have

$$\sum_{i=1}^{k} \sum_{P\in\mathcal{P}\cap P_i} \frac{1}{\sum_{e\in P} r_{e,P}} \le \sum_{i=1}^{k} R_{\mathfrak{G}}(\mathcal{P}_i)^{-1}.$$

As this holds for all $\mathcal{P}$ and all admissible $r_{e,P}$, the claim follows from (4.2.8). $\square$

We note (and this will be useful later) that, in standard treatments of electrostatic theory on graphs, the notions of effective resistance/conductance are naturally defined between subsets (as opposed to just single vertices) of the underlying network. A simplest way to reduce this to our earlier definitions is by "gluing" vertices in these sets together. Explicitly, given two non-empty disjoint sets $A, B \subseteq V(\mathfrak{G})$ consider a network $\mathfrak{G}'$ where all edges in $(A\times A)\cup(B\times B)$ have been removed and the vertices in A identified as one vertex $\langle A\rangle$ — with all edges in $\mathfrak{G}$ with exactly one endpoint in A now "pointing" to $\langle A\rangle$ in $\mathfrak{G}'$ — and the vertices in B similarly identified as one vertex $\langle B\rangle$. Then we define

$$R_{\mathfrak{G}}(A,B) := R_{\mathfrak{G}'}\bigl(\langle A\rangle,\langle B\rangle\bigr) \quad\text{and}\quad C_{\mathfrak{G}}(A,B) := C_{\mathfrak{G}'}\bigl(\langle A\rangle,\langle B\rangle\bigr). \tag{4.2.18}$$

Note that, for one-point sets, $R_{\mathfrak{G}}(\{u\},\{v\})$ coincides with $R_{\mathfrak{G}}(u,v)$, and similarly for the effective conductance. The electrostatic duality also holds, $R_{\mathfrak{G}}(A,B) = 1/C_{\mathfrak{G}}(A,B)$.

4.2.4 Self-duality

The similarity of the two formulas (4.2.2) and (4.2.9) naturally leads to the consideration of self-dual situations — i.e., those in which the resistances r_e can somehow be exchanged for the conductances c_e. An example of this is the network $\mathbb{Z}^2_\eta$ where the distributional identity $\eta \overset{\text{law}}{=} -\eta$ makes the associated resistances $\{r_e : e\in E(\mathbb{Z}^2)\}$ equidistributed to the conductances $\{c_e : e \in E(\mathbb{Z}^2)\}$. To formalize this situation, given a network $\mathfrak{G}$ we define its *reciprocal* $\mathfrak{G}^\star$ as the network with the same underlying graph but with the resistances swapped for the conductances. An edge e in network $\mathfrak{G}^\star$ thus has resistance $r^\star_e := 1/r_e$, where r_e is the resistance of e in network $\mathfrak{G}$.

Lemma 4.2.6. *Let $\mathfrak{D}$ denote the maximum vertex degree in $\mathfrak{G}$ and let $\rho_{\max}$ denote the maximum ratio of the resistances of any pair of adjacent edges in $\mathfrak{G}$. Given two pairs (A, B) and (C, D) of disjoint, nonempty subsets of $V(\mathfrak{G})$, suppose that every path between A and B shares a vertex with every path between C and D. Then*

$$R_{\mathfrak{G}}(A, B) \geq \frac{1}{4\mathfrak{D}^2 \rho_{\max} R_{\mathfrak{G}^\star}(C, D)}\,. \tag{4.2.19}$$

Proof. The proof is based on the fact that every path P between C and D defines a cutset π_P between A and B by taking π_P to be the set of all edges adjacent to any edge in P, but not including the edges in $(A \times A) \cup (B \times B)$. By the electrostatic duality we just need to show

$$C_{\mathfrak{G}}(A, B) \leq 4\mathfrak{D}^2 \rho_{\max} R_{\mathfrak{G}^\star}(A, B)\,. \tag{4.2.20}$$

To this end, given any $\mathcal{P} \in \mathfrak{P}_{C,D}$ let us pick positive numbers $\{r'_{e,P} : e \in E(\mathcal{P}), P \in \mathcal{P}\}$ such that

$$\sum_{P \in \mathcal{P}} \frac{1}{r'_{e,P}} \leq \frac{1}{c_e} \text{ for all } e \in E(\mathcal{P})\,. \tag{4.2.21}$$

For any edge e and any path $P \in \mathcal{P}$, let $N_P(e)$, $N_{\mathcal{P}}(e)$ and $N(e)$ denote the sets of all edges in $E(P)$, $E(\mathcal{P})$ and $E(\mathfrak{G})$ that are adjacent to e, respectively. For any $e \in E(\mathcal{P})$ and any $P \in \mathcal{P}$, let $\theta_{e,P} := c_e/r'_{e,P}$ and note that $\theta_{e,P}$'s are positive numbers satisfying $\sum_{P \in \mathcal{P}} \theta_{e,P} \leq 1$ for all $e \in E(\mathcal{P})$. As a consequence, if we define

$$c_{e,\pi_P} := \frac{c_e}{\sum\limits_{e' \in N_{\mathcal{P}'}(e)} \theta_{e',P}} |N_{\mathcal{P}}(e)| \tag{4.2.22}$$

then $\{c_{e,\pi_P} : e \in \bigcup_{P \in \mathcal{P}} \pi_P,\ P \in \mathcal{P}\}$ satisfies (4.2.10). Now fix a path P in $\mathcal{P}$ and compute, invoking the definitions of $\mathfrak{D}$, $\rho_{\max}$ and also Jensen's inequality in the second step:

$$\begin{aligned}\sum_{e\in\pi_P} c_{e,\pi_P} &= \sum_{e\in\pi_P} \frac{c_e}{\sum\limits_{e'\in N_{\mathcal{P}}(e)} \theta_{e',P}} |N_{\mathcal{P}}(e)| \le 2\mathfrak{D} \sum_{e\in\pi_P} \frac{c_e}{\sum\limits_{e'\in N_P(e)} \theta_{e',P}} |N_P(e)| \\ &\le 2\mathfrak{D} \sum_{e\in\pi_P} \Big(\frac{c_e}{|N_P(e)|} \sum_{e'\in N_P(e)} \frac{1}{\theta_{e',P}} \Big) \le 2\mathfrak{D} \sum_{\substack{e\in E(\mathfrak{G}), e'\in P \\ e\sim e'}} \frac{c_e}{\theta_{e',P}} \\ &= 2\mathfrak{D} \sum_{e'\in P} \frac{\sum\limits_{e\in N(e')} c_e}{\theta_{e',P}} \le 4\mathfrak{D}^2 \rho_{\max} \sum_{e'\in P} \frac{c_{e'}}{\theta_{e',P}} = 4\mathfrak{D}^2 \rho_{\max} \sum_{e'\in P} r'_{e',P} \,.\end{aligned} \tag{4.2.23}$$

Hence we get

$$C_{\mathfrak{G}}(\Pi)(A,B) \le \Big(\sum_{P\in\mathcal{P}'} \frac{1}{\sum_{e\in\pi_P} c_{e,\pi_P}} \Big)^{-1} \le 4\mathfrak{D}^2 \rho_{\max} \Big(\sum_{P\in\mathcal{P}'} \frac{1}{\sum_{e\in P} r'_{e,P}} \Big)^{-1} . \tag{4.2.24}$$

As this holds for any choice of $\mathcal{P}$ and positive numbers $\{r'_{e,P} : e \in E(\mathcal{P}), P \in \mathcal{P}\}$ satisfying (4.2.21), we get (4.2.20) as desired. □

A crucial fact underlying the proof of the previous lemma was that one could obtain a cut set for $\mathcal{P}$ from a path P in $\mathcal{P}$ by taking union of all edges adjacent to vertices in P. In the same setup, we get a corresponding result also for effective conductances. Indeed, we have:

Lemma 4.2.7. *For the same setting and notation as in Lemma 4.2.6, assume that for every cutset π between C and D, the subgraph induced by the set of all edges that are adjacent to some edge in π contains a path in $\mathfrak{P}_{A,B}$. Then*

$$C_{\mathfrak{G}}(A,B) C_{\mathfrak{G}^\star}(C,D) \ge \frac{1}{4\mathfrak{D}^2 \rho_{\max}} \,. \tag{4.2.25}$$

Proof. For any cutset π between C and D, let T_π denote the set of all edges that are adjacent to some edge in π. Thus T_π contains a path in $\mathfrak{P}_{A,B}$ by the hypothesis of the lemma. Now given any $\Pi \in \mathit{\Pi}_{C,D}$, we pick positive numbers $\{c^\star_{e,\pi} \colon e \in \bigcup_{\pi\in\Pi} T_\pi, \pi \in \Pi\}$ such that

$$\sum_{\pi\in\Pi} \frac{1}{c^\star_{e,\pi}} \le \frac{1}{r_e}\,. \tag{4.2.26}$$

Following the exact same sequence of steps as in the proof of Lemma 4.2.6, we now find $\{r_{e,T_\pi} \colon e \in \pi, \pi \in \Pi\}$ satisfying (4.2.6) such that

$$\Bigl(\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in T_\pi} r_{e,T_\pi}}\Bigr)^{-1} \le 4\mathfrak{D}^2 \rho_{\max} \Bigl(\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c^\star_{e,\pi}}\Bigr)^{-1}.$$

Proposition 4.2.2 then implies

$$R_{\mathfrak{G}}(A,B) \le \Bigl(\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in T_\pi} r_{e,T_\pi}}\Bigr)^{-1} \le 4\mathfrak{D}^2 \rho_{\max} \Bigl(\sum_{\pi\in\Pi} \frac{1}{\sum_{e\in\pi} c^\star_{e,\pi}}\Bigr)^{-1}.$$

As this holds for all choices of Π and $\{c^\star_{e,\pi} \colon e \in \pi, \pi \in \Pi\}$ satisfying (4.2.26), we get the desired inequality (4.2.25). □

4.3 Preliminaries on Gaussian processes

Before we move on to the main line of the proof, we need to develop some preliminary control on the underlying Gaussian fields. The goal of this section is to amass the relevant technical claims concerning Gaussian processes and, in particular, the GFF. An impatient, or otherwise uninterested, reader may consider only skimming through this section and returning to it when the relevant claims are used in later proofs.

4.3.1 Some standard inequalities

We start by recalling, without proof, a few standard facts about general Gaussian processes:

Lemma 4.3.1 (Theorem 7.1 in [54]). *Given a finite set A, consider a centered Gaussian* *process* $\{X_v : v \in A\}$. *Then, for* $x > 0$,

$$\mathbb{P}\Big(\Big|\max_{v\in A} X_v - \mathbb{E}\max_{v\in A} X_v\Big| \geq x\Big) \leq 2\mathrm{e}^{-x^2/2\sigma^2}\,,$$

where $\sigma^2 := \max_{v\in A}\mathbb{E}(X_v^2)$.

Lemma 4.3.2 (Theorem 4.1 in [1])**.** *Let* (S, d) *be a finite metric space with* $\max_{s,t\in S} d(s,t) = 1$. *Suppose that there are* $\beta, K_1 \in (0,\infty)$ *such that for every* $\epsilon \in (0,1]$, *the* ϵ*-covering number* $N_\epsilon(S,d)$ *of* (S,d) *obeys* $N_\epsilon(S,d) \leq K_1\epsilon^{-\beta}$. *Then for any* $\alpha, K_2 \in (0,\infty)$ *and any centered Gaussian process* $\{X_s\}_{s\in S}$ *satisfying*

$$\sqrt{\mathbb{E}(X_s - X_{s'})^2} \leq K_2\, d(s,s')^\alpha, \qquad s, s' \in S,$$

we have

$$\mathbb{E}\big(\max_{s\in A}|X_s|\big) \leq K \quad \text{and} \quad \mathbb{E}\big(\max_{s,t\in A}|X_s - X_t|\big) \leq K,$$

where $K := K_2(\sqrt{\beta\log 2} + \sqrt{\log(K_1+1)})K_\alpha$ *with* $K_\alpha := \sum_{n\geq 0} 2^{-n\alpha}\sqrt{n+1}$.

As a consequence of Lemma 4.3.2 we get the following result which we will use in the next subsection.

Lemma 4.3.3. *Let* $B_1, B_2, \ldots, B_N$ *be squares in* $\mathbb{Z}^2$ *of side lengths* $b_1, b_2, \ldots, b_N$ *respectively and let* $B := \cup_{j\in[N]} B_j$. *There exists an absolute constant* $C' > 0$ *such that, if* $\{X_v\}_{v\in B}$ *is a centered Gaussian process satisfying*

$$\mathbb{E}(X_u - X_v)^2 \leq \frac{|u-v|}{b_j}, \qquad (u,v) \in \bigcup_{j=1}^N (B_j \times B_j),$$

then

$$\mathbb{E}\max_{v\in B} X_v \leq C'\sqrt{\log N}\Big(1 + \max_{v\in B}\sqrt{\mathbb{E}X_v^2}\Big) + C'\,.$$

The following lemma, taken from [62], is the FKG inequality for Gaussian random variables. We will refer to this as the FKG in the rest of the thesis.

Lemma 4.3.4. *Consider a Gaussian process* $\mathbf{X} = \{X_v\}_{v\in A}$ *on a finite set* A*, and suppose that*

$$\mathrm{Cov}(X_u, X_v) \geq 0, \qquad u, v \in A. \tag{4.3.1}$$

Then

$$\mathrm{Cov}\big(f(\mathbf{X}), g(\mathbf{X})\big) \geq 0$$

holds for any bounded, Borel measurable functions f, g *on* $\mathbb{R}^A$ *that are increasing separately in each coordinate.*

As a corollary to FKG, we get:

Corollary 4.3.5. *Consider a Gaussian process* $\mathbf{X} = \{X_v\}_{v\in A}$ *on a finite set* A *such that* (4.3.1) *holds. If* $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_k \in \sigma(\mathbf{X})$ *are all increasing (or all decreasing), then*

$$\max_{i\in[k]} \mathbb{P}(\mathcal{E}_i) \geq 1 - \bigg(1 - \mathbb{P}\Big(\bigcup_{i\in[k]} \mathcal{E}_i\Big)\bigg)^{1/k}.$$

This is known as the "square root trick" in percolation literature (see, e.g., [46]).

4.3.2 Smoothness of harmonic averages of the GFF

Moving to the specific example of the GFF we note that one of the most important properties that makes the GFF amenable to analysis is its behavior under restrictions to a subdomain. This goes by the name Gibbs-Markov, or domain-Markov, property. In order to give a precise statement (which will happen in Lemma 4.3.6 below) we need some notations.

Given a set $A \subseteq \mathbb{Z}^2$, let ∂A denote the set of vertices in $\mathbb{Z}^2 \smallsetminus A$ that have a neighbor in A. Recall that a GFF in $A \subsetneq \mathbb{Z}^2$ with Dirichlet boundary condition is a centered Gaussian process $\chi_A = \{\chi_{A,v}\}_{v\in A}$ such that

$$\chi_{A,v} = 0 \ \text{ for } \ v \in \mathbb{Z}^2 \smallsetminus A \qquad \text{and} \qquad \mathbb{E}(\chi_{A,u}\chi_{A,v}) = G_A(u,v) \ \text{ for } \ u, v \in A,$$

where $G_A(u,v)$ is the Green function in A; i.e. the expected number of visits to v for the simple random walk on $\mathbb{Z}^2$ started at u and killed upon entering $\mathbb{Z}^2 \smallsetminus A$. We then have:

Lemma 4.3.6 (Gibbs-Markov property)**.** *Consider the GFF $\chi_A = \{\chi_{A,v}\}_{v\in A}$ on a set $A \subsetneq \mathbb{Z}^2$ with Dirichlet boundary condition and let $B \subseteq A$ be finite. Define the random fields $\chi_A^c = \{\chi_{A,v}^c\}_{v\in B}$ and $\chi_A^f = \{\chi_{A,v}^f\}_{v\in B}$ by*

$$\chi_{A,v}^c = \mathbb{E}\big(\chi_{A,v}\,\big|\,\chi_{A,u}\colon u \in A \smallsetminus B\big) \quad \textit{and} \quad \chi_{A,v}^f = \chi_{A,v} - \chi_{A,v}^c.$$

Then χ_A^f and χ_A^c are independent with $\chi_A^f \overset{\text{law}}{=} \chi_B$. Moreover, χ_A^c equals χ_A on $A \smallsetminus B$ and its sample paths are discrete harmonic on B.

Proof. This is verified directly by writing out the probability density of χ_A or, alternatively, by noting that the covariance of χ_A^c is $G_A - G_B$, which is harmonic in both variables throughout B. We leave further details to the reader. □

By way of reference to the spatial scales that these fields will typically be defined over, we refer to χ_A^f as the *fine* field and χ_A^c as the *coarse* field. However, this should not be confused with the way their actual sample paths look like. Indeed, the samples of χ_A^f will typically be quite rough (being those of a GFF), while the samples of χ_A^c will be rather smooth (being discrete harmonic on B). Our next goal is to develop a good control of the smoothness of χ_A^c precisely. A starting point is the following estimate:

Lemma 4.3.7. *There is an absolute constant $c \in (0,\infty)$ such that, given any $\emptyset \neq \tilde{B} \subseteq B \subseteq A \subsetneq \mathbb{Z}^2$ with $\tilde{B}$ connected and denoting*

$$N := \inf\big\{M \in \mathbb{N}\colon \tilde{B} + [-M,M]^2 \cap \mathbb{Z}^2 \subseteq B\big\}, \tag{4.3.2}$$

the coarse field χ_A^c on B obeys

$$\operatorname{Var}\big(\chi_{A,u}^c - \chi_{A,v}^c\big) \le c\Big(\frac{\operatorname{dist}_{\tilde{B}}(u,v)}{N}\Big)^2, \qquad u,v \in \tilde{B}, \tag{4.3.3}$$

where $\mathrm{dist}_{\tilde{B}}(x,y)$ *denotes the length of the shortest path in* $\tilde{B}$ *connecting* x *to* y.

Proof. Let $u,v\in\tilde{B}$ first be nearest neighbors and let $M:=\lfloor N/2\rfloor$. Using (f,g) to denote the canonical inner product in $\ell^2(\mathbb{Z}^2)$ with respect to the counting measure, the Gibbs-Markov property gives

$$\mathrm{Var}\big(\chi^c_{A,u}-\chi^c_{A,v}\big)=\Big(\delta_u-\delta_v,(G_A-G_B)(\delta_u-\delta_v)\Big)$$

Since $A\mapsto G_A$ is increasing (as an operator $\ell^2(\mathbb{Z}^2)\to\ell^2(\mathbb{Z}^2)$) with respect to the set inclusion, the worst case that accommodates the current setting is when A is the complement of a single point and B is the square $u+B(M)=u+[-M,M]^2\cap\mathbb{Z}^2$. Focusing on such A and B from now on and shifting the domains suitably, we may assume $A:=\mathbb{Z}^2\smallsetminus\{0\}$. Then

$$G_A(x,y)=\mathfrak{a}(x)+\mathfrak{a}(y)-\mathfrak{a}(x-y),\tag{4.3.4}$$

where $\mathfrak{a}(x)$ is the potential kernel defined, e.g., as the limit value of $G_{B(N)}(0,0)-G_{B(N)}(0,x)$ as $N\to\infty$. The relevant fact for us is that $\mathfrak{a}$ admits the asymptotic form

$$\mathfrak{a}(x)=g\log|x|+c_0+O\big(|x|^{-2}\big),\qquad |x|\to\infty,\tag{4.3.5}$$

where $g:=2/\pi$ and c_0 is a (known) constant.

There is another representation of $\mathrm{Var}\big(\chi^c_{A,u}-\chi^c_{A,v}\big)$ in terms of harmonic measures which follows from the discrete harmonicity of the coarse field. Let $H^B(x,y)$, for $x\in B$ and $y\in\partial B$, denote the harmonic measure; i.e., the probability that the simple random walk started from x first enters $\mathbb{Z}^2\smallsetminus B$ at y. Then

$$\mathrm{Var}\big(\chi^c_{A,u}-\chi^c_{A,v}\big)=\big(f,G_Af\big)$$

where

$$f(\cdot):=\sum_{z\in\partial B}\big[H^B(u,z)-H^B(v,z)\big]\delta_z(\cdot).\tag{4.3.6}$$

In order to make use of this expression, we will need are suitable estimates for the harmonic measure: There are constants $c_1, c_2 \in (0,\infty)$ such that for all $M \ge 1$, any neighbor v of u and $B := u + B(M)$, from, e.g., [53, Proposition 8.1.4] , we have

$$H^B(u,z) \le \frac{c_1}{M}, \qquad z \in \partial B, \tag{4.3.7}$$

and

$$\bigl|H^B(u,z) - H^B(v,z)\bigr| \le \frac{c_2}{M}\, H^B(u,z), \qquad z \in \partial B. \tag{4.3.8}$$

For our special choice of A, using (4.3.6) we now write

$$\begin{aligned}&\text{Var}\bigl(\chi^c_{A,u} - \chi^c_{A,v}\bigr)\\ &= \sum_{z,\tilde z \in \partial B} \bigl[H^B(u,z) - H^B(v,z)\bigr]\bigl[H^B(u,\tilde z) - H^B(v,\tilde z)\bigr]\bigl(\mathfrak{a}(z) + \mathfrak{a}(\tilde z) - \mathfrak{a}(z - \tilde z)\bigr).\end{aligned} \tag{4.3.9}$$

Since $z \mapsto H^B(u,z)$ is a probability measure for each u, the contribution of the terms $\mathfrak{a}(z)$ and $\mathfrak{a}(\tilde z)$ vanishes. For the same reason, we may replace $\mathfrak{a}(z-\tilde z)$ with $\mathfrak{a}(z-\tilde z) - g\log M$ in (4.3.9). Now we apply (4.3.8) with the result

$$\text{Var}\bigl(\chi^c_{A,u} - \chi^c_{A,v}\bigr) \le \Bigl(\frac{c_2}{M}\Bigr)^2 \sum_{z,\tilde z \in \partial B} H^B(u,z) H^B(u,\tilde z) |\mathfrak{a}(z-\tilde z) - g\log M|\,.$$

Invoking (4.3.5) and (4.3.7), the two sums are now readily bounded by a constant independent of M. This gives (4.3.3) for neighboring pairs of vertices. For the general case we just apply the triangle inequality for the intrinsic (pseudo)metric $u, v \mapsto [\text{Var}(\chi^c_{A,u} - \chi^c_{A,v})]^{1/2}$ along the shortest path in $\tilde B$ between u and v in the graph-theoretical metric. □

Using the above variance bound, we now get:

Corollary 4.3.8. *For each set $A \subseteq \mathbb{Z}^2$, let us write* $\text{diam}_A(A)$ *for the diameter A in the graph-theoretical metric on A. For each $\delta > 0$ there are constants $c, \tilde c \in (0,\infty)$ such that for* *all sets* $0 \neq \tilde{B} \subseteq B \subseteq A \subsetneq \mathbb{Z}^2$ *with* $\tilde{B}$ *connected and obeying*

$$\inf\bigl\{M \in \mathbb{N}\colon \tilde{B} + [-M, M]^2 \cap \mathbb{Z}^2 \subseteq B\bigr\} \geq \delta \operatorname{diam}_{\tilde{B}}(\tilde{B}) \tag{4.3.10}$$

and for χ_A^c *denoting the coarse field on* B *for the GFF* χ_A *on* A*, we have*

$$\mathbb{P}\Bigl(\sup_{u,v \in \tilde{B}} |\chi_{A,u}^c - \chi_{A,v}^c| > c + t\Bigr) \leq 2\mathrm{e}^{-\tilde{c}t^2} \tag{4.3.11}$$

for each $t \geq 0$*.*

Proof. The condition (4.3.10) ensures, via Lemma 4.3.7, that the variance of $\chi_{A,u}^c - \chi_{A,v}^c$ is bounded by a constant times $\operatorname{dist}_{\tilde{B}}(u, v)/N$ with N as in (4.3.2). The assumption (4.3.10) then ensures that this is at most a δ-dependent constant. Writing this constant as $2/\tilde{c}$ and denoting

$$M^\star := \sup_{u,v \in \tilde{B}} \bigl|\chi_{A,u}^c - \chi_{A,v}^c\bigr|,$$

Lemma 4.3.1 gives

$$\mathbb{P}\bigl(|M^\star - \mathbb{E}M^\star| > t\bigr) \leq 2\mathrm{e}^{-\tilde{c}t^2}.$$

It remains to show that $\mathbb{E}M^\star$ is bounded uniformly in A and B satisfying (4.3.10). For this we note that, again by Lemma 4.3.7, an ϵ-ball in the intrinsic metric $\rho(u, v) := [\operatorname{Var}(\chi_{A,u}^c - \chi_{A,v}^c)]^{1/2}$ on $\tilde{B}$ contains an order-$N\epsilon$ ball in the graph-theoretical metric on $\tilde{B}$ which itself contains an order-$(N\epsilon)^2$ ball in the ℓ^1-metric on B. Lemmas 4.3.3 thus applies with $\alpha := 1$ and $\beta := 2$. □

4.3.3 A LIL for averages on concentric annuli

The proof of the RSW estimates will require controlling the expectation of the GFF on concentric annuli, conditional on the values of the GFF on the boundaries thereof. We will conveniently represent the sequence of these expectations by a random walk. Annulus averages and the associated random walk have been central to the study of the local properties of nearly-maximal values of the GFF in [15]. However, there the emphasis was on estimating the probability that the random walk stays above a polylogarithmic curve for a majority of time, while here we are interested in a different aspect; namely, the Law of Iterated Logarithm. The conclusions derived here will be applied in the proof of Proposition 4.4.9.

