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SECURITIZATION AND LIQUIDITY CREATION IN MARKETS WITH ADVERSE SELECTION

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ABSTRACT

This paper provides an information-based theory of tranching, a practice in which sellers slice a financial asset into debt securities with different seniority. I use the competitive search framework to analyze asset-backed security markets with adverse selection and find that tranching is a robust equilibrium outcome. Tranching decomposes the asset into "information insensitive" and "information sensitive" components. The expected cash flow of the information insensitive component is independent of the seller's private signal, whereas the expected cash flow of the information sensitive component varies with the signal. When buyers are restricted to trade shares of assets, they have to purchase both components proportionally. Tranching, however, allows buyers to disproportionately purchase the information insensitive component. As a result, buyers are less concerned about adverse selection, and total trading volume in the market increases. My model also generates testable predictions on the liquidity of individual debt securities: the selling probability of a debt security increases in its seniority, which is observable to buyers, yet decreases with its performance, which is ex-ante unobservable to buyers; these predictions are supported by my empirical analysis of the non-agency MBS market. (JEL D82, G12, G32)

CHAPTER 1

INTRODUCTION

1.1 Introduction

Issuing multi-tiered debt securities is a prevalent phenomenon in securitization. Instead of trading the whole financial asset, a seller slices it into debt securities of different seniority, which are then offered to buyers. The asset-generated cash flow is later distributed according to the seniority of the debt securities, with senior debt holders being paid before subordinated debt holders. This practice, also known as tranching, dominates the non-agency mortgage-backed securities (MBS) market, which at its peak had a volume exceeding two trillion dollars. Beyond non-agency MBS, multi-tiered debt structures are also commonly used in other asset-backed securities markets. Why are multi-tiered debt securities so actively traded, and what is the benefit of creating them? The answer to these questions matters because it could change our evaluation of policies regulating securitization activities.

A popular view of tranching is that sellers package "toxic" assets into multi-tiered debt securities to create an illusion of safeness and trick unsuspecting buyers into purchasing the overpriced securities. This narrative, however, goes against recent empirical findings. First of all, highly-rated MBS are not that risky. Ospina and Uhlig (2016) show that the cumulative loss of AAA-rated non-agency MBS issued between 2006 and 2008 is less than 5 percent of the principal, which is a relatively small loss given the magnitude of the house price collapse in that period.[1] Moreover, buyers of MBS are compensated for their risk-taking. Chernenko, Hanson, and Sunderam (2016) show that the average credit spread on AAA-rated non-prime MBS is about 20 bps higher than that of AAA-rated corporate bonds between 2004Q1 and 2007Q2.[2] These findings call for an alternative explanation for the use of multi-tiered debt

1. The cumulative loss for AAA-rated non-agency MBS issued before 2004 is much lower (less than 0.5 percent).

2. As a comparison, the average credit spread on BBB corporate bonds is about 30 bps higher than that of AAA corporate bonds.

securities in securitization.

This paper argues that rather than functioning as a trick to deceive buyers, tranching mitigates adverse selection and improves allocative efficiency in the market. Impatient sellers charge actuarially fair prices for securities conditional on both public information and their private information, but many cannot sell all of their future cash flow due to buyers' adverse selection concerns. These sellers prefer selling multi-tiered debt securities because doing so allows them to sell a larger fraction of their cash flow.

To support this argument, I analyze securitization under adverse selection in a stylized model of decentralized asset-backed securities markets.[3] In this paper, I consider an economy comprised of sellers and buyers. There are gains from trade if sellers sell their assets to buyers,[4] but not all of these gains can be realized due to information asymmetry: a seller estimates the total cash flow generated by his asset more precisely than buyers. Sellers and buyers can trade securities whose payouts are contingent on the total cash flow,[5] and they do so by simultaneously posting asset-backed security contracts, each of which specifies the security they want to trade and the price. For a given contract, the trading probabilities are determined endogenously by the ratio of demand and supply in the market for that specific contract. If there is excess supply of a contract, sellers (the long side) will be rationed probabilistically. Similarly, if there is excess demand, buyers will be rationed.[6] Under reasonable assumptions regarding the set of private information and restrictions on beliefs in inactive markets, I show that issuing multi-tiered debt securities will endogenously arise.

3. My model builds on Guerrieri, Shimer, and Wright (2010); Guerrieri and Shimer (2014); Guerrieri and Shimer (2016); Chang (2017) and Williams (2016).