We begin with a quantitative version of the law of the iterated logarithm for a specific class of Gaussian random walks.

Lemma 4.3.9. *Set $\phi(x) := \sqrt{2x \log\log x}$ for $x \geq 3$ and let $Z_1, Z_2, \cdots, Z_n$ be independent random variables with $Z_i \overset{\text{law}}{=} \mathcal{N}(0, \sigma_k^2)$ for some $\sigma_i^2 > 0$. Let $s_k^2 := \sum_{1 \leq j \leq k} \sigma_j^2$ and suppose that there are $\sigma > 0$ and $d > 0$ such that*

$$\sigma^2 k - d \leq s_k^2 \leq \sigma^2 k + d, \qquad k \geq 1.$$

Then there are $c_{\sigma,d} > 0$, $C_{\sigma,d} > 0$ and $N_{\sigma,d} > 0$, depending only on d and σ, such that for all $n \geq N_{\sigma,d}$, the random walk $S_k := \sum_{1 \leq j \leq k} Z_j$ obeys

$$\mathbb{P}\Big(\#\big\{\mathrm{e}^{\sqrt{\log n}} \leq k \leq n : S_k \geq \phi(s_k^2)/2\big\} \geq c_{\sigma,d} \log\log n\Big) \geq 1 - \frac{C_{\sigma,d}}{\log\log n}. \tag{4.3.12}$$

Proof. Since ϕ is regularly varying at infinity with exponent $1/2$ and $k \mapsto s_k^2$ is within distance d of a linear function, one can find $a > 1$ and k_1 sufficiently large (and depending only on σ and d) such that

$$\phi(s_{a^k}^2 - s_{a^{k-1}}^2) \geq \frac{6}{7}\phi(s_{a^k}^2), \qquad k \geq k_1, \tag{4.3.13}$$

and

$$\phi(s_{a^{k-1}}^2) \leq \frac{2}{9}\phi(s_{a^k}^2), \qquad k \geq k_1, \tag{4.3.14}$$

hold true. Now define a sequence of random variables as

$$T_1 := S_a - S_1, \quad T_2 := S_{a^2} - S_a, \ \dots \quad T_{\lfloor \log_a n \rfloor} := S_{a^{\lfloor \log_a n \rfloor}} - S_{a^{\lfloor \log_a n \rfloor - 1}}.$$

Then $T_1, T_2, \cdots, T_{\lfloor \log_a n \rfloor}$ are independent with $T_k \overset{\text{law}}{=} \mathcal{N}(0, s_{a^k}^2 - s_{a^{k-1}}^2)$. Then, for each k with $k_1 \le k \le \lfloor \log_a n \rfloor$, the inequality (4.3.13) and a straightforward Gaussian tail estimate show

$$\mathbb{P}\big(T_k \ge \tfrac{3}{4}\phi(s_{a^k}^2)\big) \ge \mathbb{P}\big(T_k \ge \tfrac{7}{8}\phi(s_{a^k}^2 - s_{a^{k-1}}^2)\big) \ge \frac{c}{\log(s_{a^k}^2 - s_{a^{k-1}}^2)},$$

for some constant $c > 0$ depending only on σ and d. Thus, whenever n is such that $\sqrt{\lfloor \log_a n \rfloor} \ge k_1$ holds true, we have

$$\sum_{\sqrt{\lfloor \log_a n \rfloor} \le k \le \lfloor \log_a n \rfloor} \mathbb{P}\big(T_k \ge \tfrac{3}{4}\phi(s_{a^k}^2)\big) \ge c' \log\log n - c'', \tag{4.3.15}$$

for some $c', c'' > 0$. By independence of $T_1, T_2, \cdots, T_{\lfloor \log_a n \rfloor}$, the Chebyshev inequality gives

$$\mathbb{P}\bigg(\#\Big\{\sqrt{\lfloor \log_a n \rfloor} \le k \le \lfloor \log_a n \rfloor : T_k \ge \tfrac{3}{4}\phi(s_{a^k}^2)\Big\} \ge \frac{c' \log\log n}{2}\bigg) \ge 1 - \frac{\tilde{c}}{\log\log n} \tag{4.3.16}$$

for some constant $\tilde{c} \in (0, \infty)$. A computation using a Gaussian tail estimate gives

$$\mathbb{P}\big(S_{a^k} \le -\tfrac{9}{8}\phi(s_{a^k}^2)\big) \le (\log s_{a^k}^2)^{-81/64}$$

for all $k \ge 1$. Therefore

$$\mathbb{P}\bigg(\bigcup_{\sqrt{\lfloor \log_a n \rfloor} \le k \le \lfloor \log_a n \rfloor} \big\{S_{a^k} \le -\tfrac{9}{8}\phi(s_{a^k}^2)\big\}\bigg) \le \tilde{c}' (\log n)^{-17/128}, \tag{4.3.17}$$

for some constant $\tilde{c}' \in (0, \infty)$. On $\{S_{a^{k-1}} \ge -\frac{9}{8}\phi(s_{a^{k-1}}^2)\} \cap \{T_k \ge \frac{3}{4}\phi(s_{a^k}^2)\}$, (4.3.14) gives

$$S_{a^k} = S_{a^{k-1}} + T_k \ge -\tfrac{9}{8}\phi\big(s_{a^{k-1}}^2\big) + \tfrac{3}{4}\phi\big(s_{a^k}^2\big) \ge \tfrac{1}{2}\phi\big(s_{a^k}^2\big)$$

and so the bounds (4.3.16) and (4.3.17) imply (4.3.12). □

We will apply Lemma 4.3.9 to a special sequence of random variables which arise from averaging the GFF along concentric squares. For integers $N \geq 1$, $n \geq 1$ and $b \geq 2$, denote $N' := b^n N$ and, for each $k \in \{1, \ldots, n\}$, define

$$M_{n,k} := \mathbb{E}\Bigg(\chi_{N',0} \,\Bigg|\, \sigma\Big(\chi_{N',v} : v \in \bigcup_{n-k \leq j \leq n} \partial B(b^j N)\Big)\Bigg), \tag{4.3.18}$$

Notice that we can also write $M_{n,k} = \mathbb{E}\big(\chi_{N',0} | \sigma(\chi_{N',v} \colon v \in \partial B(b^{n-k}N))\big)$ due to the Gibbs-Markov property of the GFF. We then have:

Lemma 4.3.10. *For each integer $b \geq 1$ as above, there are constants $\sigma > 0$ and $d > 0$ such that for all $N \geq 1$ and all $n \geq 1$ the sequence $\{M_{n,k} - M_{n,k-1}\}_{k=1,\ldots,n-1}$ (with $M_{n,0} := 0$) satisfies the conditions of Lemma 4.3.9 with these (σ, d).*

Proof. Since the $M_{n,k} - M_{n,k-1}$'s are differences of a Gaussian martingale sequence, they are independent normals. So we only need to verify the constraints on the variances. Denoting $N'' := b^{n-k}N$, the Gibbs-Markov property of the GFF implies

$$\mathrm{Var}(M_{n,k}) = G_{B(N')}(0,0) - G_{B(N'')}(0,0). \tag{4.3.19}$$

Recalling our notation $H^B(x,y)$ for the harmonic measure, the representation

$$G_B(x,y) = -\mathfrak{a}(x-y) + \sum_{z \in \partial B} H^B(x,z)\mathfrak{a}(y-z)$$

gives

$$\mathrm{Var}(M_{n,k}) = \sum_{z \in \partial B(N')} H^{B(N')}(0,z)\mathfrak{a}(z) - \sum_{z \in \partial B(N'')} H^{B(N'')}(0,z)\mathfrak{a}(z).$$

Now substitute the asymptotic form (4.3.5) and notice that the terms arising from c_0 exactly cancel, while those from the error $O(|x|^{-2})$ are uniformly bounded. Concerning the terms arising from the term $g\log|x|$, here we note that

$$\sup_{N\ge1}\Big|\sum_{z\in\partial B(N)} H^{B(N)}(0,z)\log|z|-\log N\Big|<\infty,$$

which follows by using $\log|x+r|-\log|x|=O(|r|/|x|)$ to approximate the sum by an integral. Hence we get

$$\begin{aligned}G_{B(N')}(0,0)-G_{B(N'')}(0,0)&=g\log(N')-g\log(N'')+O(1)\\&=g\log(b)(n-k)+O(1)\end{aligned}\tag{4.3.20}$$

with $O(1)$ bounded uniformly in $N\ge1$, $n\ge1$ and $k=1,\dots,n-1$. □

Using the above setup, pick two (possibly real) numbers $1<r_1<r_2<b$ and define

$$A_{n,k}:=B\big(\lfloor r_2b^kN\rfloor\big)\smallsetminus B\big(\lceil r_1b^kN\rceil\big)^\circ.$$

The point of working with the conditional expectations of $\chi_{N'}$ evaluated at the origin is that these expectations represent very well the typical value of the same conditional expectation anywhere on $A_{n,k}$. Namely, we have:

Lemma 4.3.11. *Denote*

$$\Delta_n:=\max_{k=1,\dots,n-1}\ \max_{v\in A_{n,k}}\Big|M_{n,k}-\mathbb{E}\big(\chi_{N',v}\,\big|\,\chi_{N',v}:v\in\textstyle\bigcup_{n\ge j\ge n-k}\partial B(b^jN)\big)\Big|.$$

For each $b\ge2$ (and each r_1,r_2 as above) there are $\widetilde C>0$ and $N_0\ge1$ such that for all $N\ge N_0$ and all $n\ge1$,

$$\mathbb{P}\big(\Delta_n\ge\widetilde C\sqrt{\log n}\,\big)\le1/n^2.\tag{4.3.21}$$

Proof. Denote $A'_{n,k}:=B(b^{k+1}N)\smallsetminus B(b^kN)$ and for $v\in A'_{n,k}$ abbreviate

$$\tilde\chi_{k,v}:=\mathbb{E}\big(\chi_{N',v}\,\big|\,\chi_{N',v}\colon v\in\textstyle\bigcup_{n\ge j\ge n-k}\partial B(b^jN)\big).\tag{4.3.22}$$

From the Gibbs-Markov property we also have

$$\tilde{\chi}_{k,v} = \mathbb{E}\big(\chi_{N',v} \,\big|\, \chi_{N',v} \colon v \in \partial A'_{n,k}\big)\,, \qquad v \in A'_{n,k}.$$

As soon as N is sufficiently large, the domains $A := B(N')$, $B := A'_{n,k}$ and $\tilde{B} := A_{n,k}$ obey condition (4.3.10) with some $\delta \geq 1$ for all $n \geq 1$ and all $k \in \{1, \dots, n-1\}$. Corollary 4.3.8 then gives

$$\mathbb{P}\Big(\max_{u,v \in A_{n,k}} \big|\tilde{\chi}_{k,v} - \tilde{\chi}_{k,u}\big| > c + t\Big) \leq 2\mathrm{e}^{-\tilde{c}t^2} \tag{4.3.23}$$

for some constants $c, \tilde{c} > 0$ independent of N, n and k. This shows that the oscillation of $\tilde{\chi}_k$ on $A_{n,k}$ has a uniform Gaussian tail, so in order to bound $M_{n,k} - \tilde{\chi}_{k,v} = \tilde{\chi}_{k,0} - \tilde{\chi}_{k,v}$ uniformly for $v \in A_{n,k}$, it suffices to show that, for just one $v \in A_{n,k}$, also $\tilde{\chi}_{k,v} - \tilde{\chi}_{k,0}$ has such a tail. Since this random variable is a centered Gaussian, it suffices to estimate its variance. Here (4.3.22) gives

$$\mathrm{Var}\big(\tilde{\chi}_{k,v} - \tilde{\chi}_{k,0}\big) \leq \mathrm{Var}\big(\tilde{\chi}_{k-1,v} - \tilde{\chi}_{k-1,0}\big). \tag{4.3.24}$$

Corollary 4.3.8 can now be applied with $A := B(N')$, $B := B(b^{k+1}N)$ and $\tilde{B} := B(\lfloor r_2 b^k N \rfloor)$ to bound the right-hand side by a constant uniformly in N, n and $k = 1, \dots, n-1$. Combined with (4.3.23), the union bound shows

$$\mathbb{P}\Big(\max_{v \in A_{n,k}} \big|\tilde{\chi}_{k,v} - M_{n,k}\big| > c' + t\Big) \leq 2\mathrm{e}^{-\tilde{c}' t^2}$$

with $c', \tilde{c}' \in (0, \infty)$ independent of N, n and k. Another use of the union bound now yields (4.3.21), thus proving the claim. $\square$

4.3.4 A non-Gibbsian decomposition of GFF on a square

As a final item of concern in this section we note that, apart from the Gibbs-Markov property, our proofs will also make use of another decomposition of the GFF which is based on a suitable decomposition of the Green function. This decomposition will be of crucial importance for the development of the RSW theory in Section 4.4.

Lemma 4.3.12. *Let $\{\chi_{N,v}\}_{v\in B(N)}$ be the GFF on $B(N)$ with Dirichlet boundary condition. Then there are two independent, centered Gaussian fields $\{Y_{N,}\}_{v\in B(N)}$ and $\{Z_{N,v}\}_{v\in B(N)}$ such that the following hold:*

(a) $\chi_N = Y_N + Z_N$ *a.s.*

(b) $\mathrm{Var}(Y_{N,v}) = O(\log\log N)$ *uniformly for all* $v \in B(N)$.

(c) $\mathrm{Var}(Z_{N,v} - Z_{N,v}) = O(1/\log N)$ *uniformly for all* $u, v \in B(\lceil N/2\rceil)$ *such that* $u \sim v$.

The distribution of $\{Z_{v,N}\}_{v\in B(N)}$ is invariant under reflections and rotations that preserve $B(N)$.

Proof. Throughout the proof of the current lemma, we let $\{S_t : t \geq 0\}$ be the *lazy* discrete-time simple symmetric random walk on $\mathbb{Z}^2$ that, at each time, stays put at its current position with probability 1/2, or transitions to a uniformly chosen neighbor with the complementary probability. We denote by P^v the law of the walk with $P^v(S_0 := v) = 1$ and write E^v to denote the expectation with respect to P^v. Let τ be the first hitting time to the boundary $\partial B(N)$. It is clear that

$$\mathbb{E}(\chi_{N,v}\chi_{N,u}) = \frac{1}{2}\sum_{t=0}^{\infty} P^v(S_t = u, \tau \geq t)\,.$$

In addition, thanks to laziness of S_t, the matrix $(P^v(S_t = u, \tau \geq t))_{u,v\in B(N)}$ is non-negative definite for each $t \geq 0$. Therefore, there are independent centered Gaussian fields $\{Y_{N,v} : v \in$

$B(N)\}$ and $\{Z_{N,v}\colon v \in B(N)\}$ such that

$$\mathbb{E}(Y_{N,v}Y_{N,u}) = \frac{1}{2}\sum_{t=0}^{\lfloor \log N\rfloor^2} P^v(S_t = u, \tau \geq t)$$

and

$$\mathbb{E}(Z_{N,v}Z_{N,u}) = \frac{1}{2}\sum_{t=\lfloor \log N\rfloor^2+1}^{\infty} P^v(S_t = u, \tau \geq t)\,.$$

At this point, it is clear that we can couple the processes together so that Property (a) holds. Property (b) holds by crude computation which shows

$$\mathrm{Var}Y_{N,v} \leq \sum_{t=0}^{\lfloor \log N\rfloor^2} P^v(S_t = v) \leq O(1)\sum_{t=0}^{\lfloor \log N\rfloor^2} \frac{1}{t+1} = O(\log\log N)\,. \tag{4.3.25}$$

It remains to verify Property (c). For any $u, v \in B(\lceil N/2\rceil)$ and $u \sim v$, we have that

$$\begin{aligned}
|\mathbb{E}Z_{N,v}^2 - &\mathbb{E}Z_{N,v}Z_{N,u}| \\
&= \Big|\sum_{t=\lfloor \log N\rfloor^2+1}^{\infty} P^v(S_t = v, \tau \geq t) - \sum_{t=\lfloor \log N\rfloor^2+1}^{\infty} P^v(S_t = u, \tau \geq t)\Big| \\
&\leq \sum_{t=\lfloor \log N\rfloor^2+1}^{\infty} \Big|P^v(S_t = v) - P^v(S_t = u)\Big| + \sum_{t=0}^{\infty} E^v\Big|\,P^{S_\tau}(S_t = v) - P^{S_\tau}(S_t = u)\Big|\,.
\end{aligned} \tag{4.3.26}$$

Since

$$\Big|P^v(S_t = v) - P^v(S_t = u)\Big| = O(n^{-3/2})\,,$$

(see, e.g., [53, Exercise 2.2]), the first term on the right hand side is bounded by $O(1/\log N)$. The second term is $O(1/N)$ by [53, Theorem 4.4.6] and the fact that $u \in B(\lceil N/2\rceil)$. This completes the verification of Property (c). □

4.4 A RSW result for effective resistances

Having dispensed with preliminary considerations, we now ready to develop a RSW theory for effective resistances across rectangles. Throughout we write, for $N, M \geq 1$,

$$B(N,M) := \big([-N,N] \times [-M,M]\big) \cap \mathbb{Z}^2$$

for the rectangle of $(2N+1) \times (2M+1)$ vertices centered at the origin. Recall that $B(N,N) = B(N)$. The principal outcome of this section are Corollary 4.4.3 and Proposition 4.4.11. In Corollary 4.4.18, these yield the proof of one half of Theorem 4.1.1. The proof of the other half comes only at the very end of the chapter (in Section 4.5).

4.4.1 Effective resistance across squares

In Bernoulli percolation, the RSW theory is a loose term for a collection of methods for extracting uniform lower bounds on the probability that any rectangle of a given aspect ratio is crossed by an occupied path along its longer dimension. The starting point is a duality-based lower bound on the probability of a left-right crossing of a square. In the present context, the crossing probability is replaced by resistance across a square and duality by consideration of a reciprocal network. An additional complication is that our problem is intrinsically spatially-inhomogeneous. This means that all symmetry arguments, such as rotations and reflections, require special attention to where the underlying domain is located. In particular, it will be advantageous to work with the GFF on finite squares instead of the pinned field in all of $\mathbb{Z}^2$.

If S is a rectangular domain in $\mathbb{Z}^2$, we will write $\partial_{\text{left}} S$, $\partial_{\text{down}} S$, $\partial_{\text{right}} S$ and $\partial_{\text{up}} S$ to denote the sets of vertices in S that have a neighbor in $\mathbb{Z}^2 \smallsetminus S$ to the left, down, right and up of them, respectively. (Notice that, unlike ∂S, these "boundaries" are subsets of S.) Given any field $\chi = \{\chi_v\}_{v \in S}$ recall that S_χ denotes the network on S associated with χ. We then abbreviate

$$R_{\mathrm{LR};S,\chi} := R_{S_\chi}\big(\partial_{\mathrm{left}} S, \partial_{\mathrm{right}} S\big)$$

and

$$R_{\mathrm{UD};S,\chi} := R_{S_\chi}\big(\partial_{\mathrm{up}} S, \partial_{\mathrm{down}} S\big).$$

Our first estimate concerning these quantities is:

Proposition 4.4.1 (Duality lower bound)**.** *Let χ_M denote the* GFF *on $B(M)$ with Dirichlet boundary conditions. There is $\hat c = \hat c(\gamma) \in (0,\infty)$ and for each $\epsilon > 0$ there is $N_0 = N_0(\epsilon,\gamma)$ such that for all $N \geq N_0$ and all $M \geq 2N$,*

$$\mathbb{P}\Big(R_{\mathrm{LR};B(N),\chi_M} \leq \mathrm{e}^{\hat c \log\log M}\Big) \geq \frac{1}{2} - \epsilon\,. \tag{4.4.1}$$

The same result holds also for $R_{\mathrm{UD};B(M),\chi_M}$, which is equidistributed to $R_{\mathrm{LR};B(N),\chi_M}$.

The proof requires some elementary observations that will be useful later as well:

Lemma 4.4.2. *Let A be a finite subset of $\mathbb{Z}^2$ and $\chi_1 = \{\chi_{1,v}\}_{v\in A}$, $\chi_2 = \{\chi_{2,v}\}_{v\in A}$ be two random fields on A. Then for any $u,v \in A$ we have,*

$$R_{A_{\chi_1+\chi_2}}(u,v) \leq R_{A_{\chi_1}}(u,v) \max_{\substack{u',v'\in A\\ u'\sim v'}} \mathrm{e}^{-\gamma(\chi_{2,u'}+\chi_{2,v'})}\,. \tag{4.4.2}$$

Furthermore,

$$\mathbb{E}\big(R_{A_{\chi_1+\chi_2}}(u,v)\,\big|\,\chi_1\big) \leq R_{A_{\chi_1}}(u,v) \max_{\substack{u',v'\in A\\ u'\sim v'}} \mathbb{E}\big(\mathrm{e}^{-\gamma(\chi_{2,u'}+\chi_{2,v'})}\,\big|\,\chi_1\big) \tag{4.4.3}$$

and

$$\mathbb{E}\big(C_{A_{\chi_1+\chi_2}}(u,v)\,\big|\,\chi_1\big) \leq C_{A_{\chi_1}}(u,v) \max_{\substack{u',v'\in A\\ u'\sim v'}} \mathbb{E}\big(\mathrm{e}^{\gamma(\chi_{2,u'}+\chi_{2,v'})}\,\big|\,\chi_1\big)\,. \tag{4.4.4}$$

Proof. Let θ be a unit flow from u to v. Then (4.2.1) implies

$$R_{A_{\chi_1+\chi_2}}(u,v) \le \sum_{u',v'\in A, u'\sim v'} [\theta_{(u',v')}]^2 \mathrm{e}^{-\gamma(\chi_{1,u'}+\chi_{1,v'})} \mathrm{e}^{-\gamma(\chi_{2,u'}+\chi_{2,v'})}\,.$$

Hereby (4.4.2) follows by bounding the second exponential by its maximum over all pairs of nearest neighbors in A and optimizing over θ. The estimate (4.4.3) is obtained similarly; just take the conditional expectation before optimizing over θ. The proof of (4.4.4) exploits the similarity between (4.2.1) and (4.2.7) and is thus completely analogous. □

Proof of Proposition 4.4.1. Our aim is to use the fact that, in any Gaussian network, the resistances are equidistributed to the conductances. We will apply this in conjunction with the estimate in Lemma 4.2.7. Unfortunately, this estimate requires a hard bound on the maximal ratio of resistances at neighboring edges. These ratios would be undesirably too large if we work with the GFF network directly; instead we will invoke the decomposition of χ_M into the sum of Gaussian fields $Y_M = \{Y_{M,v}\}_{v\in B(N)}$ and $Z_M = \{Z_{M,v}\}_{v\in B(N)}$ as stated in Lemma 4.3.12 and apply Lemma 4.2.7 to the network associated with Z_M only.

We begin by estimating the oscillation of Z_M across neighboring vertices. From property (c) in the statement of Lemma 4.3.12 and a standard bound on the expected maximum of centered Gaussians, we first get

$$\sup_{N\ge 1} \mathbb{E}\Big(\max_{\substack{u,v\in B(N)\\ |u-v|_1\le 2}} (Z_{M,u} - Z_{M,v})\Big) < \infty\,.$$

Using this bound and property (c), Lemma 4.3.1 shows that for each $\epsilon > 0$ there is $c_1 \in \mathbb{R}$ such that for all $N \ge 1$,

$$\mathbb{P}\Big(\max_{\substack{u,v\in B(N)\\ |u-v|_1\le 2}} (Z_{M,u} - Z_{M,v}) \ge c_1\Big) \le \epsilon\,. \tag{4.4.5}$$

Now observe that the pairs $(\partial_{\mathrm{left}} B(N), \partial_{\mathrm{right}} B(N))$ and $(\partial_{\mathrm{up}} B(N), \partial_{\mathrm{down}} B(N))$ satisfy the conditions of Lemma 4.2.7. Using $R^\star_{\mathrm{UD};B(N),Z_M}$ to denote the top-to-bottom resistance in the reciprocal network, combining (4.2.25) with the last display yields

$$\mathbb{P}\Bigl(R_{\mathrm{LR};B(N),Z_M}R^\star_{\mathrm{UD};B(N),Z_M} \le 64\mathrm{e}^{2c_1\gamma}\Bigr) \ge 1-\epsilon\,. \tag{4.4.6}$$

A key point of the proof is that, since the law of Z_M is symmetric with respect to rotations of $B(M)$, the fact that $Z_M \overset{\text{law}}{=} -Z_M$ implies

$$R^\star_{\mathrm{UD};B(N),Z_M} \overset{\text{law}}{=} R_{\mathrm{LR};B(N),Z_M}.$$

The union bound then shows

$$\mathbb{P}\Bigl(R_{\mathrm{LR};B(N),Z_M} \le 8\mathrm{e}^{c_1\gamma}\Bigr) \ge \frac{1-\epsilon}{2}\,. \tag{4.4.7}$$

Lemma 4.4.2 and the independence of Y_M and Z_M now give

$$\mathbb{E}\bigl(R_{\mathrm{LR};B(N),\chi_M}\,\big|\,Z_M\bigr) \le R_{\mathrm{LR};B(N),Z_M} \max_{\substack{u,v\in B(N)\\ u\sim v}} \mathbb{E}\mathrm{e}^{-\gamma(Y_{M,u}+Y_{M,v})}. \tag{4.4.8}$$

Lemma 4.3.12 shows $\mathrm{Var}Y_{M,v} \le c'\log\log M$ for some constant $c' \in (0,\infty)$ and so the maximum on the right of (4.4.8) is at most $\mathrm{e}^{2c'\gamma^2\log\log M}$. Taking $\hat{c} > 2c'\gamma^2$, the desired bound (4.4.1) now follows (for N sufficiently large) from (4.4.7–4.4.8) and Markov's inequality. □

With only a minor amount of additional effort, we are able to conclude a uniform *lower* bound for the resistance across rectangles.

Corollary 4.4.3. *Let $\hat{c}$ be as in Proposition 4.4.1. For each $\epsilon > 0$ there is $N_0' = N_0'(\gamma,\epsilon)$ such that for all $N \ge N_0'$, all $M \ge 16N$ and all translates S of $B(4N,N)$ contained in $B(M/2)$, we have*

$$\mathbb{P}\bigl(R_{\mathrm{LR};S,\chi_M} \ge \mathrm{e}^{-2\hat{c}\log\log M}\bigr) \ge \frac{1}{2}-\epsilon\,. \tag{4.4.9}$$

The same applies to $R_{\mathrm{UD};S,\chi_M}$ *for any translate* S *of* $B(N,4N)$ *contained in* $B(M/2)$.

Proof. Replacing effective resistances by effective conductances in the proof of Proposition 4.4.1 (and relying on Lemma 4.2.6 instead of Lemma 4.2.7) yields

$$\mathbb{P}\Big(R_{\mathrm{LR};B(N),\chi_M} \geq \mathrm{e}^{-\hat{c}\log\log M}\Big) \geq \frac{1}{2} - \epsilon \tag{4.4.10}$$

for all $N \geq N_0$. Since

$$R_{\mathrm{LR};B(4N),\chi_M} \leq R_{\mathrm{LR};B(4N,N),\chi_M}$$

this bound extends to the rectangle $B(4N,N)$. Now consider a translate S of this rectangle that is contained in $B(M/2)$. Taking $M' := 8N$ and let $\tilde{S}$ be the translate of $B(M')$ that is centered at the same point as S. Considering the Gibbs-Markov decomposition into a fine field $\chi^f_{\tilde{S}}$ and a coarse field $\chi^c_{\tilde{S}}$ on $\tilde{S}$, we then get

$$\begin{aligned}\mathbb{P}\Big(&R_{\mathrm{LR};S,\chi_M} \geq \mathrm{e}^{\tilde{c}\gamma}\mathrm{e}^{-\hat{c}\log\log M}\Big)\\ &\geq \mathbb{P}\Big(R_{\mathrm{LR};S,\chi^f_{\tilde{S}}} \geq \mathrm{e}^{-\hat{c}\log\log M'}\Big) - \mathbb{P}\Big(\max_{u\in S}|\chi^c_{\tilde{S},u}| \leq \tilde{c}\Big).\end{aligned}$$

Since S and $\tilde{S}$ are centered at the same point, the first probability is at least $\frac{1}{2} - \epsilon$ by our extension of (4.4.10) to rectangles. The second probability can be made arbitrarily small uniformly in N by taking $\tilde{c}$ large. The claim follows. □

Remark 4.4.4. Despite our convention that constants such as $c, \tilde{c}, c'$, etc may change meaning line to line, the constant $\hat{c}$ will denote the quantity from Proposition 4.4.1 throughout the rest of this chapter.

4.4.2 Restricted resistances across squares

As noted already in the introduction, our approach to the RSW theory is strongly inspired by [77] which is itself based on inductively controlling the crossing probability (in Bernoulli

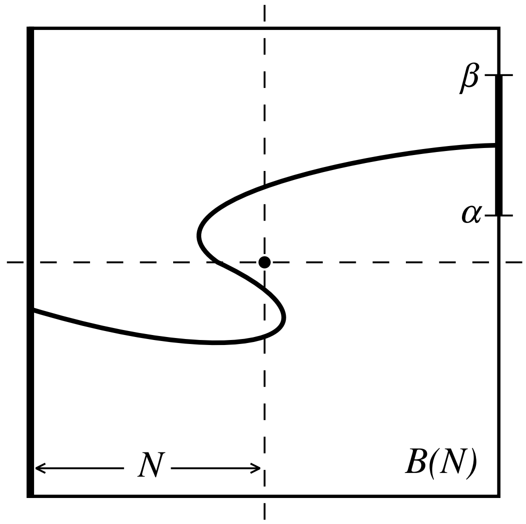

Figure 4.1: **An illustration of the geometric setting underlying the definition of the restricted effective resistance $R_{N,[\alpha,\beta],\chi}$ in (4.4.11).**

percolation) between $\partial_{\text{left}} B(N)$ and a *portion* of $\partial_{\text{right}} B(N)$. We will now setup the relevant objects and notations and prove estimates that will later serve in an argument by contradiction.

For the square $B(N)$ and $\alpha, \beta \in [-N, N] \cap \mathbb{Z}$ with $\alpha \leq \beta$, consider the subset of $\partial_{\text{right}} B(N)$ defined by

$$\partial_{\text{right}}^{[\alpha,\beta]} B(N) := (\{N\} \times [\alpha, \beta]) \cap \mathbb{Z}^2$$

Let $\mathcal{P}_{N;[\alpha,\beta]}$ denote the set of paths in $B(N)$ that use only the vertices in $((-N, N) \times [-N, N]) \cap \mathbb{Z}^2$ except for the initial vertex, which lies in $\partial_{\text{left}} B(N)$, and the terminal vertex, which lies in $\partial_{\text{right}}^{[\alpha,\beta]} B(N)$. With these notions in place, we now introduce the shorthand

$$R_{N,[\alpha,\beta],\chi} := R_{B(N)_\chi}\big(\mathcal{P}_{N;[\alpha,\beta]}\big) = R_{B(N)_\chi}\big(\partial_{\text{left}} B(N), \partial_{\text{right}}^{[\alpha,\beta]} B(N)\big). \tag{4.4.11}$$

Our first goal is to define a quantity α_N which will mark, in rough terms, the point of transition of $\alpha \mapsto R_{N,[0,\alpha],\chi_{2N}}$ from large to small values.

We first need a couple of simple observations. Note that $\mathcal{P}_{N;[0,N]} \cup \mathcal{P}_{N;[-N,0]}$ includes all paths starting on $\partial_{\text{left}} B(N)$ and terminating on $\partial_{\text{right}} B(N)$. Lemma 4.2.5 then shows

$$\frac{1}{R_{\text{LR};B(N),\chi_{2N}}} \le \frac{1}{R_{N,[0,N],\chi_{2N}}} + \frac{1}{R_{N,[-N,0],\chi_{2N}}}$$

while the symmetry of both the law of χ_{2N} and the square $B(N)$ with respect to the reflection through the x axis implies $R_{N,[0,N],\chi_{2N}} \overset{\text{law}}{=} R_{N,[-N,0],\chi_{2N}}$. By Proposition 4.4.1, there is N_0 such that

$$\mathbb{P}\big(R_{\text{LR};B(N),\chi_{2N}} > \mathrm{e}^{\hat{c}\log\log(2N)}\big) \le 2/3$$

as soon as $N \ge N_0$. The square-root trick in Corollary 4.3.5 then shows

$$\mathbb{P}\big(R_{N,[0,N],\chi_{2N}} > 2\mathrm{e}^{\hat{c}\log\log(2N)}\big) \le \sqrt{2/3} < 0.82 \tag{4.4.12}$$

as soon as $N \ge N_0$.

Next we note that, by Lemma 4.3.7,

$$\sup_{N\ge1} \max_{\substack{v\in B(3N/2)\\ u\sim v}} \text{Var}(\chi_{2N,v} - \chi_{2N,u}) < \infty.$$

Hence, there is $C' \in (0,\infty)$ such that $\chi := \chi_{2N}$ obeys

$$\max_{v\in B(N)} \mathbb{P}\Big(\max\{\chi_{v-e_2} - \chi_{v+e_1}, \chi_{v-e_2+e_1} + \chi_{v-e_2} - \chi_v - \chi_{v+e_1}\} \ge C'\Big) \le 0.005 \tag{4.4.13}$$

for all $N \ge 1$. Now set $C_1 := 2(2\mathrm{e}^{C'\gamma} + 1)$, define $\phi_N \colon \{0,\dots,N\} \to [0,1]$ by

$$\phi_N(\alpha) := \mathbb{P}\big(R_{N,[\alpha,N],\chi_{2N}} > (4 + C_1)\,\mathrm{e}^{\hat{c}\log\log(2N)}\big)$$

and, noting that $\alpha \mapsto \phi_N(\alpha)$ is non-decreasing with $\phi_N(0) < 0.82$ (cf (4.4.12)), let

$$\alpha_N := \begin{cases} \min\bigl\{\alpha \in \{0,\dots,\lfloor N/2\rfloor\} \colon \phi_N(\alpha) > 0.99\bigr\} & \text{if } \phi_N(\lfloor N/2\rfloor) > 0.99, \\ \lfloor N/2\rfloor, & \text{otherwise}\,. \end{cases}$$

This definition implies the following inequalities:

Lemma 4.4.5. *For C' as in* (4.4.13)*, define $C_2 := 4(2\mathrm{e}^{C'\gamma}+1)^2$ and let $\hat{c}$, N_0 and C_1 be as above. Then the following two properties hold for all $N \ge N_0$:*

(P1) *For all $\alpha \in \{0,\dots,\alpha_N\}$,*

$$\mathbb{P}\bigl(R_{N,[\alpha,N],\chi_{2N}} \le 5C_2\mathrm{e}^{\hat{c}\log\log(2N)}\bigr) \ge 0.005. \tag{4.4.14}$$

(P2) *If $\alpha_N < \lfloor N/2\rfloor$, then for all $\alpha \in \{\alpha_N,\dots,N\}$,*

$$\mathbb{P}\bigl(R_{N,[\alpha,N],\chi_{2N}} \ge (4+C_1)\,\mathrm{e}^{\hat{c}\log\log(2N)}\bigr) > 0.99 \tag{4.4.15}$$

and

$$\mathbb{P}\bigl(R_{N,[0,\alpha],\chi_{2N}} \le 4\mathrm{e}^{\hat{c}\log\log(2N)}\bigr) \ge 0.17\,. \tag{4.4.16}$$

Proof. We begin with (P1). Since $\phi_N(\alpha) \le 0.99$ for $\alpha \in \{0,\dots,\alpha_N-1\}$, for all such α we have

$$\mathbb{P}\bigl(R_{N,[\alpha,N],\chi_{2N}} \le (4+C_1)\,\mathrm{e}^{\hat{c}\log\log(2N)}\bigr) \ge 0.01\,. \tag{4.4.17}$$

In order to deal with $\alpha = \alpha_N$, we will will need:

Lemma 4.4.6. *For $\chi := \chi_{2N}$ and v being the point with coordinates $(N-1,\alpha_N)$, we have*

$$\begin{aligned} &\bigl\{R_{N,[\alpha_N,N],\chi_{2N}} > C_1 R_{N,[\alpha_N-1,N],\chi_{2N}}\bigr\} \\ &\subseteq \Bigl\{\max\bigl\{\chi_{v-e_2}-\chi_{v+e_1},\, \chi_{v-e_2+e_1}+\chi_{v-e_2}-\chi_v-\chi_{v+e_1}\bigr\} \ge C'\Bigr\}. \end{aligned} \tag{4.4.18}$$

Deferring the proof of this lemma until after this proof, we now combine (4.4.17) for $\alpha := \alpha_N - 1$ with (4.4.13) to get

$$\begin{aligned}\mathbb{P}\Big(R_{N,[\alpha_N,N],\chi_{2N}} \le\ &(4+C_1)C_1\mathrm{e}^{\hat{c}\log\log(2N)}\Big)\\ &\ge \mathbb{P}\Big(R_{N,[\alpha_N-1,N],\chi_{2N}} \le (4+C_1)\,\mathrm{e}^{\hat{c}\log\log(2N)},\ R_{N,[\alpha_N,N],\chi_{2N}} \le C_1 R_{N,[\alpha_N-1,N]}\Big)\\ &\ge 0.01 - 0.005 = 0.005\,.\end{aligned} \tag{4.4.19}$$

Since $(4+C_1)C_1 \le 5C_2$, the bound (4.4.14) holds for $\alpha := \alpha_N$ as well. Thanks to the upward monotonicity of $\alpha \mapsto R_{N,[\alpha,N],\chi_{2N}}$, the inequality then extends to all $\alpha \le \alpha_N$.

The first inequality in (P2) evidently holds by our choice of α_N. As for the second inequality, Lemma 4.2.5 shows

$$\frac{1}{R_{N,[0,N],\chi_{2N}}} \le \frac{1}{R_{N,[0,\alpha],\chi_{2N}}} + \frac{1}{R_{N,[\alpha,N],\chi_{2N}}}$$

and this then implies

$$\begin{aligned}\big\{R_{N,[0,N],\chi_{2N}} \le 2\mathrm{e}^{\hat{c}\log\log(2N)}, R_{N,[\alpha,N],\chi_{2N}} > (4+C_1)\,\mathrm{e}^{\hat{c}\log\log(2N)}\big\}&\\ \subseteq \big\{R_{N,[0,\alpha],\chi_{2N}} \le 4\mathrm{e}^{\hat{c}\log\log(2N)}\big\}&.\end{aligned}$$

Invoking (4.4.12) and the definition of α_N, the probability of the event on the right is than at most $0.99 - 0.82 = 0.17$. □

We still owe to the reader:

Proof of Lemma 4.4.6. Suppose χ is such that the complementary event to that on the right of (4.4.18) occurs. We will show that then the complement of the event on the left occurs as well. For this, let θ be the optimal flow realizing the effective resistivity in (4.4.11) and let $\theta(x,y)$ denote its value on edge (x,y). To reduce clutter of indices, write $r(x,y)$ for the resistance of edge (x,y). Abbreviate $t := v + e_1$, $u := v - e_2$ and $w := u + e_1 =$

$(N, \alpha_N - 1)$. Our aim is to reroute $\theta(v,t)$ through u to w. Define a flow $\tilde{\theta}$ by setting $\tilde{\theta}(v,u) := \theta(v,u) + \theta(v,t)$, $\tilde{\theta}(u,w) := \theta(u,w) + \theta(v,t)$ and $\tilde{\theta}(v,t) := 0$ and letting $\tilde{\theta}_e := \theta_e$ for all other edges e. The only edges where $\tilde{\theta}$ might expend more energy than θ are the edges (v,u) and (u,w). To bound the change in energy, we note

$$\begin{aligned} r(v,u)\tilde{\theta}(v,u)^2 &\leq r(v,u)\big[\theta(v,u) + \theta(v,t)\big]^2 \\ &\leq 2r(v,u)\theta(v,u)^2 + 2r(v,t)\,\mathrm{e}^{C'\gamma}\theta(v,t)^2 \end{aligned} \tag{4.4.20}$$

with the second inequality due to the containment in the complement of the event on the right of (4.4.18). Similarly we have

$$r(u,w)\tilde{\theta}(u,w)^2 \leq 2r(u,w)\theta(u,w)^2 + 2r(v,t)\,\mathrm{e}^{C'\gamma}\theta(v,t)^2.$$

Hence we get $R_{N,[\alpha_N-1,N],\chi_{2N}} \leq (2 + 4\mathrm{e}^{C'\gamma})R_{N,[\alpha_N,N],\chi_{2N}} = C_1 R_{N,[\alpha_N,N],\chi_{2N}}$, thus proving (4.4.18). □

4.4.3 From squares to rectangles

We now move to bounds on resistance across rectangular domains. As in Bernoulli percolation, a fundamental tool in this endeavor is the FKG inequality which, in our case, will be used in the following form:

Lemma 4.4.7. *Consider a finite $S \subseteq \mathbb{Z}^2$ and a Gaussian process $\{\chi_v\}_{v\in\mathcal{R}}$ with $\mathrm{Cov}(\chi_u, \chi_v) \geq 0$ for all $u, v \in S$. Suppose that $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$ are collections of paths in S that satisfy the conditions of Lemma 4.2.4 for a pair of disjoint subsets (A, B) of S. Then for any $r > 0$, we have*

$$\mathbb{P}\big(R_{S_\chi}(A,B) \leq nr\big) \geq \prod_{i=1}^{n} \mathbb{P}\big(R_{S_\chi}(\mathcal{P}_i) \leq r\big).$$

Proof. This is an immediate consequence of Lemma 4.2.4, the monotonicity of $R_{S_\chi}(\mathcal{P}_i)$ in individual edge resistances, and the FKG inequality in Lemma 4.3.4. □

The principal outcome of this subsection is:

Proposition 4.4.8. *There are $c_0, C_3 \in (0,\infty)$ such that for all $N \ge N_0$ for which $\alpha_N \le 2\alpha_{\lfloor 4N/7 \rfloor}$ holds, all $M \ge 8N$ and any shift S of $B(4N,N)$ satisfying $S \subseteq B(M/2)$,*

$$\mathbb{P}\bigl(R_{\mathrm{LR};S,\chi_M} \le C_3 \mathrm{e}^{\hat{c}\log\log M}\bigr) \ge c_0\,. \tag{4.4.21}$$

The same applies to $R_{\mathrm{UD};S,\chi_M}$ for any shift S of $B(N,4N)$ that obeys $S \subseteq B(M/2)$.

By Proposition 4.4.1 the bound holds for left-to-right resistance of centered squares. We will employ a geometric argument combined with the FKG inequality to extend the bound from squares to rectangular domains. The main technical tool is Lemma 4.2.4 which, in a sense, permits us to bound resistance by path-connectivity considerations only. We will actually use a different argument depending on whether α_N equals, or is less than $\lfloor N/2 \rfloor$.

Proof of Proposition 4.4.8, case $\alpha_N = \lfloor N/2 \rfloor$. Here we will need the bound (4.4.14), but for the underlying domain not necessarily centered at the box which defines the underlying field. Thus, for S a translate of the square $B(N)$ such that $S \subseteq B(M/2)$, let $R_{S,[\alpha,\beta],\chi_M}$ denote the quantity corresponding to $R_{N,[\alpha,\beta],\chi_M}$ for the square S and the underlying field given by χ_M. In light of (4.4.14), Corollary 4.3.8 and Lemma 4.4.7 show that, for some constant $C_3' \in (0,\infty)$ depending only on C_1 and C_2,

$$\mathbb{P}\bigl(R_{S,[\alpha_N,N],\chi_M} \le C_3' \mathrm{e}^{\hat{c}\log\log M}\bigr) \ge 0.001 \tag{4.4.22}$$

holds for all $N \ge N_0$, all $M \ge 8N$ and all squares S as above that are contained in $B(M/2)$. Thanks to invariance of the law of χ_M under rotations of $B(M)$, the same bound holds also for the "rotated" quantities; namely, those dealing with "up-down' resistivities.

Now let S be a translate by $x \in \mathbb{Z}^2$ of the rectangle $B(4N,N)$ such that $S \subseteq B(M/2)$

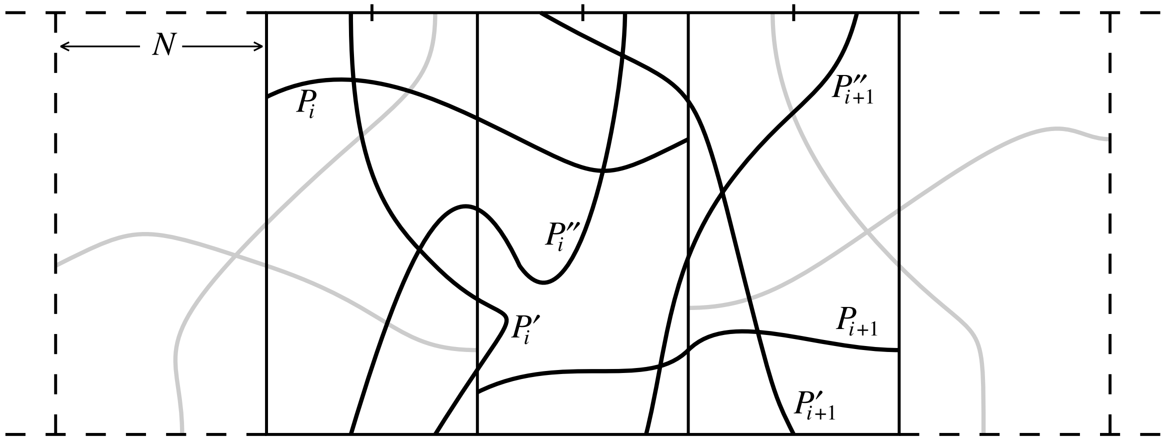

Figure 4.2: **The setting of the proof of Proposition 4.4.8, case $\alpha_N = \lfloor N/2 \rfloor$.** The collection of paths shown suffices to ensure a left-to right crossing through the four shown translates of $B(N)$. The key points to observe are that P_i intersects both P'_i and P''_i while P''_i intersects P'_{i+1}, for each i.

and let us regard S as the union of the squares

$$S_i := x + (i-5)Ne_1 + B(N), \qquad i = 1, \dots, 7.$$

For each $i \in \{1, \dots, 7\}$, consider the following collections of paths: First, let $\mathcal{P}_i$ be the set of all paths in S_i that cross S_i left to right (with only the initial and terminal point visiting the left and right boundaries of S_i). Then (referring to parts of the boundary as if S_i were the square $B(N)$), let $\mathcal{P}'_i$ be the collection of paths that connects the bottom of the square to the $[-N, -\alpha_N]$ portion of the top boundary, and let $\mathcal{P}''_i$ be the path between the bottom of the square to the $[\alpha_N, N]$ portion of the top boundary. The key point (implied by the fact that $\alpha_N = \lfloor N/2 \rfloor$) is now that, for any choice of paths $P_i \in \mathcal{P}_i$, $P'_i \in \mathcal{P}'_i$ and $P''_i \in \mathcal{P}''_i$ and any $i = 1, \dots, 7$, the graph union of the triplet of paths (P_i, P'_i, P''_i) is connected and, for each $i = 1, \dots, 6$, the graph union of (P_i, P'_i, P''_i) is connected to the graph union of $(P_{i+1}, P'_{i+1}, P''_{i+1})$; see Fig. 4.2.

It follows that the graph union of the seven triplets of paths contains a left-to-right crossing of the rectangle S and, by Lemma 4.2.4, we thus get

$$R_{\mathrm{LR};S,\chi_M} \le \sum_{i=1}^{7} \Bigl(R_{S_i,\chi_M}(\mathcal{P}_i) + R_{S_i,\chi_M}(\mathcal{P}_i') + R_{S_i,\chi_M}(\mathcal{P}_i'') \Bigr).$$

In light of the definition (4.4.11) (and, for simplicity of computation, restricting $\mathcal{P}_i$ to paths that terminate only at the top $[\alpha_N, N]$ portion of the right boundary), (4.4.22) and the FKG inequality now give (4.4.21) with $C_3 := 21C_3'$ and $c_0 := 10^{-63}$. $\square$

Proof of Proposition 4.4.8, case $\alpha_N < \lfloor N/2 \rfloor$. Here, in addition to (4.4.15) which, as before, we bring to the form (4.4.22), we will also need (4.4.16) — this is why we need $\alpha_N < \lfloor N/2 \rfloor$ — which we extend using Corollary 4.3.8 and Lemma 4.4.7 to the form

$$\mathbb{P}\bigl(R_{S,[0,\alpha_N],\chi_M} \le C_3'' \mathrm{e}^{\hat{c} \log\log M}\bigr) \ge 0.01 \tag{4.4.23}$$

for some $C_3'' \in (0,\infty)$, all $N \ge N_0$ and all translates S of $B(N)$ such that $S \subseteq B(M/2)$. The same bound holds also for all rotations and reflections of these quantities.

Abbreviate $K := \lfloor 4N/7 \rfloor$ and note that $K < N < 2K$ for N large enough. Let us first deal with S being a translate of the rectangle $([-N, 3N-2K] \times [-N, N]) \cap \mathbb{Z}^2$ by some $x \in \mathbb{Z}^2$ subject to the restriction $S \subseteq B(4K)$. Consider the squares

$$S_1 := x + B(N), \quad S_2 := x + 2(N-K)e_1 + B(N)$$

and

$$S_3 := x + (N-K)e_1 + \alpha_K e_2 + [-K,K]^2 \cap \mathbb{Z}^2$$

and note that $S_1 \cup S_2 = S$ and $S_3 \subseteq S_1 \cap S_2$; see Fig. 4.2. Define the following collections of paths: First, let $\mathcal{P}_1$ be all paths in S_1 from the left side to the $[0, \alpha_N]$ portion of the right side. Similarly, let $\mathcal{P}_2$ be all paths in S_2 from the $[0, \alpha_N]$ portion of the left side to the right side of S_2. Next we define the following collections of paths in S_3:

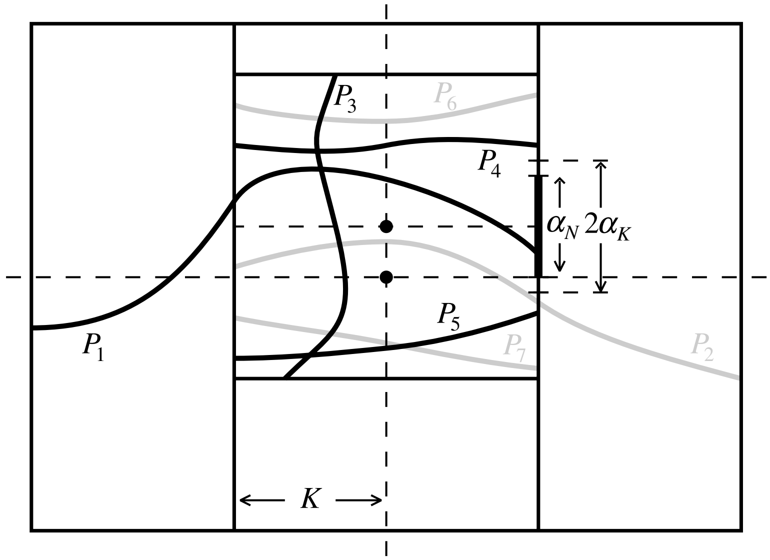

Figure 4.3: **An illustration of the geometric setting underlying the key argument in the proof of Proposition 4.4.8, case** $\alpha_N < \lfloor N/2 \rfloor$**.** Here $K := \lfloor 4N/7 \rfloor$ and $\alpha_N \leq 2\alpha_K$. Examples of paths $P_1 \in \mathcal{P}_1$, $P_3 \in \mathcal{P}_3$, $P_4 \in \mathcal{P}_4$ and $P_5 \in \mathcal{P}_5$ are shown in black. Together with any choice of paths $P_2 \in \mathcal{P}_2$, $P_6 \in \mathcal{P}_6$ and $P_7 \in \mathcal{P}_7$ (shown in gray), these enforce a left-to-right crossing of the rectangle.

(1) the set $\mathcal{P}_3$ of all paths from the top to the bottom sides of S_3,

(2) the set $\mathcal{P}_4$ of all paths from the left side of S_3 to the $[\alpha_K, K]$ portion of the right side,

(3) the set $\mathcal{P}_5$ of all paths from the left side of S_3 to the $[-K, -\alpha_K]$ portion of the right side,

(4) the set $\mathcal{P}_6$ of all paths from the $[\alpha_K, K]$ portion of the left side of S_3 to the right side, and

(5) the set $\mathcal{P}_7$ of all paths from the $[-K, -\alpha_K]$ portion of the left side of S_3 to the right side.

The key point is that, thanks to the assumption $\alpha_N \leq 2\alpha_K$, for any choice of paths $P_i \in \mathcal{P}_i$, the graph union of these paths will contain a left-to-right path crossing S; see Fig. 4.2. By

Lemma 4.2.4,

$$R_{\mathrm{LR},S,\chi_M} \leq \sum_{i=1}^{7} R_{S_i,\chi_M}(\mathcal{P}_i),$$

where $S_4 = \cdots = S_7 := S_3$. From here we get (4.4.21) for all $2(2N-K)\times 2N$ rectangles $S \subseteq B(M/2)$ with $C_3 := 21\max\{C_3', C_3''\}$ and $c_0 := 10^{-14}$.

In order to prove the desired claim, consider a translate S of $B(4N, N)$ by $x \in \mathbb{Z}^2$ entirely contained in $B(M/2)$ and note that, letting $k := \lceil \frac{4N}{N-K} \rceil$, and we can cover S by the family of rectangles $S_0', \dots, S_k'$ and $S_1'', \dots, S_{k-1}''$ defined as follows:

$$S_j' := x_j + \big([0, 2(2N-K)] \times [-N, N]\big) \cap \mathbb{Z}^2, \quad j = 0, \dots, k,$$

where $x_j := x + 2(N-K)je_1$ for all $j = 0, \dots, k-1$ and $x_k := x + [8N - 2k(N-K)]e_1$, which ensures that all S_i' lie inside S (and thus inside $B(M/2)$), and

$$S_j'' := y_j + \big([-N, N] \times [0, 2(2N-K)]\big) \cap \mathbb{Z}^2, \quad j = 1, \dots, k-1,$$

where $y_j - x_j$ are such that all S_j'' lie in $B(M/2)$ (this is possible because $2(2N-K) < 16N$) and such that $S_j' \cap S_j'' \subseteq S_{j+1}'$ for each $j = 1, \dots, k-1$. Assuming each S_j' and S_j'' contains a path connecting the shorter sides of the rectangle, the graph union of these paths then contains a left-to-right crossing of S. Lemma 4.2.4 then gives

$$R_{\mathrm{LR},S,\chi_M} \leq \sum_{j=0}^{k} R_{\mathrm{LR},S_j',\chi_M} + \sum_{j=1}^{k-1} R_{\mathrm{UD},S_j'',\chi_M}.$$

In light of our earlier proof of (4.4.21) for rectangles of dimensions $2N \times 2(2N-K)$, we get (4.4.21) for $2N \times 8N$ rectangles as well with $C_3 := 21(2k+1)\max\{C_3', C_3''\}$ and $c_0 = 10^{-14(2k+1)}$. $\square$

4.4.4 Bounding the growth of α_N

It appears that Proposition 4.4.8 could be more than sufficient for proving uniform upper bound on resistance across rectangles, provided we can somehow guarantee that $N \mapsto \alpha_N$ does not grow faster than exponentially with N. This is the content of:

Proposition 4.4.9. *For each $c_0 \in (0,1)$ and each $C_3 \in (0,\infty)$, there exists an integer $C_5 > 8$ such that if, for some $N \geq 1$,*

$$\mathbb{P}\big(R_{\mathrm{LR};S,\chi_{16N}} \leq C_3 \mathrm{e}^{\hat{c}\log\log(16N)}\big) \geq c_0 \tag{4.4.24}$$

holds all translates or rotates S of $B(4N,N)$ contained in $B(8N)$, then we have $\alpha_{N'} \geq N$ for at least one $N' \in \{8N,\dots,C_5N\}$.

The proof will be based on the following lemma:

Lemma 4.4.10. *Suppose that, for some $c_0, C_3 \in (0,\infty)$ and some $N \geq 1$,* (4.4.24) *holds for all translates and rotates of $B(4N,N)$ contained in $B(8N)$. There are c_1 and C_4, depending only on c_0 and C_3, respectively, such that whenever $K > 2N$ is such that $\alpha_K \leq N$ and $M \geq 16K$,*

$$\mathbb{P}\big(R_{\mathrm{LR},S,\chi_M} \leq C_4 \mathrm{e}^{\hat{c}\log\log M}\big) \geq c_1 \tag{4.4.25}$$

holds for all translates and rotates of $B(4K,K)$ contained in $B(8K)$.

Proof. We will first prove this for rectangles S of the form $B(2K,K)$. Consider the squares $S_1 := -Ke_1 + [-K,K]^2 \cap \mathbb{Z}^2$ and $S_2 := Ke_1 + B(K)$ and let $S_1',\dots,S_4'$ be the four maximal rectangles of dimensions $N \times 4N$, labeled counterclockwise starting from the one at the bottom, contained in the annulus $B(2N) \smallsetminus B(N)^\circ$. Let P_1 be a path in S_1 connecting the left-hand side to the $[0,\alpha_K]$ portion of the right-hand side and, similarly, P_2 is the path in S_2 connecting the $[0,\alpha_K]$-portion of the left-hand side to the right hand side. Let $P_1',\dots,P_4'$ be paths (in $S_1',\dots,S_4'$, respectively) between the shorter sides of $S_1',\dots,S_4'$, respectively.

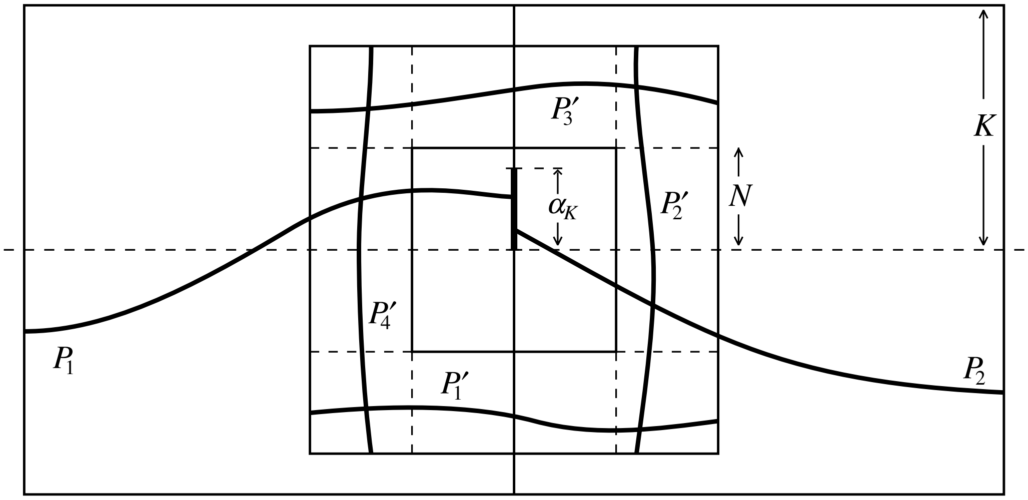

Figure 4.4: **The geometric setup for the proof of Lemma 4.4.10.** The graph union of paths $P_1, P_2, P_1', \dots, P_4'$ contains a left-to-right crossing of the $4K \times 2K$-rectangle.

Then the assumption $\alpha_K \le N$ implies that the graph union of $P_1, P_2, P_1', \dots, P_4'$ contains a path in S connecting the left side to the right side; see Fig. 4.4. Combining (4.4.25) with (4.4.23) (in which N is replaced by K), we get the claim for S with $C_4 := 2C_3'' + 4C_3$ and $c_1 := 10^{-4}(c_0)^4$.

To extend this to rectangles S of the form $B(4K, K)$, we note that these can be covered by four translates and two rotates of $B(2K, K)$ such that the existence of a crossing between the shorter sides in each of these rectangles forces a crossing of S. Thanks to Lemma 4.4.7, the desired bound then holds for S as well; we just need to multiply the above C_4 by 6 and raise the above c_1 to the sixth power. □

We are now ready to give:

Proof of Proposition 4.4.9. The proof is by way of contradiction; indeed, we will prove that if such N' does not exist, then we will ultimately violate the first inequality in (P2) in Lemma 4.4.5 for a sufficiently large square. This will be done by showing that a path from the left side of the square $B(N')$ to the $[0, \alpha_{N'}]$ part of the right side can be re-routed to instead terminate in the $[\alpha_{N'}, N']$-part of the right side. The re-routing will be achieved by showing existence of a path winding around an annulus of inner "radius" at least $\alpha_{N'}$ centered at the point $\varnothing_{N'} := (N', 0)$.

We will focus on N' of the form $N' := b^n N$, where $b := 8$ and $n \geq 1$. Fix such an n (and thus N') and, for $k = 1, \dots, n$, let $B_{n,k} := \varnothing_{N'} + B(b^k N)$. Consider also the annulus $A_{n,k} := \varnothing_{N'} + B(4b^k N) \smallsetminus B(2b^k N)^\circ$ and define the conditional field

$$\chi_{4N',k;v} := \chi_{4N',v} - \mathbb{E}\biggl(\chi_{4N',v} \biggm| \sigma\Bigl(\chi_{4N',u} \colon u \in \bigcup_{n-k \leq j \leq n} \partial B_{n,j}\Bigr)\biggr).$$

By the Gibbs-Markov property of the GFF, $\{\chi_{4N',k;v} \colon v \in A_{n,k}\}$ has the law of the values on $A_{n,k}$ of the GFF in $B(b^{k+1}N) \smallsetminus B(b^k N)^\circ$ with Dirichlet boundary condition. Let $R_{A_{n,k};\chi_{4N',k}}$ denote the sum of the resistances between the shorter sides of the four maximal rectangles contained in $A_{n,k}$, in the field $\chi_{4N',k}$.

Assuming $\alpha_{N'} \leq N$, Lemma 4.4.10 in conjunction with Corollary 4.3.8 and Lemma 4.4.7 show that, for some $C_4' \in (0, \infty)$ and $c_2 > 0$:

$$\mathbb{P}\bigl(R_{A_{n,k};\chi_{4N',k}} \leq C_4' \mathrm{e}^{\hat{c} \log \log N'}\bigr) \geq c_2\,. \tag{4.4.26}$$

Let m be the smallest integer such that $(1 - c_2)^m \leq 0.01$, let C_1 be as in the first inequality in (P2) in Lemma 4.4.5 and let $\widetilde{C}$ be the constant from Lemma 4.3.10. Define

$$\widetilde{M}_{n,k} := \min_{v \in A_{n,k}} \mathbb{E}\biggl(\chi_{4N',v} \biggm| \sigma\Bigl(\chi_{N',u} \colon u \in \bigcup_{n-k \leq j \leq n} \partial B_{j,n}\Bigr)\biggr).$$

Lemma 4.3.10 (dealing with the LIL for the sequence $M_{n,k}$) and Lemma 4.3.11 (dealing with the deviations Δ_n) tell us that there is a positive integer $m' > 100$ satisfying

$$\begin{aligned}\mathbb{P}\biggl(\#\Bigl\{k = 1, \dots, m' - 1 \colon \gamma \widetilde{M}_{k,m'} \geq 0.5 \log \frac{C_4'}{C_1} + \log 5 + \widetilde{C}\gamma\sqrt{\log m'}\Bigr\} < m\biggr)& \\ \leq 0.01 + 0.01 = 0.02\,.&\end{aligned} \tag{4.4.27}$$

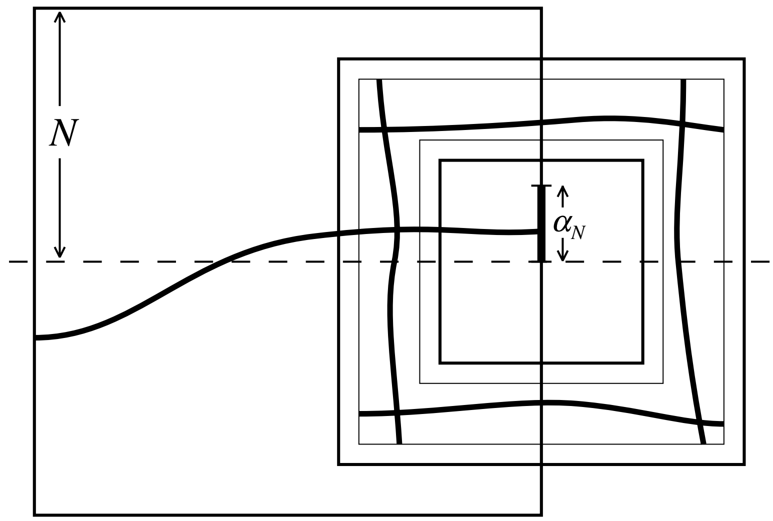

Figure 4.5: **The geometric setting for a key argument in the proof of Proposition 4.4.9.** Once α_N is less than the inner radius of the depicted annulus, $R_{B(N)_\chi}(\mathcal{P}_{N;[\alpha_N,N]})$ is bounded by $R_{B(N)_\chi}(\mathcal{P}_{N;[0,\alpha_N]})$ plus the sum of the resistances between the shorter sides of the four maximal rectangles contained in the annulus.

Putting together (4.4.26), (4.4.27), the choices of m and m' along with Lemmas 4.3.11 and 4.4.2 we get for all N such that $c \log\big(1 + \frac{(m'+1)\log 8}{\log N}\big) \leq \log 5$,

$$\mathbb{P}\big(\exists k \in \{1,\dots,m'\}\colon R_{A_{k,m'};\chi_{C_5 N}} \geq C_1 \mathrm{e}^{\hat{c}\log\log N}\big) \leq 0.02 + 0.01 = 0.03\,, \tag{4.4.28}$$

where $C_5 := 8^{m'+1}$.

We are now ready to derive the desired contradiction. Lemma 4.2.4 gives us that if $\alpha_{N'} \leq N$ for all $8N \leq N' \leq C_5 N$, then

$$\mathbb{P}\Big(R_{B(N')_{\chi_{N'}}}(\mathcal{P}_{N';[\alpha_{N'},N']}) \leq R_{B(N')_{\chi_{N'}}}(\mathcal{P}_{N';[0,\alpha_{N'}]}) + C_1 \mathrm{e}^{\hat{c}\log\log N}\Big) \geq 1 - 0.03 = 0.97\,. \tag{4.4.29}$$

From the second inequality in (P2) in Lemma 4.4.5 we have

$$\mathbb{P}\big(R_{B(N')_{\chi_{N'}}}(\mathcal{P}_{N';[0,\alpha_{N'}]}) \leq 4\mathrm{e}^{\hat{c}\log\log N'}\big) \geq 0.17\,.$$

The last two displays and the FKG imply

$$\mathbb{P}\Big(R_{B(N')_{\chi_{N'}}}(\mathcal{P}_{N';[\alpha_{N'},N']}) \le (4+C_1)\,\mathrm{e}^{\hat{c}\log\log N'}\Big) > 0.17 \times 0.97 > 0.16\,.$$

in contradiction with the first inequality in (P2) in Lemma 4.4.5. The claim follows. □

4.4.5 Resistance across rectangles and annuli

As a consequence of the above arguments, we are now ready to state our first *unrestricted* general upper bound on the effective resistance across rectangles:

Proposition 4.4.11. *There are constants $C_6, c_3 \in (0,\infty)$ and $N_1 \ge 1$ such that for all $N \ge N_1$, all $M \ge 16N$ and for every translate S of $B(4N,N)$ contained in $B(M/2)$, we have*

$$\mathbb{P}\big(R_{\mathrm{LR};S,\chi_M} \le C_6 \mathrm{e}^{\hat{c}\log\log(M)}\big) \ge c_3\,. \tag{4.4.30}$$

The same applies to $R_{\mathrm{UD};S,\chi_M}$ for translates S of $B(N,4N)$ with $S \subseteq B(M/2)$.

We begin by showing that (4.4.24) holds (with the same constants) along an exponentially growing sequence of N. This is where Proposition 4.4.8 and Proposition 4.4.9 come together.

Lemma 4.4.12. *Let c_0 and C_3 be as in Proposition 4.4.8. There is $c \in (0,\infty)$ and an increasing sequence $\{N_k \colon k \ge 1\}$ of positive integers such that, for each $k \ge 1$, we have*

$$14N_k - 1 \le N_{k+1} \le cN_k \tag{4.4.31}$$

and the bound

$$\mathbb{P}\big(R_{\mathrm{LR};S,\chi_{16N_k}} \le C_3 \mathrm{e}^{\hat{c}\log\log(16N_k)}\big) \ge c_0 \tag{4.4.32}$$

holds for all translates S of $B(4N_k,N_k)$ contained in $B(8N_k)$.

Proof. We will construct $\{N_k \colon k \ge 1\}$ by induction. Suppose that $N_1,\dots,N_k$ have already been defined. Since (4.4.32) holds for N_k, Proposition 4.4.9 shows the existence of an $L \in$

$[8N_k, C_5N_k]$ with $\alpha_L \geq N_k$. Define a sequence $\{L_j : j \geq 0\}$ by $L_0 := L$ and $L_{j+1} := \min\{L \in \mathbb{N} : \lfloor 4L/7 \rfloor = L_j\}$ and note that $L_j \leq c(7/4)^j L$ for some numerical constant $c' \in (0,\infty)$. Now if $\alpha_{L_{i+1}} > 2\alpha_{L_i}$ is true for $i = 0, \dots, j-1$, then

$$2^j N_k \leq 2^j \alpha_L < \alpha_{L_j} \leq L_j \leq c'(7/4)^j L \leq c'(7/4)^j C_5 N_k. \tag{4.4.33}$$

The fact that $7/4 < 2$ implies that this must fail once j is sufficiently large; i.e., for some $j \in \{0, \dots, C_5'\}$, where C_5' depends only on C_5. We thus let $j \geq 1$ be the smallest such that $\alpha_{L_j} \leq 2\alpha_{L_{j-1}}$ and set $N_{k+1} := L_j$. Then (4.4.31) holds by the inequality on the right of (4.4.33) and the fact that $N_{k+1} \geq L_1 \geq (7/4)L - 1 \geq 14N_k - 1$. The bound (4.4.32) is implied by Proposition 4.4.8.

To start the induction, we just take the above sequence $\{L_j\}$ with $L := 1$ and find the first index j for which $\alpha_{L_j} \leq 2\alpha_{L_{j-1}}$. Then we set $N_1 := L_j$ and argue as above. □

From here we now conclude:

Proof of Proposition 4.4.11. Let $\{N_k\}$ be the sequence from Lemma 4.4.12. Invoking Corollary 4.3.8 and Lemma 4.4.7, the bound (4.4.32) shows that, for each $M \geq 16N_k$ and any translate S of $B(4N_k, N_k)$ contained in $B(M/2)$,

$$\mathbb{P}\big(R_{\mathrm{LR};S,\chi_M} \leq C_3' \mathrm{e}^{\hat{c}\log\log(M)}\big) \geq c_0'. \tag{4.4.34}$$

holds with some constants $C_3', c_0' \in (0,\infty)$ independent of k and M. By invariance of the law of χ_M with respect to rotations of $B(M)$, the same holds for the resistance $R_{\mathrm{UD};S,\chi_M}$ for all rotations of $B(4N_k, N_k)$ contained in $B(M/2)$.

Now pick $N \geq N_1$ and let k be such that $N_k \leq N < N_{k+1}$. For $M \geq 16N \geq 16N_k$, consider a translate S of $B(4N, N)$ contained in $B(M/2)$. Let $m := \min\{r \in \mathbb{N} : (3r+1)N \geq N_{k+1}\}$; by (4.4.31) this m is bounded uniformly in k. We then find rectangles S_i, $i = 1, \dots, m$ that are translates of $B(4N_k, N_k)$ such that $S_{i+1} = 3Ne_1 + S_i$ for each $i = 1, \dots, m-1$ and are centered along the same horizontal line as S and positioned in such a way that they all lie inside $B(M/2)$. Next we find translates $S'_1, \dots, S'_{m-1}$ of $B(N_k, 4N_k)$ such that $S_i \cap S_{i+1}$, which is a translate of $B(N)$, is contained in S'_i for each $i = 1, \dots, m-1$. We can again position these so that $S'_i \subseteq B(M/2)$ for each i.

It is clear from the construction that if, for each $i = 1, \dots, m$, we are given a path in S_i and, for each $i = 1, \dots, m-1$, a path in S'_i and these paths connect the shorter sides of the rectangle they lie in, then the graph union of all these paths contains a path in S between the left side and right side thereof. Lemma 4.2.4 then gives

$$R_{\mathrm{LR};S,\chi_M} \le \sum_{i=1}^{m} R_{\mathrm{LR};S_i,\chi_M} + \sum_{i=1}^{m-1} R_{\mathrm{UD};S'_i,\chi_M}. \tag{4.4.35}$$

All of the rectangles lie in $B(M/2)$ and so (4.4.34) applies to the resistivities on the right of (4.4.35). Lemma 4.4.7 then readily gives (4.4.32) with $C_6 := (2m-1)C'_3$ and $c_3 := (c'_0)^{2m-1}$. $\square$

In addition to resistance across rectangles, the proofs in Section 5.3 will also require an lower bound for resistances across annuli. For $N < M$, let $A(N, M) := B(M) \smallsetminus B(N)^\circ$ and denote

$$\partial^{\mathrm{in}} A(N, M) := \partial B(N) \quad \text{and} \quad \partial^{\mathrm{out}} A(N, M) := \partial B(M)^\circ$$

Note that $\partial^{\mathrm{in}} A(N, M) \subset A(N, M)$ as well as $\partial^{\mathrm{out}} A(N, M) \subset A(N, M)$. We have:

Lemma 4.4.13. *There $C_7, c_4 \in (0, \infty)$ such that for all N sufficiently large and $A := A(N, 2N)$,*

$$\mathbb{P}\Big(R_{A_{\chi_{4N}}}(\partial^{\mathrm{in}} A, \partial^{\mathrm{out}} A) \ge C_7 \mathrm{e}^{-3\hat{c} \log\log(4N)}\Big) \ge c_4.$$

Proof. Let S_1, S_2, S_3, S_4 denote the four maximal rectangles contained in A. We assume that the rectangles are labeled clockwise starting from the one on the right. Now observe that every path in A from $\partial^{\mathrm{in}} A$ to $\partial^{\mathrm{out}} A$ contains a path that is contained in, and connects the longer sides of, one of the rectangles S_1, S_2, S_3, S_4. It follows that

$$R_{A_{\chi_{4N}}}(\partial^{\text{in}} A, \partial^{\text{out}} A) \geq R_{\text{LR},S_1,\chi_{4N}} + R_{\text{UD},S_2,\chi_{4N}} + R_{\text{LR},S_3,\chi_{4N}} + R_{\text{UD},S_4,\chi_{4N}}\,.$$

The claim will follow from the FKG inequality if we can show that, for some $p > 0$ and $C_7' > 0$,

$$\mathbb{P}\bigl(R_{\text{LR},S,\chi_{4N}} \geq C_7' \mathrm{e}^{-3\hat{c}\log\log(4N)}\bigr) \geq p \tag{4.4.36}$$

holds for all translates S of $([0,N]\times[0,4N])\cap\mathbb{Z}^2$ contained in $B(2N)$ and all N sufficiently large. (Indeed, then $c_4 := p^4$ and $C_7 := 4C_7'$.)

We will show this using the duality in Lemma 4.2.6, but for that we will first need to invoke the decomposition $\chi_{4N} = Y_{4N} + Z_{4N}$ from Lemma 4.3.12. First, for any $r, A > 0$,

$$\mathbb{P}\bigl(R_{\text{LR},S,\chi_{4N}} \geq r\bigr) \geq \mathbb{P}\bigl(R_{\text{LR},S,Z_{4N}} \geq r/A\bigr) - \mathbb{P}\bigl(R_{\text{LR},S,\chi_{4N}} < AR_{\text{LR},S,Z_{4N}}\bigr)$$

Passing over to conductances, from Lemma 4.4.2 we then get, as before,

$$\mathbb{P}\bigl(R_{\text{LR},S,\chi_{4N}} < AR_{\text{LR},S,Z_{4N}}\bigr) \leq \frac{1}{A}\mathrm{e}^{\hat{c}\log\log(4N)},$$

while the duality in Lemma 4.2.6 gives, as in the proof of Proposition 4.4.1,

$$\mathbb{P}\Bigl(R_{\text{LR},S,Z_{4N}}\, R^\star_{\text{UD},S,Z_{4N}} \geq \mathrm{e}^{-2\gamma c_1}/64\Bigr) \geq 1-\epsilon.$$

Finally, we use Lemma 4.4.2 one more time to get

$$\mathbb{P}\bigl(R^\star_{\text{UD},S,Z_{4N}} \leq \tilde{r}\bigr) \geq \mathbb{P}\bigl(R_{\text{UD},S,\chi_{4N}} \leq \tilde{r}/A\bigr) - \frac{1}{A}\mathrm{e}^{\hat{c}\log\log(4N)}.$$

If we set $\tilde{r}/A := C_6 \mathrm{e}^{\hat{c}\log\log(4N)}$, Proposition 4.4.11 bounds the first probability below by c_3. Now take $A := C\mathrm{e}^{3\hat{c}\log\log(4N)}$ for C large and work your way back to get (4.4.36). □

4.4.6 Gaussian concentration and upper bound on point-to-point resistances

In order to get the tail estimate on the effective resistance in Theorem 4.1.1, we need to invoke a concentration-of-measure argument for the quantity at hand. Recall the notation $R_{A_\chi}(\mathcal{P})$ for the effective resistance in network A_χ restricted to the collection of paths in $\mathcal{P}$.

Proposition 4.4.14. *Suppose χ is a Gaussian field on $B(N)$ with $\operatorname{Var}(\chi_x) \le c_1 \log N$ for all $u \in B(N)$ and c_1 independent of N. Let A_χ be a subnetwork of $B(N)_\chi$ and let $\mathcal{P}$ be a finite collection of paths within A between some given source and destination. There is a constant $c_2 \in (0,\infty)$ such that for all $N \ge 1$, all $t \ge 0$ and all $\gamma > 0$,*

$$\mathbb{P}\Big(\big|\log R_{A_\chi}(\mathcal{P}) - \mathbb{E}\log R_{A_\chi}(\mathcal{P})\big| \ge t\sqrt{\log N}\Big) \le 2\mathrm{e}^{-c_2\gamma^{-2}t^2}\,.$$

For the proof, we will need:

Lemma 4.4.15. *Let A be a subnetwork of $B(N)$ and $\mathcal{P}$ be a finite collection of paths within A between some given source and destination. Let $g\colon \mathbb{R}^{V(A)} \to \mathbb{R}$ be defined by*

$$g(\mathbf{x}) := \max_{\mathbf{q}\in\mathcal{Q}} \log\Big(\sum_{P\in\mathcal{P}} \frac{1}{\sum\limits_{e\in P} \mathrm{e}^{-\gamma(x_{e_-}+x_{e_+})} q_{e,P}}\Big)\,,$$

where $\mathcal{Q}$ is the set of all $\mathbf{q} = (q_{e,P})_{e\in E(A),P\in\mathcal{P}} \in \mathbb{R}_+^{E(A)\times\mathcal{P}}$ such that

$$\sum_{P\in\mathcal{P}} \frac{1}{q_{e,P}} \le 1 \text{ for all } e \in E(A)\,.$$

Then g is a Lipschitz function relative to the L_∞ norm on $\mathbb{R}^{V(A)}$ with Lipschitz constant 2γ.

Proof. Define a new real-valued function, also denoted by g, on $\mathbb{R}^{V(A)} \times \mathbb{R}_+^{E(A)\times\mathcal{P}}$ via

$$g(\mathbf{x},\mathbf{q}) := \log\Big(\sum_{P\in\mathcal{P}} \frac{1}{\sum_{e\in P} \mathrm{e}^{-\gamma(x_{e_-}+x_{e_+})} q_{e,P}}\Big).$$

Then for any $\mathbf{q}\in\mathcal{Q}$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}^{V(A)}$ it is clear that

$$|g(\mathbf{x},\mathbf{q}) - g(\mathbf{y},\mathbf{q})| \le 2\gamma||\mathbf{x}-\mathbf{y}||_\infty .$$

Hence $g(\mathbf{x}) = \max_{\mathbf{q}\in\mathcal{Q}} g(\mathbf{x},\mathbf{q})$ is 2γ- Lipschitz relative to the L_∞ norm as well. □

Proof of Proposition 4.4.14. This follows directly from the Gaussian concentration inequality (see [76, 18]) and Lemma 4.4.15. □

We are now ready to give a version of the upper bound in Theorem 4.1.1, albeit for a network arising form a GFF on a finite subset of $\mathbb{Z}^2$:

Lemma 4.4.16. *There is $c_1 \in (0,\infty)$ depending only on γ and a constant $c'' \in (0,\infty)$ such that*

$$\mathbb{P}\Big(R_{B(N)_{\chi_M}}(u,v) \ge c_1(\log M)\,\mathrm{e}^{t\sqrt{\log M}}\Big) \le 2c_1(\log M)\,\mathrm{e}^{-c''t^2}$$

holds for all $N \ge 1$, all $M \ge 32N$ and all $t \ge 0$.

Proof. Combining Proposition 4.4.11 with Corollary 4.4.3, for each $\epsilon > 0$ there is $N_0'' = N_0''(\gamma,\epsilon)$ such that if $N \ge N_0''$, $M \ge 32N$ and S is a translate of $B(4N,N)$ contained in $B(M/2)$, then we have

$$\mathbb{P}\Big(\big|\log R_{\mathrm{LR};S,\chi_{2M}}\big| \le 2\hat{c}\log\log(2M) + \log C_6\Big) \ge \epsilon. \tag{4.4.37}$$

Decomposing χ_{2M} on $B(M)$ into a fine field χ_M^f and a coarse field χ_M^c, the fact that

$$|\log R_{\mathrm{LR};S,\chi_{2M}}| \ge |\log R_{\mathrm{LR};S,\chi_M^f}| - 2\gamma\max_{u\in S}|\chi_M^c|$$

along with $\chi_M^f \overset{\text{law}}{=} \chi_M$ shows

$$\mathbb{P}\Big(\big|\log R_{\text{LR};S,\chi_M}\big| \le 2\hat{c}\log\log(2M) + \log C_6 + 2\tilde{c}\gamma\Big) \ge \epsilon - \mathbb{P}\Big(\max_{u\in S}\big|\chi_M^c\big| > \tilde{c}\Big).$$

The last probability tends to zero as $\tilde{c} \to \infty$ uniformly in $M \ge 1$ and so, by choosing $\tilde{c}$ large, there is a constant $C_7 \in (0,\infty)$ such that, for all $N \ge N_0''$,

$$\mathbb{P}\Big(\big|\log R_{\text{LR};S,\chi_M}\big| \le 2\hat{c}\log\log(2M) + \log C_7\Big) \ge \epsilon/2 \tag{4.4.38}$$

holds for all $M \ge 32N$ and all translates of $B(4N,N)$ contained anywhere in $B(M)$.

Since (4.4.38) gives us an interval of width of order $\log\log M$ where $\big|\log R_{\text{LR};S,\chi_M}\big|$ keeps a uniformly positive mass, the Gaussian concentration in Proposition 4.4.14 shows that, for some constants $c', c'' \in (0,\infty)$,

$$\mathbb{E}\big|\log R_{\text{LR};S,\chi_M}\big| \le c'\sqrt{\log M}$$

and also

$$\mathbb{P}\Big(\big|\log R_{\text{LR};S,\chi_M}\big| > t\sqrt{\log M}\Big) \le 2\mathrm{e}^{-c''t^2} \tag{4.4.39}$$

hold for every $t \ge 0$. The proof has so far assumed $N \ge N_0''$; to eliminate this assumption we note that $\text{Var}(\chi_{M,v}) \le \tilde{c}\log M$ uniformly in $v \in B(M)$ and so the union bound gives

$$\mathbb{P}\big(\max_{v\in S}|\chi_{M,v}| > t\sqrt{\log M}\big) \le 2|S|\mathrm{e}^{-\frac{1}{2}\tilde{c}^{-2}t^2}.$$

Since $|S| \le (4N_0''+1)^2$ while $|\log R_{\text{LR};S,\chi_M}|$ is at most $2\gamma\max_{v\in S}|\chi_{M,v}|$ times an N_0''-dependent constant, by adjusting c'' we make (4.4.39) hold for all $N \ge 1$. Due to rotation symmetry, the same bound holds also for $R_{\text{UD};S,\chi_M}$ and any translate S of $B(N,4N)$ contained in $B(M)$.

Now fix $M \ge 32$ and let $u, v \in B(M)$. Then one can find a collection of rectangles of the form $B(N,4N)$ or $B(4N,N)$ with $32N \le M$ that are contained in $B(M)$ and satisfy:

(1) There are at most $c_1 \log M$ of such rectangles with $c_1 \in (0,\infty)$ independent of M.

(2) If a path is chosen connecting the shorter sides in each of these rectangles, then the graph union of these paths contains a path from u to v.

By Lemma 4.2.4, this construction dominates $R_{B(N)_{\chi_M}}(u,v)$ by the sum of the resistances between the shorter sides of these rectangles. The FKG inequality, (4.4.39) and a union bound then imply

$$\mathbb{P}\Big(R_{B(N)_{\chi_M}}(u,v) \ge c_1(\log M)\,\mathrm{e}^{t\sqrt{\log M}}\Big) \le 2c_1(\log M)\,\mathrm{e}^{-c''t^2}$$

This is the desired claim. □

In order to extend this to the network with the underlying field η, we first note:

Lemma 4.4.17. *Let η denote the GFF on $\mathbb{Z}^2$ pinned at the origin. There are $C_1, c_1 \in (0,\infty)$ and $N_1 \ge 1$ such that for all $N \ge N_1$, all $M \ge 16N$ and for every translate S of $B(4N,N)$ contained in $B(M/2)$, we have*

$$\mathbb{P}\big(R_{\mathrm{LR};S,\eta} \le C_1 \mathrm{e}^{2\hat{c}\log\log(M)}\big) \ge c_1\,. \tag{4.4.40}$$

The same applies to $R_{\mathrm{UD};S,\chi_M}$ for translates S of $B(N,4N)$ with $S \subset B(M/2)$.

Proof. We will assume that M is the minimal integer such that $S \subset B(M/2)$. Note that this means that M/N is bounded. We proceed in two steps, first reducing η to the GFF in $\Lambda := B(M) \smallsetminus \{0\}$ and then relating this field to χ_M. Using the Gibbs-Markov property, the field η can be written as $\chi_\Lambda + \chi^c$, where χ_Λ, the fine field, has the law of the GFF on Λ while the coarse field χ^c is η conditional on its values outside of $B(M)$. Now pick an $x \in B(M) \smallsetminus B(M/2)^\circ$ such that x is at least $M/6$ lattice steps from both $B(M/2)$

and $B(M)^{\rm c}$. For any $r, A > 0$ we then have

$$\begin{aligned}\mathbb{P}\big(R_{\mathrm{LR};S,\eta} \le r\big) &\ge \mathbb{P}\big(R_{\mathrm{LR};S,\eta} \le r,\, \eta^c(x) \ge 0\big)\\ &\ge \mathbb{P}\big(R_{\mathrm{LR};S,\eta_\Lambda} \le r/A,\, \eta^c_x \ge 0\big) - \mathbb{P}\big(R_{\mathrm{LR};S,\eta} > AR_{\mathrm{LR};S,\eta_\Lambda},\ \eta^c_x \ge 0\big)\end{aligned} \tag{4.4.41}$$

Noting that both events are increasing functions of η, for the first probability on the right we get

$$\mathbb{P}\big(R_{\mathrm{LR};S,\eta_\Lambda} \le r/A,\, \eta^c_x \ge 0\big) \ge \frac{1}{2}\mathbb{P}\big(R_{\mathrm{LR};S,\eta_\Lambda} \le r/A\big)$$

using the FKG inequality. For the second probability we set

$$\varphi_u := \eta^c_u - \frac{\mathrm{Cov}(\eta^c_u, \eta^c_x)}{\mathrm{Var}(\eta^c_x)}\eta^c_x, \qquad u \in B(M/2),$$

and note, since $\mathrm{Cov}(\eta^c_u, \eta^c_x) \ge 0$, we have

$$R_{\mathrm{LR};S,\eta} \le R_{\mathrm{LR};S,\eta_\Lambda+\varphi} \quad \text{on } \big\{\eta^c_x \ge 0\big\}.$$

But the above definition ensures that φ is independent of η^c_x and a calculation using the explicit form of the law of η^c gives that $\max_{v\in\Lambda}\mathrm{Var}(\varphi_v)$ is bounded by a constant independent of M. Markov's inequality and (4.4.3) then bound the last probability in (4.4.41) by c'/A for some constant $c' \in (0,\infty)$ independent of A or M.

Next let $\mathfrak{g}_M\colon \mathbb{Z}^2 \to [0,1]$ be discrete harmonic on Λ with $\mathfrak{g}_M(0) := 1$ and $\mathfrak{g}_N(u) := 0$ whenever $u \notin B(M)$. Let $\tilde\chi$ have the law of $\chi_M(0)\mathfrak{g}(\cdot)$ but assume that $\tilde\chi$ is independent of χ_Λ. The Gibbs-Markov property shows

$$\tilde\chi + \chi_\Lambda \overset{\text{law}}{=} \chi_M.$$

A direct use of Lemma 4.4.2 is hampered by the fact that $\mathrm{Var}(\tilde\chi(0))$ is of order $\log M$. However, this is not a problem when S is at least distance δM from the origin because then

$\mathfrak{g}_N(x) = O(1/\log M)$. Letting $K := \lfloor N/3 \rfloor$, we now note that each translate S of $B(4N, N)$ contains a translate $\tilde{S}$ of $B(4N, K)$ which is at least distance N from the origin and is aligned with one of the longer side of S. Lemma 4.4.2 then gives, for any $b \in \mathbb{R}$,

$$R_{\text{LR};\tilde{S},\chi_\Lambda+\tilde{\chi}} \geq \mathrm{e}^{-c''b} R_{\text{LR};\tilde{S},\chi_\Lambda} \geq \mathrm{e}^{-c''b} R_{\text{LR};S,\chi_\Lambda}, \qquad \text{on } \{\tilde{\chi}(0) \leq b \log N\}$$

for some $c'' > 0$. Hence

$$\mathbb{P}\big(R_{\text{LR};S,\chi_\Lambda} \leq r/A\big) \geq \mathbb{P}\big(R_{\text{LR};\tilde{S},\chi_\Lambda+\tilde{\chi}} \leq \mathrm{e}^{-\tilde{c}b} r/A\big) - \mathbb{P}\big(\tilde{\chi}(0) > b \log N\big). \tag{4.4.42}$$

Now set $r := C_1 \mathrm{e}^{2\hat{c}\log\log(M)}$, $A := \mathrm{e}^{\hat{c}\log\log(M)}$ and pick any $b > 0$. Then the last probability in both (4.4.41) and (4.4.42) tends to zero as $N \to \infty$, while, as soon as C_1 is large enough, the first probability on the right of (4.4.42) is uniformly positive by Proposition 4.4.11 and a routine use of the FKG inequality (to get us from rectangles of the form $B(4N, K)$ to those with aspect ratio 4). The claim follows. $\square$

Using exactly the same argument as in the proof of Lemma 4.4.16, we then get:

Corollary 4.4.18. *Let η be the GFF in $\mathbb{Z}^2 \smallsetminus \{0\}$. There are $C, C' \in (0, \infty)$ such that*

$$\mathbb{P}\Big(R_{B(N)_\eta}(u, v) \geq C\,\mathrm{e}^{Ct\sqrt{\log N}}\Big) \leq C'\mathrm{e}^{-t^2} \log N$$

holds for all $N \geq 1$ and all $t \geq 0$.

This is one half of Theorem 4.1.1; the other half will be shown in the next section.

4.5 Asymptotic growth rate of $\log R_{\mathbb{Z}^2_\eta}(0, B(N)^c)$

Proof of Theorem 4.1.1. Here the bound (4.1.1) has already been shown in Corollary 4.4.18, so we just have to focus on (4.1.2–4.1.3). We will use a decomposition of η from [15, Proposition 3.12]. Let $b := 8$ and consider the annuli $A'_k := B(b^{k+1}) \smallsetminus B(b^k)^\circ$ and

$A_k := B(4b^k) \smallsetminus B(2b^k)$ for all $k \geq 0$. Then

$$\eta_v = \sum_{k\geq 0} \Big[\mathfrak{b}_k(v) X_k + \psi_{k,v} + \eta^f_{k,v} \Big], \tag{4.5.1}$$

where $\mathfrak{b}_k \colon \mathbb{Z}^2 \to \mathbb{R}$ is a function such that

$$\mathfrak{b}_k(v) = -1 \text{ if } v \notin B(b^k) \quad \text{and} \quad \big|\mathfrak{b}_k(v)\big| \leq cb^{\ell-k} \text{ if } v \in B(b^\ell) \subseteq B(b^k), \tag{4.5.2}$$

while $\{X_k \colon k \geq 0\}$ are random variables and $\{\psi_k \colon k \geq 0\}$ and $\{\eta^f_k \colon k \geq 0\}$ are random fields (all measurable with respect to η) that are independent of one another and distributed as centered Gaussian with the specifics of the law determined as follows:

(1) $\lim_{k\to\infty} \mathrm{Var}(X_k) = g \log b$,

(2) writing χ^c_k for the coarse field obtained as the conditional expectation of the GFF on $B(b^k)$ given its values on $\partial B(b^{k-1})$, we have

$$\psi_k \overset{\text{law}}{=} \chi^c_k - \mathbb{E}(\chi^c_k | \chi^c_{k,0}),$$

(3) η^f_k is the fine field on A'_k.

For ψ_k, we in addition have the following variance estimate,

$$\mathrm{Var}(\psi_{k,v}) \leq cb^{\ell-k}, \qquad v \in B(b^\ell) \subseteq B(b^k). \tag{4.5.3}$$

See [15, Lemma 3.7] for (4.5.2) and [15, Lemma 3.8] for (4.5.3).

Clearly, only one of the fine fields χ^f_k can contribute in (4.5.1) for each given v and $\chi_{k,v} = 0$ unless $v \in B(b^k)$. Setting (with some abuse of our earlier notation),

$$\Delta_k := \max_{v\in A_k} \Big| \sum_{j>k} \mathfrak{b}_j X_j + \sum_{j\geq k} \psi_{j,v} \Big|$$

[15, Lemma 3.8] shows that, for some constants $c, c' \in (0,\infty)$,

$$\mathbb{P}\big(\Delta_k \ge c + t\big) \le \mathrm{e}^{-c't^2}, \qquad t \ge 0. \tag{4.5.4}$$

The first half of (4.5.2) then lets us write

$$\eta_v + \sum_{j=0}^{k} X_k - \chi^f_{k,v} \le \Delta_k, \qquad v \in A_k.$$

We now set $S_k := \sum_{j=0}^{k} X_k$ and note that the Nash-Williams estimate and Lemma 4.4.2 imply

$$R_{B(N+1)_\eta}\big(0, \partial B(N)\big) \ge \max_{k=1,\dots,n-1} \Big[\mathrm{e}^{-2\gamma(\Delta_k - S_k)}\, R_{A_k,\eta^f_k}\big(\partial^{\mathrm{in}} A_{n,k}, \partial^{\mathrm{out}} A_{n,k}\big)\Big] \tag{4.5.5}$$

where $n := \max\{k \ge 0 \colon b^k \le N\}$.

Our aim is to study the maximum in (4.5.5) and show that it grows at least as exponential of $\sqrt{n}/(\log n)^{1+\delta}$. To this end, we define the sequence of record values of the sequence S_n as follows: Set $\tau_0 := 0$ and for $m \ge 1$ let

$$\tau_m := \inf\big\{k > \tau_{m-1} \colon S_k \ge S_{\tau_{m-1}} + 1\big\}.$$

Then we have:

Lemma 4.5.1. $\{\tau_m - \tau_{m-1} \colon m \ge 1\}$ *are independent with a uniform bound on their tail,*

$$\mathbb{P}\big(\tau_m - \tau_{m-1} > t\big) \le \frac{c}{\sqrt{t}}, \qquad t \ge 1, \tag{4.5.6}$$

for some constant $c > 0$. In particular, for each $\delta > 0$ there is $c' \in (0,\infty)$ such that

$$\mathbb{P}(\tau_m > t) \le \frac{c'm}{\sqrt{t}}, \qquad t \ge 1. \tag{4.5.7}$$

holds for all $m \geq 1$.

Postponing the proof temporarily, we note that (4.5.7) shows

$$\mathbb{P}\Big(\tau_m > m^2(\log m)^{2+2\delta}\Big) \leq \frac{c'}{(\log m)^{1+\delta}}, \qquad m \geq 2. \tag{4.5.8}$$

A Borel-Cantelli argument then gives

$$\sup_{m\geq 1} \frac{\tau_m}{m^2(\log m)^{2+2\delta}} < \infty, \qquad \text{a.s.} \tag{4.5.9}$$

(This is first proved for m running along powers of 2 and then extended by monotonicity of both numerator and denominator.) In particular, for n large enough, the sequence $S_1, \dots, S_n$ will see at least $\sqrt{n}/(\log n)^{1+\delta}$ record values as defined above. If it were not for the terms Δ_k and $R_{A_k,\eta_k^f}(\partial^{\text{in}} A_{n,k}, \partial^{\text{out}} A_{n,k})$, this observation would bound the maximum in (4.5.5) by what we want, so we have to ensure that these terms do not spoil this.

Consider the events

$$E_k := \big\{\Delta_k \leq \log\log k\big\}$$

and

$$F_k := \Big\{R_{A_k,\eta_k^f}\big(\partial^{\text{in}} A_{n,k}, \partial^{\text{out}} A_{n,k}\big) \geq C\mathrm{e}^{-3\hat{c}\log\log(b^k)}\Big\}.$$

By (4.5.4) and Markov's inequality, there is an a.s. finite n_0 such that

$$\sum_{k=1}^n 1_{E_k^{\mathrm{c}}} \leq \frac{1}{2}\frac{\sqrt{n}}{(\log n)^{1+\delta}}, \qquad n \geq n_0.$$

(Again, we prove this for n running along powers of 2 and then fill the gaps by monotonicity.) This means that at least half of the record values by time n occur at indices where E_k occurs, i.e.,

$$\sum_{m\geq 1} 1_{E_{\tau_m}\cap\{\tau_m\leq n\}} \geq \frac{1}{2}\frac{\sqrt{n}}{(\log n)^{1+\delta}}$$

as soon as n is large enough. But the events F_k are independent of each other and of *all* of E_j's and τ_m's and, since Lemma 4.4.13 tells us $\inf_{k\ge1}\mathbb{P}(F_k)>0$ for some $C>0$, the longest run of 1's in the sequence $\{1_{F^{\rm c}_{\tau_m}}:\tau_m\le n\}$ has length at most $\tilde c\log n$. It follows that, for n large, the event $E_k\cap F_k$ occurs for some k of the form $k=\tau_m$ for some $m=m(n)\ge1$ subject to $\tau_m\le n$ and $\tau_{m'}>n$ for $m':=m-\lceil c\log n\rceil$. This shows $m=n^{1/2+o(1)}$ and so $m\ge m'/2$ once n is large enough. From (4.5.9) we now conclude

$$S_{\tau_m}\ge m\ge\frac{m'}{2}\ge c\frac{\tau_{m'}}{(\log\tau_{m'})^{1+\delta}}\ge c'\frac{\sqrt{n}}{(\log n)^{1+\delta}}$$

for some constants $c,c'\in(0,\infty)$ as soon as n is large enough. Since also $E_k\cap F_k$ occur for $k:=\tau_m$, using this in (4.5.5) yields

$$\begin{aligned}\log R_{B(N+1)_\eta}&\bigl(0,\partial B(N)\bigr)\\&\ge 2\gamma c'\frac{\sqrt{n}}{(\log n)^{1+\delta}}-2\gamma\log\log n-3\hat c\log\log(b^n)+\log C\,.\end{aligned}$$

The bound (4.1.3) follows. □

Proof of Lemma 4.5.1. We will follow the proof of [15, Lemma 4.16]. Since the sequence $\{S_n\colon n\ge1\}$ has independent (centered) Gaussian increments, we can embed it into a path of standard Brownian motion by putting $S_n=B_{t_n}$ where $t_n:=\mathrm{Var}(S_n)$. By property (1) above, we have $t_n-t_{n-1}\to g\log b$ as $n\to\infty$. Consider the process $W^{(k)}$ which is zero outside the interval $[t_k,t_{k+1}]$ and on this interval,

$$W^{(k)}(s):=\frac{t_{k+1}-s}{t_{k+1}-t_k}B_{t_k}+\frac{s-t_k}{t_{k+1}-t_k}B_{t_{k+1}}-B_s,\qquad t_k\le s\le t_{k+1}.$$

The independence of increments of Brownian motion now gives

$$\begin{aligned}
&\mathbb{P}\Big(B_{t_k+s} - B_{t_k} \le 2 + \log(1+s)\colon s + t_k \in [t_k, t_n]\Big) \\
&\quad \ge \mathbb{P}\Big(B_{t_j} - B_{t_k} \le 1\colon j = t_k, \dots, t_n\Big) \prod_{j=k}^{n-1} \mathbb{P}\Big(\max_{s\in[t_j,t_{j+1}]} W^{(j)}(s) \le 1 + \log(1 + t_j - t_k)\Big).
\end{aligned}$$

Since $W^{(k)}$ are Brownian bridges on intervals of bounded length, and maxima thereof thus have a uniformly Gaussian tail, the product on the right-hand side is positive uniformly in n. It follows that, for c^{-1} being a uniform lower bound on the product,

$$\mathbb{P}\big(\tau_m - \tau_{m-1} \ge t\big) \le c\mathbb{P}^0\big(B_s \le 2 + \log(1+s)\colon s \le \tilde{c}t\big)$$

where $\tilde{c} := \inf_{n\ge1}(t_n - t_{n-1})$. The probability on the right is at most $c'/\sqrt{t}$ by, e.g., [15, Proposition 4.9]. This proves (4.5.6). The bound (4.5.7) now follows from standard estimates of sums of independent heavy-tailed random variables. □

CHAPTER 5

SOME PROPERTIES OF THE RANDOM WALK DRIVEN BY PLANAR GFF

5.1 Recurrence, return probability and subdiffusivity

In this chapter we will study the random walk defined in Section 1.2 using the effective resistance estimates from previous chapter. For convenience of the reader, we will discuss the setup once again before presenting our main results.

Let $\eta = \{\eta_v\}_{v\in\mathbb{Z}^2}$ denote a sample of the discrete GFF on $\mathbb{Z}^2$ pinned to 0 at the origin. Thus $\{\eta_v\}_{v\in\mathbb{Z}^2}$ is a centered Gaussian process such that

$$\eta_0 = 0 \quad \text{and} \quad \mathbb{E}(\eta_u\eta_v) = G_{\mathbb{Z}^2\setminus\{0\}}(u,v) \text{ for all } u,v\in\mathbb{Z}^2\,,$$

where $G_{\mathbb{Z}^2\setminus\{0\}}(u,v)$ is the Green function in $\mathbb{Z}^2\setminus\{0\}$. For $\gamma > 0$ and conditional on the sample η of the GFF, let $\{X_t\}_{t\geq 0}$ be a discrete-time Markov chain with transition probabilities given by

$$p_\eta(u,v) := \frac{\mathrm{e}^{\gamma(\eta_v-\eta_u)}}{\sum_{w:|w-u|_1=1}\mathrm{e}^{\gamma(\eta_w-\eta_u)}}\mathbf{1}_{|v-u|_1=1}\,, \tag{5.1.1}$$

where $|\cdot|_1$ denotes the ℓ^1-norm on $\mathbb{Z}^2$. We will write P_η^x for the law of the above random walk such that $P_\eta^x(X_0 = x) = 1$ and use E_η^x to denote the corresponding expectation. We also write $\mathbb{P}$ for the law of the GFF and use $\mathbb{E}$ (as above) to denote the expectation with respect to $\mathbb{P}$.

The transition kernel p_η depends only on the differences $\{\eta_x-\eta_y\colon x,y\in\mathbb{Z}^2\}$ whose law is, as it turns out, invariant and ergodic with respect to the translates of $\mathbb{Z}^2$. The Markov chain $\{X_t\}_{t\geq 0}$ is thus an example of a random walk in a stationary random environment. The main conclusion we prove about this random walk is then:

Theorem 5.1.1. *For each $\gamma > 0$ and each $\delta > 0$,*

$$\lim_{T\to\infty} \mathbb{P}\Big(\mathrm{e}^{-(\log T)^{1/2+\delta}} T^{-1} \le P_\eta^0(X_{2T}=0) \le \mathrm{e}^{(\log T)^{1/2+\delta}} T^{-1}\Big) = 1\,. \tag{5.1.2}$$

Furthermore, $\{X_t\}_{t\ge 0}$ is recurrent for $\mathbb{P}$-almost every η.

We also prove a version of subdiffusivity for the expected exit time from large balls:

Theorem 5.1.2. *Let $\tau_{B(N)^{\mathrm{c}}}$ denote the first exit time of $\{X_t \colon t \ge 0\}$ from $B(N) := [-N,N]^2 \cap \mathbb{Z}^2$. For each $\delta > 0$, we then have*

$$\lim_{N\to\infty} \mathbb{P}\Big(N^{\psi(\gamma)} \mathrm{e}^{-(\log N)^{1/2+\delta}} \le E_\eta^0 \tau_{B(N)^{\mathrm{c}}} \le N^{\psi(\gamma)} \mathrm{e}^{(\log N)^{1/2+\delta}}\Big) = 1\,, \tag{5.1.3}$$

where

$$\psi(\gamma) := \begin{cases} 2 + 2(\gamma/\gamma_{\mathrm{c}})^2, & \textit{if } \gamma \le \gamma_{\mathrm{c}} := \sqrt{\pi/2}, \\ 4\gamma/\gamma_{\mathrm{c}}, & \textit{otherwise.} \end{cases} \tag{5.1.4}$$

The bounds on the expected hitting time indicate that $|X_T|$ should scale as $T^{\frac{1}{\psi(\gamma)}+o(1)}$ for large T. Although we expect this to be true, we have so far only been able to prove a corresponding lower bound:

Theorem 5.1.3. *For $\mathbb{P}$-almost every η and each $\delta > 0$,*

$$P_\eta^0\Big(|X_T| \ge \mathrm{e}^{-(\log T)^{1/2+\delta}} T^{\frac{1}{\psi(\gamma)}}\Big) \underset{T\to\infty}{\longrightarrow} 1 \quad \textit{in probability,} \tag{5.1.5}$$

where $\psi(\gamma)$ is as in (5.1.4).

We note that Theorems 5.1.2 and 5.1.3 are consistent with the predictions in [21, 22] for general log-correlated fields. In particular, (5.1.5) confirms the prediction for the diffusive exponent of the walk from [21, 22] as a lower bound. The reason why the bounds in (5.1.3) are not sufficient is that we do not know whether $\tau_{B(N)^{\mathrm{c}}}$ scales with N proportionally to its expectation. A full proof of subdiffusive behavior thus remains elusive.

As mentioned several times before, the technical approach that makes our analysis possible stems from the following simple rewrite of the transition kernel,

$$p_\eta(u,v) = \frac{\mathrm{e}^{\gamma(\eta_v+\eta_u)}}{\sum_{w:|w-u|_1} \mathrm{e}^{\gamma(\eta_w+\eta_u)}} \mathbf{1}_{|v-u|_1=1}. \tag{5.1.6}$$

This represents $\{X_t\}_{t\geq 0}$ as a random walk among random conductances where the edge (u,v) is given a conductance $\mathrm{e}^{\gamma(\eta_u+\eta_v)}$. As it turns out, the change of the behavior of the expected exit time at the critical point γ_{c} (see Theorem 5.1.2) arises, in its entirety, from the asymptotic

$$\pi_\eta\big(B(N)\big) = N^{\psi(\gamma)+o(1)}, \qquad N \to \infty\,. \tag{5.1.7}$$

This is because, as a consequence of Theorem 4.1.1, point-to-point effective resistances in the associated random conductance network $\mathbb{Z}^2_\eta$ behave, for points at distance N, as $N^{o(1)}$ for every $\gamma > 0$.

5.1.1 *Proof strategy*

Apart from Theorem 4.1.1 which is a key ingredient of our proofs, substantial work is needed on the random walk side as well. The upper bound on the return probability is proved in Section 5.3.1 using the methods drawn from [52]. The lower bound on the return probability is more subtle as it requires showing that the effective resistivity from 0 to v in $B(N)$ is bounded by the sum of the resistances from 0 to $\partial B(N)$ and from v to $\partial B(N)$. This amounts to bounding a *difference* of effective resistances, which is not immediate from the estimates obtained in Chapter 4.

We approach this by invoking a concentric decomposition of the GFF along a sequence of annuli, which permits representing of the typical value of the resistance as an exponential of a random walk. The Law of the Iterated Logarithm then shows that the natural fluctuations of the effective resistance (which are of order $\mathrm{e}^{O(\sqrt{\log N})}$) can be beaten in at least one of the annuli. These key steps are the content of Proposition 5.3.8 and Lemma 5.3.9. As an immediate consequence, we then get recurrence.

5.2 Cardinality of the level sets of GFF

In this section, we estimate the cardinality of the sets of points where the GFF equals (roughly) a prescribed multiple of its absolute maximum. The main use of this is to prove Lemma 5.3.2 where we obtain a lower bound on $\pi_\eta(B(N))$ (see (5.1.7)).

Recall that from [20, 19] we know that the family of random variables

$$\max_{v\in B(N)} \chi_{N,v} - 2\sqrt{g}\log N - \frac{3}{4}\sqrt{g}\log\log N \tag{5.2.1}$$

is tight as $N\to\infty$. The level sets we are interested in are of the form

$$\mathcal{A}_{N,\alpha} := \Bigl\{v\in B(\lfloor N/2\rfloor)\colon \chi_{N,v}\in(\alpha\tilde{m}_N, \alpha\tilde{m}_N+1)\Bigr\}, \tag{5.2.2}$$

where $\tilde{m}_N := 2\sqrt{g}\log N$ and $\alpha\in(0,1)$. Our conclusion about these is as follows:

Theorem 5.2.1. *For any $\alpha_0\in(0,1)$ there are $c=c(\alpha_0)>0$ and $\kappa=\kappa(\alpha_0)>0$ such that for all $0\le\alpha_N\le\alpha_0$ and all $\delta\ge \mathrm{e}^{-(\log N)^{1/4}}$ the bound*

$$\mathbb{P}\bigl(|\mathcal{A}_{N,\alpha_N}|\le\delta\mathbb{E}|\mathcal{A}_{N,\alpha_N}|\bigr)\le c\delta^\kappa \tag{5.2.3}$$

holds for all N sufficiently large. The same statement holds also for the GFF on $B(N)\smallsetminus\{0\}$.

The exponent linking the cardinality of the level set to the linear size of the underlying domain has been computed in [25] building on [17] where the leading-order growth-rate of the absolute maximum was determined. While much progress on the maxima of the GFF has been made recently, notably with the help of modified branching random walk (MBRW) introduced in [20], the methods used in these studies do not seem to be of much use here. Indeed, in order to make use of the modified branching random walk one needs to invoke a comparison between the GFF and MBRW, which is conveniently available for the maximum (using Slepian's lemma [75]), but does not seem to extend to the cardinality of the level sets.

Another possible approach to consider is the intrinsic dimension of the level sets (see [24]), but this would not give a sharp estimate as we desire. Our approach to Theorem 5.2.1 is much simpler, being a combination of the second moment method (which directly applies to GFF) and the "sprinkling method" which was employed in [28] in the context of the GFF. We remark that the second moment method has recently been used to prove that a suitably-scaled size of the whole level set admits a non-trivial distributional limit [16].

Proof of Theorem 5.2.1. The proof is actually quite easy when $\alpha < 1/\sqrt{2}$, but becomes more complicated in the complementary regime of α. This is due to well known failure of the second-moment method in these problems and the need for a suitable truncation to make it work again. The first half of the proof thus consists of the set-up, and control, of the truncation.

Pick $N \geq 1$ large and let $n := \max\{k \colon 2^k < N/8\}$. For $v \in B(\lfloor N/2 \rfloor)$, write $B(v, L) := v + B(L)$ and, for $k = 1, \dots, n$, set, abusing of our earlier notation, $A_{n,k}(v) := B(v, 2^{k+1}) \smallsetminus B(v, 2^k)$. Note that $A_{n,k}(v) \subset B(\lfloor 3N/4 \rfloor)$ for all $k = 1, \dots, n$. Then for all $x, y \in A_{n,k}(v)$ and with $g := 2/\pi$,

$$\mathbb{E}(\chi_{N,v}\chi_{N,x}) = g(\log 2)(n-k) + O(1)$$

and

$$\mathbb{E}(\chi_{N,x}\chi_{N,y}) \geq g(\log 2)(n-k) + O(1)$$

hold with $O(1)$ uniformly bounded in N and x, y as above. Next denote

$$\bar{\chi}_{N,k,v} := \frac{1}{|A_{n,k}(v)|} \sum_{u \in A_{n,k}(v)} \chi_{N,u} \,.$$

A straightforward calculation then shows that

$$\mathrm{Var}(\bar{\chi}_{N,k,v}) = g(\log 2)(n-k) + O(1) \tag{5.2.4}$$

and

$$\mathbb{E}(\bar{\chi}_{N,k,v}\chi_{N,v}) = g(\log 2)(n-k) + O(1),$$

again, with $O(1)$ uniform in N. It follows that there are numbers $a_x = a_{N,k,v,x}$ with $|a_x - 1| = O(1/(n-k))$ and a Gaussian process $Y_x = Y_{N,k,v,x}$ which is independent of $\bar{\chi}_{N,k,v}$ and obeys $\mathrm{Var}(Y_x) = g(\log 2)k + O(1)$ such that

$$\chi_{N,x} = a_x\bar{\chi}_{N,k,v} + Y_x, \qquad x \in \{v\} \cup A_{n,k}(v).$$

Further, we have that

$$\max_{x \in A_{n,k}(v)} \mathbb{E}(Y_v Y_x) = O(1) \tag{5.2.5}$$

again with $O(1)$ uniform in N.

For $\epsilon > 0$, $r > 0$ and $0 \leq \alpha_N \leq \alpha_0$, define the event

$$E_{v,\epsilon,r,\alpha_N} := \big\{\chi_{N,v} \in (\alpha_N\tilde{m}_N, \alpha_N\tilde{m}_N + 1)\big\} \\ \cap \bigcap_{k=1}^{n}\Big\{\bar{\chi}_{N,k,v} \leq \alpha_N\frac{n-k}{n}\tilde{m}_N + \epsilon[k \wedge (n-k)] + r\Big\}.$$

We claim that for $\epsilon := \frac{(1-\alpha_0)}{10}$ and $r := r_{\alpha_0}$ sufficiently large, we have

$$\mathbb{P}\big(E_{v,\epsilon,r,\alpha_N}\big) \geq \tfrac{1}{2}\mathbb{P}\big(\chi_{N,v} \in (\alpha_N\tilde{m}_N, \alpha_N\tilde{m}_N + 1)\big). \tag{5.2.6}$$

In order to prove (5.2.6), note that by (5.2.4)

$$\mathbb{E}\big(\bar{\chi}_{N,k,v} \,\big|\, \chi_{N,v} \in (\alpha_N\tilde{m}_N, \alpha_N\tilde{m}_N + 1)\big) = \alpha_N\frac{n-k}{n}\tilde{m}_N + O(1)$$

and

$$\mathrm{Var}\big(\bar{\chi}_{N,k,v}\,\big|\,\chi_{N,v}\big) \le \frac{4(n-k)k}{n}\,.$$

Abbreviating $s_k := \alpha_N \frac{n-k}{n}\tilde{m}_N + \epsilon[k\wedge(n-k)] + r$, from these observations we have

$$\sum_{k=1}^{n} \mathbb{P}\Big(\bar{\chi}_{N,k,v} \ge s_k \,\Big|\, \chi_{N,v} \in (\alpha_N \tilde{m}_N, \alpha_N \tilde{m}_N + 1)\Big)$$
$$\le \sum_{k=1}^{n} \mathrm{e}^{-\epsilon((n-k)\wedge k + r + O(1))/100} \le 1/2\,,$$

where the last inequality holds for all $r \ge r(\alpha_0)$ where $r(\alpha_0) \in (0,\infty)$. This yields (5.2.6).

Now we are ready to apply the second moment method. We will work with

$$\mathcal{Z} := \sum_{v\in B(\lfloor N/2\rfloor)} \mathbf{1}_{E_{v,\epsilon,r,\alpha_N}}$$

From (5.2.6) and a calculation for the Gaussian distribution we get

$$\mathbb{E}\mathcal{Z} \ge \frac{1}{2}\mathbb{E}|\mathcal{A}_{N,\alpha_N}| \ge \frac{c}{\sqrt{n}} 4^{(1-\alpha_N^2)n} \tag{5.2.7}$$

for some constant $c>0$. Our next task is a derivation of a suitable upper bound on $\mathrm{Var}\mathcal{Z}$. From (5.2) and (5.2.5) we get that, for any $v \in B(\lfloor N/2\rfloor)$ and with $c_r > 0$ a constant depending on r but not on v or N,

$$\begin{aligned}
&\sum_{u\in B(\lfloor N/2\rfloor)} \mathbb{P}(E_{u,\epsilon,r,\alpha_N} \cap E_{v,\epsilon,r,\alpha_N}) \\
&\qquad \le \sum_{k=1}^{n} \sum_{x\in A_{n,k}(v)} \mathbb{P}\Big(\chi_{N,u}, \chi_{N,v} \in (\alpha_N \tilde{m}_N, \alpha_N \tilde{m}_N + 1), \bar{\chi}_{N,k,v} \le x_k\Big) \\
&\qquad \le \sum_{k=1}^{n} \sum_{u\in A_{n,k}(v)} \int_{-\infty}^{x_\ell} \mathbb{P}\Big(Y_v \wedge Y_u \ge \alpha_N \tilde{m}_N - s\Big)\mathbb{P}(\bar{\chi}_{N,k,v} \in \mathrm{d}s) \\
&\qquad \le c_r \sum_{k=1}^{n} \frac{1}{\sqrt{n-k}} \Big(\frac{1}{\sqrt{k}}\Big)^2 4^{-\alpha_N^2(n-k)} 4^{(1-2\alpha_N^2)k} 4^{2\epsilon\alpha_N[(n-k)\wedge k]}\,.
\end{aligned} \tag{5.2.8}$$

Here the last inequality follows from the fact that, once we write the integral using the explicit form of the law of $\bar{\chi}_{N,k,v}$, the integrand is maximized at $s := s_k$ and decays exponentially when s is away from s_k. Combined with (5.2.7), the preceding inequality implies that

$$\frac{\mathrm{Var}\mathcal{Z}}{(\mathbb{E}\mathcal{Z})^2} \le c_r \sum_{k=1}^{n} \frac{n}{\sqrt{n-k}} \Big(\frac{1}{\sqrt{k}}\Big)^2 4^{-(1-\alpha_N^2)(n-k)} 4^{2\epsilon\alpha_N[(n-k)\wedge k]} = O(1)\,.$$

This implies

$$\mathbb{P}(\mathcal{Z} \ge \mathbb{E}\mathcal{Z}) \ge c \tag{5.2.9}$$

for some $c = c(\alpha_0) > 0$ sufficiently small uniformly in $N \ge N_1$ for some N_1 large.

It remains to enhance the lower bound in (5.2.9) to a number sufficiently close to one. To this end, pick an integer M with $N_1 \le M \le \mathrm{e}^{(\log N)^{1/4}}$, let $L := \lfloor N/(2M) \rfloor$ and consider a collection of boxes $V_1, \dots, V_{L^2}$ of the form $V_i := v_1 + B(M)$ contained in $B(\lfloor N/2 \rfloor)$. For $u \in V_i$, $i = 1, \dots, L^2$, define the coarse fields

$$\chi^c_{N,i,u} = \mathbb{E}\big(\chi_{N,u} \,\big|\, \chi_{N,x} \colon x \in \partial V_i\big)\,. \tag{5.2.10}$$

By Lemma 4.3.3 and [19, Lemma 3.10], we get that

$$\mathbb{E} \max_{v \in V_i} |\chi^c_{N,i,v} - \chi^c_{N,i,v_i}| \le O(1)\,.$$

In addition, as is easy to check, $\mathrm{Var}\chi^c_{N,i,v_i} \le 4\log M$. Introducing the event

$$\mathcal{E} := \big\{\chi^c_{N,i,v} \ge -40\log M \colon v \in V_i, 1 \le i \le L^2\big\},$$

we obtain that

$$\mathbb{P}(\mathcal{E}^c) = O(M^{-1})\,. \tag{5.2.11}$$

Conditioning on $\mathcal{E}$ and on the values $\{\chi_{N,v} \colon v \in \partial V_i, 1 \le i \le L^2\}$, the GFF in each square of V_i are independent of each other. Further, the Gaussian field on V_i dominates the field obtained from subtracting $40\log M$ from the GFF on V_i with Dirichlet boundary condition on ∂V_i. Write

$$\mathcal{A}_{N,\alpha_N,i} := \big\{v \in V_i \colon \chi_{N,v} \in (\alpha_N \tilde{m}_N, \alpha_N \tilde{m}_N + 1)\big\}.$$

By a straightforward first moment computation, we see that

$$\mathbb{E}|\mathcal{A}_{N,\alpha_N}| \leq M^{400}\mathbb{E}|\mathcal{A}_{N,\alpha_N+40\log M/\tilde{m}_N,i}|\,.$$

Therefore, applying (5.2.9) to V_i we get that

$$\mathbb{P}\big(|\mathcal{A}_{N,\alpha_N,i}| \geq M^{-400}\mathbb{E}|\mathcal{A}_{N,\alpha_N}|\,\big|\,\mathcal{E}\big) \geq c\,.$$

By conditional independence, we then get that

$$\mathbb{P}\big(\max_{1\leq i\leq L^2} |\mathcal{A}_{N,\alpha_N,i}| \geq M^{-400}\mathbb{E}|\mathcal{A}_{N,\alpha_N}|\,\big|\,\mathcal{E}) \geq 1-(1-c)^{L^2}\,.$$

Combined with (5.2.11), it gives

$$\mathbb{P}\big(|\mathcal{A}_{N,\alpha_N}| \geq M^{-400}\mathbb{E}|\mathcal{A}_{N,\alpha_N}|\big) \geq 1-O(M^{-1})-(1-c)^{L^2}.$$

Choosing M so large that $\delta < M^{-400} < 2\delta$ (assuming that δ is sufficiently small), this readily gives the claim for the GFF on $B(N)$ with Dirichlet boundary condition.

In the case that the GFF on $B(N) \smallsetminus \{0\}$, the same calculation goes through by considering instead the level set restricted to the square $(\lfloor N/4\rfloor, 0) + B(\lfloor N/2\rfloor)$ and replacing χ_N in (5.2.10) by η. We leave further details to the reader. □

5.3 Proofs of the main results

Here prove our main results. We begin with some preparatory claims; the actual proofs start to appear in Section 5.3.2.

5.3.1 Points with moderate resistance to origin

Our proofs will require restricting to subsets of $\mathbb{Z}^2$ of points with only a moderate value of the effective resistance to the origin and/or the boundary of a box centered there in. Here we give the needed bounds on cardinalities of such sets.

Lemma 5.3.1. *Denote $A(N,2N) := B(2N) \smallsetminus B(N)^\circ$. For any $\delta > 0$, we have*

$$\mathbb{P}\Bigl(\sum_{v\in A(N,2N)} \pi_\eta(v)\, 1_{\{R_{B(N)_\eta}(0,v)>\mathrm{e}^{(\log N)^{1/2+\delta}}\}} > N^{\psi(\gamma)}\mathrm{e}^{-(\log N)^\delta}\Bigr) \le \mathrm{e}^{-(\log N)^\delta} \tag{5.3.1}$$

as soon as N is sufficiently large.

Proof. Abbreviate, as in (4.3.5), $g := 2/\pi$. We will proceed by a straightforward first-moment estimate, but first we have to localize the problem to a finite box. Write $\eta = \eta^f + \eta^c$ where η^f is the fine field on the box $B(4N)$. Since $\mathrm{Var}(\eta^c_v) \le \mathrm{Var}(\eta_v)$, the variance of η^c is bounded by a constant times $\log N$ uniformly on $B(N)$ and so, combining Corollary 4.3.8 with a bound at one vertex,

$$\mathbb{P}\Bigl(\min_{v\in A(N,2N)} \eta^c_v \le -(\log N)^{1/2+\delta/2}\Bigr) \le c\mathrm{e}^{-\tilde c(\log N)^\delta}.$$

On the event when $\eta^c \ge -(\log N)^{1/2+\delta/2}$ we have

$$R_{B(N)_\eta}(0,v) \le R_{B(N)^f_\eta}(0,v)\,\mathrm{e}^{2\gamma(\log N)^{1/2+\delta/2}}$$

and so comparing this with the restriction on the effective resistivity in (5.3.1) we may as well estimate the probability in (5.3.1) for η replaced by χ_{4N}.

Here we will still need to employ a truncation to keep the field χ_{4N} below its typical maximum scale. The following crude estimate based on a union bound is sufficient,

$$\mathbb{P}\Bigl(\max_{v\in B(N)} \chi_{4N,v} \ge 2\sqrt{g}\log N + (\log N)^\delta\Bigr) \le c\mathrm{e}^{-\tilde c(\log N)^\delta}$$

for some constants $c, \tilde c \in (0,\infty)$. Writing F_N for the complementary event and inserting F_N

in the probability in (5.3.1) with η replaced by χ_{4N}, Markov's inequality bounds the result by

$$N^{-\psi(\gamma)}\mathrm{e}^{(\log N)^\delta}\sum_{v\in A(N,2N))}\mathbb{E}\Bigl(\pi_{\chi_{4N}}(v)1_{\{R_{B(N)_{\chi_{4N}}}(0,v)>\mathrm{e}^{(\log N)^{1/2+\delta}}\}}\Bigm| F_N\Bigr). \tag{5.3.2}$$

Now $\eta\mapsto\pi_\eta(v)$ is increasing while $\{R_{B(N)_\eta}(0,v)>\mathrm{e}^{(\log N)^\delta}\}$ is a decreasing event. Since the conditioning on F_N preserves the FKG inequality, the quantity in (5.3.2) is no larger than

$$\frac{1}{\mathbb{P}(A_N)^2}N^{-\psi(\gamma)}\mathrm{e}^{(\log N)^\delta}\sum_{v\in B(N)}\mathbb{E}\bigl(\pi_{\chi_{4N}}(v);\,F_N\bigr)\mathbb{P}\Bigl(R_{B(N)_{\chi_{4N}}}(0,v)>\mathrm{e}^{(\log N)^{1/2+\delta}}\Bigr)$$

Corollary 4.4.18 bounds the last probability by $\mathrm{e}^{-\tilde{c}(\log N)^{2\delta}}$ so we just have to compute the sum of the expectations of $\pi_{\eta_{4N}}(v)$'s.

Pick a pair of nearest neighbors u and v, with $v\in A(N,2N)$, and let $X:=\chi_{4N,u}+\chi_{4N,v}$. Disregarding the event F_N, a straightforward moment computation using $\mathrm{Var}(\chi_{4N,v})\le g\log N+c$ for $v\in A(N,2N)$ shows

$$\mathbb{E}\bigl(\mathrm{e}^{\gamma X}\bigr)=\mathrm{e}^{\frac12\gamma^2\mathrm{Var}(X)}\le cN^{2\gamma^2 g},\qquad v\in A(N,2N). \tag{5.3.3}$$

On the other hand, a change of measure argument gives

$$\begin{aligned}\mathbb{E}\bigl(\mathrm{e}^{\gamma X};F_N\bigr)&\le\mathrm{e}^{\frac12\gamma^2\mathrm{Var}(X)}\mathbb{P}\Bigl(X\le4\sqrt{g}\log N+2(\log N)^\delta-\gamma\mathrm{Var}(X)\Bigr)\\&\le cN^{2\gamma^2 g}\,\mathbb{P}\Bigl(X\le4(\sqrt{g}-\gamma g)\log N+3(\log N)^\delta\Bigr)\end{aligned} \tag{5.3.4}$$

For $\gamma>\gamma_\mathrm{c}:=1/\sqrt{g}$, the probability itself decays as $N^{-2(1-\gamma/\gamma_\mathrm{c})^2}\mathrm{e}^{c'(\log N)^\delta}$. Invoking the definition of $\psi(\gamma)$ in (5.1.4), the inequalities (5.3.3–5.3.4) thus give

$$\mathbb{E}\bigl(\pi_{\chi_{4N}}(v);\,F_N\bigr)\le cN^{\psi(\gamma)-2}\mathrm{e}^{c'(\log N)^\delta},\qquad v\in A(N,2N).$$

Summing over $v \in A(N,2N)$, the claim follows. $\square$

Consider now the set

$$\Xi_N := \{0\} \cup \Bigl\{ v \in A(N,2N) \colon R_{B(4N)_\eta}(0,v) \le \mathrm{e}^{(\log T)^{1/2+\delta}} \Bigr\}. \tag{5.3.5}$$

With the help of the above lemma we then show:

Lemma 5.3.2. *For each $\delta > 0$, there is $c > 0$ such that for all N sufficiently large,*

$$\mathbb{P}\Bigl(\pi_\eta(\Xi_N) \le N^{\psi(\gamma)} \mathrm{e}^{-(\log N)^\delta} \Bigr) \le \frac{c}{(\log N)^2}.$$

Proof. In light of Lemma 5.3.1, it suffices to show that

$$\mathbb{P}\biggl(\sum_{v \in A(N,2N)} \pi_\eta(v) \le 3N^{\psi(\gamma)} \mathrm{e}^{-(\log N)^\delta} \biggr) \le \frac{c}{(\log N)^2} \tag{5.3.6}$$

Thanks to the Gibbs-Markov property, it actually suffices to show this (with δ replaced by $\delta/2$) for η replaced by χ_N and $A(N,2N)$ replaced by a box $B(N)$. (Indeed, we just need to take a translate B of $B(N)$ with $B \subset A(N,2N)$ and then use the Gibbs-Markov property on a translate of $B(\lfloor 3N/2 \rfloor)$ centered at the same point as B. The contribution of the coarse field is estimated using Corollary 4.3.8.)

The argument for (5.3.6) is different depending on the relation between γ and γ_{c}. For $\gamma \ge \gamma_{\mathrm{c}}$ we use that the maximum of the GFF has doubly-exponential lower tails (see [28]). Invoking the Gibbs-Markov property we then conclude that, with probability at least $\mathrm{e}^{-(\log N)^c}$, for some $c > 0$, there is at least one point u where

$$\chi_{N,u} \ge 2\sqrt{g}\log N - \hat{c}\log\log N \tag{5.3.7}$$

for some large enough $C > 0$. As $\chi_{N,u} - \chi_{N,v}$, for u and v neighbors, have bounded (in fact, stationary) variances, a union bound shows that (5.3.7) will hold also for the neighbors of u.

On this event, and denoting by v a neighbor of u,

$$\sum_{v\in B(N)} \pi_{\chi_N}(v) \ge \mathrm{e}^{\gamma(\chi_{N,u}+\chi_{N,v})} = N^{4\sqrt{g}\gamma}\mathrm{e}^{-c'\log\log N}.$$

Since $4\sqrt{g}\gamma = 4(\gamma/\gamma_{\rm c})$ equals $\psi(\gamma)$ for $\gamma \ge \gamma_{\rm c}$, we are done here.

Concerning $\gamma < \gamma_{\rm c}$, here will will apply Theorem 5.2.1 for $\alpha := \gamma/\gamma_{\rm c}$. Recall the notation $\mathcal{A}_{N,\alpha}$ for the level set in (5.2.2). A straightforward computation using the explicit form of the Gaussian probability density shows

$$\mathbb{P}\big(x \in \mathcal{A}_{N,\alpha}\big) \ge \frac{c}{\log N}\, N^{-2\alpha^2},$$

and so $\mathbb{E}(|\mathcal{A}_{N,\alpha}|) \ge cN^{\psi(\gamma)}/\log N$. Theorem 5.2.1 now guarantees that $|\mathcal{A}_{N,\alpha}| \ge \delta\mathbb{E}(|\mathcal{A}_{N,\alpha}|)$ occurs with probability $O(\delta^c)$. This statements permits even setting $\delta := 1/(\log N)^{c'}$, whereby the claim readily follows. □

We also record an upper estimate on the total volume of π_η:

Lemma 5.3.3. *For any $\delta > 0$, we have*

$$\mathbb{P}\biggl(\,\sum_{v\in B(N)} \pi_\eta(v) \, > N^{\psi(\gamma)}\mathrm{e}^{(\log N)^\delta}\biggr) \le \mathrm{e}^{-(\log N)^\delta} \tag{5.3.8}$$

as soon as N is sufficiently large.

Proof. This follows directly from the Markov inequality and the calculations in (5.3.3–5.3.4). □

5.3.2 Upper bound on heat-kernel and exit time

The starting point of our proofs is an upper bound on the return probability for the random walk. We remark that numerous methods exist in the literature to derive such bounds. Some of these are based on geometric properties of the underlying Markov graph such as isoperimetry and volume growth, others are based on resistance estimates. The most natural approach to use would be that of [5] (see also [52]); unfortunately, this does not seem possible due to our lack of required uniform control of the resistance growth. Instead, we base our presentation on the general strategy outlined in [55, Chapter 21.5]. We begin by restating, and proving, one half of Theorem 5.1.1:

Lemma 5.3.4. *For each $\delta > 0$,*

$$\lim_{T\to\infty} \mathbb{P}\Big(P_\eta^0(X_{2T}=0) \le \mathrm{e}^{(\log T)^{1/2+\delta}} T^{-1}\Big) = 1\,. \tag{5.3.9}$$

Proof. Pick $\delta > 0$ and a large integer T, and recall the notation Ξ_T for the set in (5.3.5). Consider the random walk $\{\widetilde{X}_t \colon t \ge 0\}$ on the network $B(4T)_\eta$; this walk starts at 0 and moves around $B(4T)$ indefinitely using the transition probabilities (1.2.2) that are modified on the boundary of $B(4T)$ so that jumps outside $B(4T)$ are suppressed. Let $\{Y_t \colon t \ge 0\}$ record the successive visits of $\widetilde{X}$ to Ξ_T. Then Y is a Markov chain on Ξ_T with stationary distribution

$$\nu(x) := \frac{\pi_\eta(x)}{\pi_\eta(\Xi_T)}. \tag{5.3.10}$$

Let $\tau_0 := 0$, τ_1, τ_2, etc be the times of the successive visits of Y to 0. Define

$$\hat{\sigma} := \inf\big\{k \ge 1 \colon \tau_k \ge T \text{ and } Y_k = 0\big\}.$$

Then we have

$$TP^0(\widetilde{X}_T = 0) \le E^0\Big(\sum_{t=0}^{T-1} 1_{\{\widetilde{X}_t=0\}}\Big) \le E^0\Big(\sum_{k=0}^{T-1} 1_{\{Y_k=0\}}\Big) \le E^0\Big(\sum_{k=0}^{\hat{\sigma}-1} 1_{\{Y_k=0\}}\Big), \tag{5.3.11}$$

where the first inequality comes from the monotonicity of $T \mapsto P^0(\widetilde{X}_T = 0)$ and the second inequality reflects the fact that $0 \in \Xi_T$. Since $Y_{\hat\sigma} = 0$, by, e.g., [55, Lemma 10.5] we have

$$E^0\Bigl(\sum_{k=0}^{\hat\sigma-1} 1_{\{Y_k=x\}}\Bigr) = E^0(\hat\sigma)\nu(x). \tag{5.3.12}$$

(This is proved by noting that the object on the left is a stationary measure for the walk Y of total mass $E^0(\hat\sigma)$.) By conditioning on Y_T we further estimate

$$E^0(\hat\sigma) \le T + \max_{u\in\Xi_T} E^u(\sigma_0),$$

where $\sigma_0 := \inf\{k \ge 0\colon Y_k = 0\}$ and note that

$$E^u(\sigma_0) \le \pi_\eta(\Xi_T)\, R_{B(4T)_\eta}(0,u) \le \pi_\eta(\Xi_T)\, \mathrm{e}^{(\log T)^{1/2+\delta}}, \qquad u \in \Xi_T,$$

by the commute-time identity of [23] (cf [57, Corollary 2.21]). Combining this with (5.3.11–5.3.12) and (5.3.10) we then get

$$P^0(\widetilde X_T = 0) \le \frac{1}{T}\, \pi_\eta(0)\, \mathrm{e}^{(\log T)^{1/2+\delta}},$$

which proves (5.3.9) because, due to the jumps being only to nearest neighbors, the walk $\widetilde X$ coincides with the walk X up to time at least $4T$. □

This now permits to give:

Proof of Theorem 5.1.3. A standard calculation based on reversibility and the Cauchy-Schwarz inequality yields

$$\begin{aligned} P^0(X_{2T} = 0) &\ge \sum_{x\in B(N)} P^0(X_T = x) P^x(X_T = 0) \\ &= \pi_\eta(0) \sum_{x\in B(N)} \frac{P^0(X_T = x)^2}{\pi_\eta(x)} \ge \pi_\eta(0) \frac{P^0\bigl(X_T \in B(N)\bigr)^2}{\pi_\eta\bigl(B(N)\bigr)}. \end{aligned} \tag{5.3.13}$$

Invoking the upper bound on the heat-kernel and Lemma 5.3.3, we get that with probability tending rapidly to one as N and T tend to infinity, we have

$$P^0\bigl(X_T \in B(N)\bigr) \le \Bigl[\frac{1}{T}\mathrm{e}^{(\log T)^{1/2+\delta}}\,N^{\psi(\gamma)}\mathrm{e}^{(\log N)^{\delta}}\Bigr]^2. \tag{5.3.14}$$

Setting $T := N^{\psi(\gamma)}\mathrm{e}^{(\log N)^{1/2+2\delta}}$ gives the desired claim. □

The same conclusion could in fact be inferred from the following claim which constitutes one half of Theorem 5.1.2:

Lemma 5.3.5. *For each $\delta > 0$ and all N sufficiently large,*

$$\mathbb{P}\Bigl(E^0(\tau_{B(N)^{\mathrm{c}}}) > N^{\psi(\gamma)}\mathrm{e}^{(\log N)^{1/2+\delta}}\Bigr) \le \mathrm{e}^{-(\log N)^{\delta}}.$$

Proof. By the hitting time identity (or, alternatively, the commute time identity)

$$E^0(\tau_{B(N)^{\mathrm{c}}}) \le R_{B(N+1)_\eta}\bigl(0, \partial B(N)\bigr)\pi_\eta\bigl(B(N)\bigr)$$

The claim then follows from Corollary 4.4.18 and Lemma 5.3.3. □

5.3.3 Bounding the voltage from below

We now move to the proofs of the requisite lower bounds. Here the focus will be trained on the expected exit time which we write using the hitting time identity as

$$E^0(\tau_{B(N)^{\mathrm{c}}}) = R_{B(N+1)_\eta}\bigl(0, \partial B(N)\bigr) \sum_{v\in B(N)} \pi_\eta(v)\phi(v), \tag{5.3.15}$$

where, using our convention that $\partial B(N)$ is the external boundary of $B(N)$,

$$\phi(v) := P^v\bigl(\tau_0 < \tau_{\partial B(N)}\bigr)$$

is the electrostatic potential, a.k.a. voltage, in $B(N)$ with $\phi(0) = 1$ and ϕ vanishing on $\partial B(N)$. Estimating (5.3.15) from below naturally requires finding a sufficiently good lower bound on ϕ. The idea is to recast the problem using a simple electric network and invoke suitable effective resistance estimates. The following computation will be quite useful:

Lemma 5.3.6. *Consider a resistor network with three nodes, $\{1,2,3\}$, and for each i,j let c_{ij} denote the conductance of the edge (i,j). Let R_{ij} denote the effective resistance between node i and node j. Then,*

$$\frac{c_{12}}{c_{12}+c_{13}} = \frac{R_{13}+R_{23}-R_{12}}{2R_{23}}. \tag{5.3.16}$$

Proof. Let us represent the network by an equivalent network, now with nodes $\{0,1,2,3\}$ whose only edges are from 0 to each of $1,2,3$. Denoting the conductances of these edges by c_1, c_2, c_3 respectively, the Y-Δ transform shows

$$c_{ij} = \frac{c_i c_j}{c_1+c_2+c_3}, \qquad 1 \le i < j \le 3.$$

Next let us introduce the associate resistances $r_i := 1/c_i$. The Series Law then gives $R_{ij} = r_i + r_j$ for all $1 \le i < j \le 3$. A computation shows that, for all cyclic permutations (i,j,k) of $(1,2,3)$,

$$r_i = \frac{1}{2}(R_{ij}+R_{ik}-R_{jk}).$$

Some algebra then shows that the ratio on the left of (5.3.16) equals $\frac{r_3}{r_2+r_3}$. This is then checked to agree with the right-hand side. □

Using this lemma we then get:

Corollary 5.3.7. *For any $v \in B(N) \smallsetminus \{0\}$ and ϕ as above,*

$$\begin{aligned} &2R_{B(N+1)_\eta}\big(0,\partial B(N)\big)\phi(v) \\ &\quad = R_{B(N+1)_\eta}\big(0,\partial B(N)\big) + R_{B(N+1)_\eta}\big(v,\partial B(N)\big) - R_{B(N+1)_\eta}(0,v). \end{aligned} \tag{5.3.17}$$

Proof. As $v \notin \{0\} \cup \partial B(N)$, we may apply the network reduction principle to represent the problem on an effective network of three nodes, with node 1 labeling v, node 2 marking the origin and node 3 standing for $\partial B(N)$. Since ϕ is harmonic on $B(N) \smallsetminus \{0\}$, it is also harmonic on the effective network. But there $\phi(v)$ is just the probability that the random walk at v jumps right to 0 in the first step. Using conductances, this probability is exactly the expression on the left of (5.3.16). Plugging in the effective resistances, the claim follows. □

A key point is to bound the expression involving effective resistances on the right of (5.3.17) from below. This is the subject of:

Proposition 5.3.8. *Let* $D_{N,\eta}(v)$ *denote the difference on the right of* (5.3.17)*. For any* $\delta \in (0,1)$*, we then have*

$$\lim_{N\to\infty} \mathbb{P}\Big(\min_{v \in B(\lfloor N \mathrm{e}^{-(\log N)^\delta} \rfloor)} D_{N,\eta}(v) \geq \log N \Big) = 1. \tag{5.3.18}$$

For the proof we recall the annulus decomposition of the GFF from Section 4.3.2. Let $b := 8$ and for a given $N \geq 1$ and $n \in \mathbb{N}$, set $N' := b^n N$. Define the annuli

$$A'_{n,k} := B(b^{n-k+1}N) \smallsetminus B(b^{n-k}N)^\circ, \qquad k = 1, \dots, n-1. \tag{5.3.19}$$

and

$$A_{n,k} := B(4b^{n-k}N) \smallsetminus B(2b^{n-k}N)^\circ, \qquad k = 1, \dots, n-1. \tag{5.3.20}$$

Note that $A_{n,k} \subset A'_{n,k}$. Write $\eta = \eta^c + \chi_{2N'}$, where η^c is the coarse field on $B(2N')$ and $\chi_{2N'}$ is the corresponding fine field. Denote

$$\Delta' := \max_{v \in B(N')} |\eta^c_v|.$$

Define $M_{n,k}$ as in (4.3.18) and for $1 \le \ell < m \le n$ let

$$\Delta_{\ell,m} := \max_{k=\ell,\dots,m-1} \max_{v\in A_{n,k}} \Bigl| M_{n,k} - \mathbb{E}\bigl(\chi_{N',v} \,\big|\, \chi_{N',v} : v \in \textstyle\bigcup_{n\ge j\ge n-k} \partial B(b^j N)\bigr)\Bigr|.$$

(Both objects are measurable with respect to η.) Similarly to Lemma 4.3.11 we get

$$\mathbb{P}\Bigl(\Delta_{\ell,m} \ge \widetilde{C}\sqrt{m-\ell}\Bigr) \le \frac{1}{(m-\ell)^2} \tag{5.3.21}$$

as soon as $m-\ell$ is sufficiently large.

Let $\chi^f_{k,v}$ denote the fine field on $A'_{n,k}$,

$$\chi^f_{k,v} := \mathbb{E}\Bigl(\chi_{2N',v} \Bigm| \chi_{2N',u} \colon u \in \partial A_{n,k}\Bigr), \qquad v \in A_{n,k},$$

(we think of χ^f_k as set to zero outside $A'_{n,k}$) and $\chi^c_k := \chi_{2N'} - \chi^f_k$ be the corresponding coarse field. The definitions ensure

$$\max_{k=\ell,\dots,m} \max_{v\in A_{n,k}} \bigl|\eta_v - (\chi^f_{k,v} + M_{n,k})\bigr| \le \Delta_{\ell,m} + \Delta'.$$

Note also that $M_{n,k}$ and $\chi^f_{k'}$ are independent as long as $k \ge k'$.

Next recall that $R_{A,\eta}$, for A an annulus in $\mathbb{Z}^2$, denotes the sum of the effective resistances in network A_η between the shorter sides of the four maximal rectangles contained in A. Recall also that $R_{A,\eta}(\partial^{\text{in}}A, \partial^{\text{out}}A)$ denotes the effective resistance in A_η between the inner and outer boundaries of annulus A. We define the events:

$$\mathcal{E}^\star_{n,k} := \Bigl\{R_{A_{n,k},\chi^f_k}(\partial^{\text{in}}A_{n,k}, \partial^{\text{out}}A_{n,k}) \ge \mathrm{e}^{-3\hat{c}\log\log(b^{-k}N')}\Bigr\} \cap \Bigl\{M_{n,k} \le -C^\star\sqrt{k\log\log(k)}\Bigr\} \tag{5.3.22}$$

and

$$\mathcal{E}^{\star\star}_{n,k} := \Bigl\{R_{A_{n,k},\chi_k^f} \le \mathrm{e}^{\hat{c}\log\log(b^{-k}N')}\Bigr\} \cap \Bigl\{\min_{v\in A_{n,k}} \eta^c_{k,v} \ge -\log\log(N')\Bigr\}. \tag{5.3.23}$$

Here $\hat{c}$ is the constant Proposition 4.4.1 and $C^\star$ is fixed via:

Lemma 5.3.9. *For each $\delta > 0$ there are $n_0 \ge 1$, $N_0 \ge 1$, $c_1 \in (0,\infty)$ such that one can choose $C^\star \in (0,\infty)$ in the definitions of $\mathcal{E}^\star_{n,k}$ and $\mathcal{E}^{\star\star}_{n,k}$ so that, for all $N \ge N_0$ and all $n \ge n_0$,*

$$\mathbb{P}\Bigl(\exists k^\star, k_\star\colon \mathrm{e}^{\sqrt{\log n}} < k^\star < k_\star < n,\ \mathcal{E}^\star_{n,k^\star} \cap \mathcal{E}^{\star\star}_{n,k_\star} \text{ occurs}\Bigr) \ge 1 - \frac{c_1}{\log\log n}.$$

Proof. Abbreviate by $E^\star_k$ the first event on the right of (5.3.22). This event is measurable with respect to χ_k^f and so $\{E_k\colon k = 1,\dots,n\}$ are independent. By Lemma 4.4.13, $\mathbb{P}(E^\star_k) \ge p$ holds for some $p > 0$ and all k as soon as $N \ge N_0$. We are first interested in a simultaneous occurrence of $E^\star_k$ and $\{M_{n,k} \le -C^\star\sqrt{k\log\log(k)}\}$.

Recalling that $k \mapsto M_{n,k}$ is a random walk, define the stopping time

$$T_n := \inf\bigl\{k\colon \mathrm{e}^{\sqrt{\log(n)}} \le k \le n,\ M_{n,k} \le -2C^\star\sqrt{k\log\log(k)}\bigr\}.$$

Then, for $C^\star$ sufficiently small, Lemma 4.3.9 shows

$$\mathbb{P}\bigl(T_n > n/4\bigr) \le \frac{c_1}{\log\log n}$$

for some constant $c_1 \in (0,\infty)$. Since the increments of $M_{n,k}$ are independent centered Gaussians with a uniform bound on their tail, for the event

$$\mathcal{G}_{n,k} := \Bigl\{M_{n,k+j+1} - M_{n,k+j} \le \log(k)\colon 0 \le j \le \log(k)^2\Bigr\}$$

the fact that $T_n \ge \mathrm{e}^{\sqrt{\log(n)}}$ yields

$$\mathbb{P}\Big(\{T_n \le n/4\} \cap \mathcal{G}_{n,T_n}\Big) \ge 1 - \frac{2c_1}{\log\log n}$$

as soon as n is larger than a positive constant. Under a similar restriction on n, we then also have

$$\{T_n \le n/4\} \cap \mathcal{G}_{n,T_n} \subseteq \bigcap_{T_n \le k \le T_n + (\log T_n)^2} \{M_{n,k} \le -C^\star \sqrt{k \log\log(k)}\}$$

Therefore, on the event on the left, $E_k^\star \cap \{M_{n,k} \le -C^\star \sqrt{k\log\log(k)}\}$ will not occur for some $k < n/2$ only if the sequence $\{1_{E_k^{\mathrm{c}}} \colon 1 \le k \le n\}$ contains a run of 1's of length at least $\log(n)^2$. This has probability $n(1-p)^{\lfloor \log(n) \rfloor^2}$. As $p > 0$, we get

$$\mathbb{P}\Big(\bigcup_{1 \le k < n/2} E_k^\star \cap \{M_{n,k} \le -C^\star \sqrt{k\log\log(k)}\}\Big) \ge 1 - \frac{2c_1}{\log\log n}$$

as soon as n is larger than some positive constant.

For event $\mathcal{E}_{n,k}^{\star\star}$, the fact that the coarse field η^c on $A_{n,k}$ has uniformly bounded variances implies, via Corollary 4.3.8,

$$\mathbb{P}\Bigg(\bigcup_{0 \le k - n/2 \le (\log n)^2} \Big\{\min_{v \in A_{n,k}} \eta^c_{k,v} \ge -\log\log(N')\Big\}\Bigg) \ge 1 - c'(\log n)^2 \mathrm{e}^{-c''(\log\log N')^2}$$

for some $c', c'' > 0$. Proposition 4.4.11 in turn shows that the first event on the right of (5.3.23) has a uniformly positive probability. The claim then follows as before. □

Now we can complete:

Proof of Proposition 5.3.8. Fix $N' \ge 1$ large and, given $\delta \in (0,1)$, let n be the largest integer such that $N := b^{-n}N' > N'\mathrm{e}^{-(\log N')^\delta}$. (We are assuming the setting of Lemma 5.3.9.)

Abbreviate $k_n := \mathrm{e}^{\sqrt{\log n}}$ and suppose that the event

$$\mathcal{E}^{\star}_{n,k^\star} \cap \mathcal{E}^{\star\star}_{n,k_\star} \cap \{\Delta' \le \log\log(N')\} \cap \bigcap_{k_n \le k \le n} \{\Delta_{k_n,k} \le \widetilde{C}\sqrt{k}\} \tag{5.3.24}$$

occurs for some $k^\star, k_\star$ with $k_n \le k^\star < k_\star \le n$. Then

$$\begin{aligned} R_{A_{n,k^\star},\eta}\big(\partial^{\mathrm{in}} A_{n,k^\star}, \partial^{\mathrm{out}} A_{n,k^\star}\big) &\ge \mathrm{e}^{-2\gamma(\Delta' + M_{n,k^\star} + \Delta_{k_n,k^\star})} R_{A^\star_{n,k^\star},\chi^f_{k^\star}}\big(\partial^{\mathrm{in}} A_{n,k^\star}, \partial^{\mathrm{out}} A_{n,k^\star}\big) \\ &\ge \mathrm{e}^{2\gamma[C^\star\sqrt{k^\star \log\log k^\star} - \widetilde{C}\sqrt{k^\star} - \log\log(N')]}\, \mathrm{e}^{-3\hat{c}\log\log(N')} \\ &\ge \mathrm{e}^{\tilde{c}\sqrt{k^\star \log\log k^\star}} \end{aligned} \tag{5.3.25}$$

holds for some constant $\tilde{c} > 0$, where we used that $k^\star \ge k_n$ implies $\sqrt{k^\star} \gg \log\log(N')$ as soon as N' is sufficiently large. Similarly, abbreviating $\mathfrak{m}_{n,k} := \min_{v \in A_{n,k}} \eta^c_{k,v}$, we get

$$\begin{aligned} R_{A_{n,k_\star},\eta} \le \mathrm{e}^{-2\gamma(\mathfrak{m}_{n,k_\star} - \Delta')} R_{A_{n,k_\star},\chi^f_{k_\star}} &\le \mathrm{e}^{4\gamma\log\log(N')}\, \mathrm{e}^{\hat{c}\log\log(N')} \\ &\le R_{A_{n,k^\star},\eta}\big(\partial^{\mathrm{in}} A_{n,k^\star}, \partial^{\mathrm{out}} A_{n,k^\star}\big) - \log(N') \end{aligned} \tag{5.3.26}$$

where we again used that $\sqrt{k^\star} \gg \log\log(N')$.

Now observe that if $v \in B(N)$, then the Nash-Williams estimate implies

$$R_{B(N'),\eta}\big(v, \partial B(N')\big) \ge R_{B(N'),\eta}\big(v, \partial^{\mathrm{in}} A_{n,k^\star}\big) + R_{A_{n,k^\star},\eta}\big(\partial^{\mathrm{in}} A_{n,k^\star}, \partial^{\mathrm{out}} A_{n,k^\star}\big) \tag{5.3.27}$$

while the Series Law gives

$$R_{B(N'),\eta}(0,v) \le R_{B(N'),\eta}\big(0, \partial^{\mathrm{out}} A_{n,k_\star}\big) + R_{B(N'),\eta}\big(v, \partial^{\mathrm{out}} A_{n,k_\star}\big) + R_{A_{n,k_\star},\eta}$$

Since $k^\star < k_\star$ implies that $A_{n,k^\star}$ lies outside $A_{n,k_\star}$, we also have

$$R_{B(N'),\eta}\big(v, \partial^{\mathrm{in}} A_{n,k^\star}\big) \ge R_{B(N'),\eta}\big(v, \partial^{\mathrm{out}} A_{n,k_\star}\big) \tag{5.3.28}$$

Combining (5.3.26–5.3.28) we thus get that $D_{N',\eta}(v) \ge \log N'$ for all $v \in B(N)$ as soon as the event in (5.3.24) occurs. The claim now follows (for N replaced by N') from (5.3.21) and Lemma 5.3.9. ☐

5.3.4 Proofs of the main results

We will now move to prove the remaining part of our main results. Fix $\delta \in (0,\infty)$ small, abbreviate $N_\delta := N\mathrm{e}^{(\log N)^\delta}$ and consider the set

$$\Xi_N^\star := \{0\} \cup \partial B(N) \\ \cup \Big\{ v \in A(N_\delta, 2N_\delta) \colon R_{B(N+1)_\eta}(0,v) \vee R_{B(N+1)_\eta}(v, \partial B(N)) \le \mathrm{e}^{(\log N)^{1/2+\delta}} \Big\}.$$

We again claim:

Lemma 5.3.10. *For each $\delta > 0$, there is $c > 0$ such that for all N sufficiently large,*

$$\mathbb{P}\Big(\pi_\eta(\Xi_N^\star) \le N^{\psi(\gamma)} \mathrm{e}^{-(\log N)^\delta} \Big) \le \frac{c}{(\log N)^2}. \tag{5.3.29}$$

Proof. Using the same proof, Lemma 5.3.1 applies also for resistivity $R_{B(N)_\eta}(v, \partial B(N))$. In light of

$$R_{B(N+1)_\eta}\big(v, \partial B(N)\big) \le R_{B(N+1)_\eta}(v,u), \qquad u \in \partial B(N),$$

Corollary 4.4.18 applies to $R_{B(N+1)_\eta}(v, \partial B(N))$ just as well. Combining this with (5.3.6), we now proceed as in the proof of Lemma 5.3.2 to get the result. ☐

We are now ready to give:

Proof of Theorem 5.1.2. The upper bound has already been shown in Lemma 5.3.5, so we just need to derive the corresponding lower bound. For this we write (5.3.15) as a bound and apply (5.3.17) with Proposition 5.3.8 to get that, with probability tending to one as $N \to \infty$,

$$\mathbb{E}^0(\tau_{B(N)^c}) \geq R_{B(N+1)_\eta}\big(0, \partial B(N)\big) \sum_{v \in \Xi_N^\star} \pi_\eta(v)\phi(v) \geq \pi_\eta(\Xi_N^\star)\log(N) \tag{5.3.30}$$

The claim then follows from Lemma 5.3.10. □

We then use the lower bound on the expected exit time to also get:

Proof of Theorem 5.1.1. The upper bound on the return probability has already been proved in Lemma 5.3.4, so we will focus on the lower bound and recurrence. Consider again the random walk $\widetilde{X}$ on $B(N+1)$ and let Y be its trace on $\Xi_N^\star$. Let $\hat\tau_{\partial B(N)} := \inf\{k \geq 0 \colon Y_k \in \partial B(N)\}$. Then

$$\begin{aligned}\mathbb{E}^0(\hat\tau_{\partial B(N)}) &\leq T\mathbb{P}^0(\hat\tau_{\partial B(N)} \leq T) + \mathbb{P}^0(\hat\tau_{\partial B(N)} > T)\big(T + \max_{v \in \Xi_N^\star \smallsetminus \partial B(N)} \mathbb{E}^v(\hat\tau_{\partial B(N)})\big) \\ &= T + \mathbb{P}^0(\hat\tau_{\partial B(N)} > T) \max_{v \in \Xi_N^\star \smallsetminus \partial B(N)} \mathbb{E}^v(\hat\tau_{\partial B(N)})\end{aligned} \tag{5.3.31}$$

The hitting time estimate in conjunction with the definition of $\Xi_N^\star$ gives

$$\mathbb{E}^v(\hat\tau_{\partial B(N)}) \leq \pi_\eta(\Xi_N^\star)\,\mathrm{e}^{(\log N)^{1/2+\delta}}, \qquad v \in \Xi_N^\star \smallsetminus \partial B(N)$$

whereby we get

$$\mathbb{P}^0(\hat\tau_{\partial B(N)} > T) \geq \pi_\eta(\Xi_N^\star)^{-1}\mathrm{e}^{-(\log N)^{1/2+\delta}}\big(\mathbb{E}^0(\hat\tau_{\partial B(N)}) - T\big).$$

Since (5.3.30) applies also for the expectation of $\hat\tau_{\partial B(N)}$, the choice $N := T^{1/\psi(\gamma)}\mathrm{e}^{(\log N)^\delta}$ implies $\mathbb{E}^0(\hat\tau_{\partial B(N)}) \geq 2T$ and thus, using (5.3.30) one more time,

$$\mathbb{P}^0(\hat\tau_{\partial B(N)} > T) \geq \mathrm{e}^{-(\log N)^{1/2+\delta}}.$$

But $\hat{\tau}_{\partial B(N)} \le \tau_{\partial B(N)} := \inf\{k \ge 0\colon X_k \in \partial B(N)\}$ and so we get

$$\mathbb{P}^0\bigl(X_T \in B(N)\bigr) \ge \mathbb{P}^0(\tau_{\partial B(N)} > T) \ge \mathrm{e}^{-(\log N)^{1/2+\delta}}$$

as well. Using this in (5.3.13), the desired lower bound then follows from, e.g., (5.3.6).

It remains to show recurrence. Here we note that (5.3.25) and (5.3.27) along with Lemma 5.3.9 imply that $R_{B(N),\eta}(0, \partial B(N)) \to \infty$ in probability along a sufficiently rapidly growing deterministic sequence of N's. Since the sequence of resistances is increasing in N, the convergence holds almost surely. By a well known criterion, this implies recurrence. $\square$

REFERENCES

[1] R. J. Adler. An introduction to continuity, extrema and related topics for general gaussian processes. 1990. Lecture Notes - Monograph Series. Institute Mathematical Statistics, Hayward, CA.

[2] J. Ambjørn and T. G. Budd. Geodesic distances in Liouville quantum gravity. *Nuclear Phys. B*, 889:676–691, 2014.

[3] Jan Ambjørn, Jakob L. Nielsen, Juri Rolf, Dimitrij Boulatov, and Yoshiyuki Watabiki. The spectral dimension of 2d quantum gravity. *Journal of High Energy Physics*, 1998(02):010, 1998.

[4] Antonio Auffinger, Michael Damron, and Jack Hanson. 50 years of first passage percolation. Preprint, available at http://arxiv.org/abs/1511.03262.

[5] Martin T. Barlow, Thierry Coulhon, and Takashi Kumagai. Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. *Comm. Pure Appl. Math.*, 58(12):1642–1677, 2005.

[6] Vincent Beffara and Hugo Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probab. Theory Related Fields*, 153(3-4):511–542, 2012.

[7] Itai Benjamini. Random planar metrics. In *Proceedings of the International Congress of Mathematicians. Volume IV*, pages 2177–2187. Hindustan Book Agency, New Delhi, 2010.

[8] Nathanaël Berestycki. Diffusion in planar liouville quantum gravity. to appear in *Ann. Inst. Henri Poincaré Probab. Stat.*

[9] Nathanaël Berestycki. Introduction to the gaussian free field and liouville quantum gravity. Available at http://www.statslab.cam.ac.uk/~beresty/Articles/oxford4.pdf.

[10] Nathanaël Berestycki, Christophe Garban, Rémi Rhodes, and Vincent Vargas. KPZ formula derived from liouville heat kernel. Preprint, arXiv:1406.7280.

[11] Kenneth A. Berman and Mokhtar H. Konsowa. Random paths and cuts, electrical networks, and reversible Markov chains. *SIAM J. Discrete Math.*, 3(3):311–319, 1990.

[12] Marek Biskup. Recent progress on the random conductance model. *Probab. Surv.*, 8:294–373, 2011.

[13] Marek Biskup, Jian Ding, and Subhajit Goswami. Return probability and recurrence for the random walk driven by two-dimensional gaussian free field. Preprint, available at https://arxiv.org/abs/1611.03901.

[14] Marek Biskup and Roman Kotecký. Phase coexistence of gradient Gibbs states. *Probab. Theory Related Fields*, 139(1-2):1–39, 2007.

[15] Marek Biskup and Oren Louidor. Full extremal process, cluster law and freezing for two-dimensional discrete Gaussian Free Field. 2016. Preprint, available at `http://arxiv.org/abs/1606.00510`.

[16] Marek Biskup and Oren Louidor. On intermediate level sets of two-dimensional discrete Gaussian Free Field. 2016. arXiv:1612.01424.

[17] Erwin Bolthausen, Jean-Dominique Deuschel, and Giambattista Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):1670–1692, 2001.

[18] Christer Borell. The Brunn-Minkowski inequality in Gauss space. *Invent. Math.*, 30(2):207–216, 1975.

[19] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 69(1):62–123, 2016.

[20] Maury Bramson and Ofer Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65:1–20, 2011.

[21] David Carpentier and Pierre Le Doussal. Glass transition of a particle in a random potential, front selection in nonlinear rg and entropic phenomena in liouville and sinhgordon models. *Phys. rev. E.*, 63:026110, 2001.

[22] H E Castillo and P Le Doussal. Freezing of dynamical exponents in low dimensional random media. *Phys. Rev. Lett.*, 86:4859, 2001.

[23] Ashok K. Chandra, Prabhakar Raghavan, Walter L. Ruzzo, Roman Smolensky, and Prasoon Tiwari. The electrical resistance of a graph captures its commute and cover times. *Comput. Complexity*, 6(4):312–340, 1996/97.

[24] S. Chatterjee, A. Dembo, and J. Ding. On level sets of gaussian fields. Preprint, available at `http://arxiv.org/abs/1310.5175`.

[25] Olivier Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3):962–986, 2006.

[26] François David and Michel Bauer. Another derivation of the geometrical KPZ relations. *J. Stat. Mech. Theory Exp.*, (3):P03004, 9, 2009.

[27] J. Ding, R. Roy, and O. Zeitouni. Convergence of the centered maximum of log-correlated gaussian fields. Preprint, available at `http://arxiv.org/abs/1503.04588`.

[28] Jian Ding. Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):285–299, 2013.

[29] Jian Ding and Alex Dunlap. Liouville first passage percolation: subsequential scaling limits at high temperatures. Preprint, available at `http://arxiv.org/abs/1605.04011`.

[30] Jian Ding and Subhajit Goswami. Liouville first passage percolation: the weight exponent is strictly less than 1 at high temperature. Preprint, available at https://arxiv.org/abs/1605.08392.

[31] Jian Ding and Subhajit Goswami. First passage percolation on the exponential of two-dimensional branching random walk. 2015. Preprint, available at http://arxiv.org/abs/1511.06932.

[32] Jian Ding and Subhajit Goswami. Upper bounds on Liouville first passage percolation and Watabiki's prediction. 2016. arXiv:1610.09998.

[33] Jian Ding and Li Li. Chemical distances for level-set percolation of two-dimensional discrete Gaussian free fields. Preprint, available at http://arxiv.org/abs/1605.04449.

[34] Jian Ding, Ofer Zeitouni, and Fuxi Zhang. On the liouville heat kernel for k-coarse mbrw and nonuniversality. *arXiv preprint arXiv:1701.01201*, 2017.

[35] Jian Ding and Fuxi Zhang. Liouville first passage percolation: geodesic dimension is strictly larger than 1 at high temperatures. Preprint, available at https://arxiv.org/abs/1610.02766.

[36] Jian Ding and Fuxi Zhang. Non-universality for first passage percolation on the exponential of log-correlated gaussian fields. Preprint, available at http://arxiv.org/abs/1506.03293.

[37] Julien Dubédat. SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*, 22(4):995–1054, 2009.

[38] Hugo Duminil-Copin, Clément Hongler, and Pierre Nolin. Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Comm. Pure Appl. Math.*, 64(9):1165–1198, 2011.

[39] Hugo Duminil-Copin, Ioan Manolescu, and Vincent Tassion. An RSW theorem for Gaussian free field. In preparation.

[40] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. A new computation of the critical point for the planar random-cluster model with $q \geq 1$. 2016. Preprint, arXiv 1604.03702.

[41] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion. Continuity of the phase transition for planar random-cluster and potts models with $1 \leq q \leq 4$. 2015. Preprint, arXiv 1505.04159.

[42] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Critical Gaussian multiplicative chaos: convergence of the derivative martingale. *Ann. Probab.*, 42(5):1769–1808, 2014.

[43] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[44] Christophe Garban, Rémi Rhodes, and Vincent Vargas. Liouville Brownian motion. Preprint, available at http://arxiv.org/abs/1301.2876.

[45] Christophe Garban, Rémi Rhodes, and Vincent Vargas. On the heat kernel and the Dirichlet form of Liouville Brownian motion. *Electron. J. Probab.*, 19:no. 96, 25, 2014.

[46] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[47] Geoffrey R. Grimmett and Harry Kesten. Percolation since Saint-Flour. In *Percolation theory at Saint-Flour*, Probab. St.-Flour, pages ix–xxvii. Springer, Heidelberg, 2012.

[48] Ewain Gwynne, Nina Holden, and Xin Sun. A distance exponent for Liouville quantum gravity. 2016. Preprint, available at http://arxiv.org/abs/1606.01214.

[49] Xiaoyu Hu, Jason Miller, and Yuval Peres. Thick points of the Gaussian free field. *Ann. Probab.*, 38(2):896–926, 2010.

[50] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985.

[51] Richard Kenyon. Dominos and the Gaussian free field. *Ann. Probab.*, 29(3):1128–1137, 2001.

[52] Takashi Kumagai. *Random walks on disordered media and their scaling limits*, volume 2101 of *Lecture Notes in Mathematics*. Springer, Cham, 2014. Lecture notes from the 40th Probability Summer School held in Saint-Flour, 2010, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[53] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[54] M. Ledoux. The concentration of measure phenomenon, volume 89 of mathematical surveys and monographs. 2001. American Mathematical Society, Providence, RI.

[55] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[56] Titus Lupu and Wendelin Werner. The random pseudo-metric on a graph defined via the zero-set of the gaussian free field on its metric graph. Preprint, available at https://arxiv.org/abs/1607.06424.

[57] Russell Lyons and Yuval Peres. *Probability on Trees and Networks*. Cambridge University Press, 2016. Available at http://pages.iu.edu/~rdlyons/.

[58] Thomas Madaule. Maximum of a log-correlated Gaussian field. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(4):1369–1431, 2015.

[59] Pascal Maillard, Rémi Rhodes, Vincent Vargas, and Ofer Zeitouni. Liouville heat kernel: regularity and bounds. 2014. Preprint, available at `http://arxiv.org/abs/1406.0491`.

[60] Jason Miller and Scott Sheffield. Quantum Loewner evolution. 2015. Preprint, available at `http://arxiv.org/abs/1409.7055`.

[61] Jason Miller and Scott Sheffield. Liouville quantum gravity and the brownian map III: the conformal structure is determined. 2016. Preprint, available at `https://arxiv.org/abs/1608.05391`.

[62] Loren D. Pitt. Positively correlated normal variables are associated. *Ann. Probab.*, 10(2):496–499, 1982.

[63] A. M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103(3):207–210, 1981.

[64] Rémi Rhodes and Vincent Vargas. Lecture notes on gaussian multiplicative chaos and liouville quantum gravity. Preprint, available at `http://arxiv.org/abs/1602.07323`.

[65] Rémi Rhodes and Vincent Vargas. KPZ formula for log-infinitely divisible multifractal random measures. *ESAIM Probab. Stat.*, 15:358–371, 2011.

[66] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probab. Surv.*, 11:315–392, 2014.

[67] Brian Rider and Bálint Virág. The noise in the circular law and the Gaussian free field. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm006, 33, 2007.

[68] Lucio Russo. A note on percolation. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 43(1):39–48, 1978.

[69] Lucio Russo. On the critical percolation probabilities. *Z. Wahrsch. Verw. Gebiete*, 56(2):229–237, 1981.

[70] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field [mr2486487]. In *Selected works of Oded Schramm. Volume 1, 2*, Sel. Works Probab. Stat., pages 1031–1147. Springer, New York, 2011.

[71] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013.

[72] P. D. Seymour and D. J. A. Welsh. Percolation probabilities on the square lattice. *Ann. Discrete Math.*, 3:227–245, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[73] Alexander Shamov. On Gaussian multiplicative chaos. *J. Funct. Anal.*, 270(9):3224–3261, 2016.

[74] Scott Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007.

[75] David Slepian. The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.*, 41:463–501, 1962.

[76] V. N. Sudakov and B. S. Cirel′son. Extremal properties of half-spaces for spherically invariant measures. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 41:14–24, 165, 1974. Problems in the theory of probability distributions, II.

[77] Vincent Tassion. Crossing probabilities for voronoi percolation. Preprint, available at `http://arxiv.org/abs/1410.6773`.

[78] Yoshiyuki Watabiki. Analytic study of fractal structure of quantized surface in two-dimensional quantum gravity. *Progress of Theoretical Physics Supplement*, 1993(114):1–17, 1993.

[79] Wendelin Werner and Hao Wu. On conformally invariant CLE explorations. *Comm. Math. Phys.*, 320(3):637–661, 2013.