4. For MBS market, sellers will be investment banks, and buyers will be mutual funds, insurance companies, hedge funds and CDO managers. The gain from trade comes from heterogeneity in cost of funds among financial intermediaries. The assets are mortgage pools underlying the MBS.

5. This includes the straight trading of the asset as a special case.

6. There is no search friction in my model because I assume a Leontief matching function and a zero search cost. As a result, rationing and retention is isomorphic here. One can imagine an alternative model where buyers and sellers can also contract on the retention of a security. As long as buyers of a security cannot contract with a seller on how much the seller will retain other securities, the trading pattern in the alternative model will be identical to my model. Separation of different types of sellers there can work through rationing, retention, or a combination of both.

In my model, the choice of security matters because securities differ in their liquidity profiles.[7] In a separating equilibrium, buyers induce sellers to post actuarially fair prices for their securities through rationing. Suppose there are only two types of sellers in the economy, bad and good sellers. Because the alternative of not being able to trade is more costly to bad sellers who value their securities less, buyers ration sellers who post high prices (the fair prices for good sellers) in order to screen out bad sellers.[8] Consequently, although all securities are liquid at the low prices (the fair prices for bad sellers), they have various degrees of illiquidity at the high prices. The liquidity profile of a security is determined by its information sensitivity, which measures the deviating incentive for bad sellers. When a security is more information sensitive, bad sellers have more to gain by selling the security at the high price. Consequently, good sellers have to endure a much lower trading probability so that the bad sellers' incentive compatibility constraint can be satisfied.

Good sellers in my model prefer to sell multi-tiered debt securities because it reduces a bad seller's incentive to deviate and allows them to sell a larger fraction of their cash flow. Debt securities of different seniority are the extremes of all monotone securities: any intermediate security is a bundle of debt securities. For good sellers, trading debt securities in a bundle is dominated by trading them separately. When debt securities are traded in a bundle, buyers have to purchase them in a fixed proportion. But when debt securities are traded separately, buyers can disproportionately purchase debt securities with lower information sensitivity,[9] thus reducing the average information sensitivity of successfully sold securities. With lower information sensitivity, bad sellers have less incentive to deviate, and good sellers can sell a larger fraction of their cash flow. In aggregate, the use of multi-tiered debt securities, or tranching, increases aggregate trading volume and reduces inefficiency from information

7. Here I follow the competitive search literature in modeling liquidity as selling probability. In the finance literature, it also is common to model liquidity as underpricing.

8. Mathematically, a seller's payoff satisfies decreasing difference property in selling probability and his type. Thus, the optimal selling probability for him decreases with his type.

9. Under model assumptions, senior securities have lower information sensitivity.

frictions.

My model produces testable predictions on the association between liquidity and safeness in the cross-section of securities. Liquidity positively correlates with seniority, which is observable to buyers, yet negatively correlates with performance, which is ex-ante unobservable to buyers.[10] For the first part, information sensitivity, or sellers' deviating incentive, decreases with seniority in the model. Consequently, sellers can trade a more senior debt with a higher probability. For the second part, like standard models of adverse selection and rationing, selling probability in my model decreases with the unobservable quality of the underlying asset. Since securities collateralized by better quality assets are more likely to perform, the selling probability of securities which perform will be lower than that of securities which default.

These theoretical predictions are supported by my empirical analysis of non-agency MBS issued between 2000 and 2007. Specifically, subordinated MBS overall are less likely to be sold than senior MBS; conditioning on seniority, debt securities that perform are less likely to be sold ex-ante compared with securities that default. This evidence suggests that sellers do have private information, and such information systematically affected the trading pattern in the market.

Acknowledging that tranching improves allocative efficiency helps to frame the policy discussion on securitization. For example, mandatory risk retention has been adopted by regulators in various countries.[11] Under this policy, a seller must retain a fraction of the credit risk of the assets that collateralize an issuance. Proposers argue that mandatory risk retention mitigates agency problems related to securitization activities. To evaluate this claim, I extend my baseline model by allowing some bad sellers to change their type

10. Guerrieri, Shimer, and Wright (2010) and Guerrieri and Shimer (2014) also find that liquidity is negatively associated with asset quality. However, the correlation is not testable since econometricians do not observe sellers' private information. My paper connects private information with the default status of debt securities, thus generating several testable results.

11. These include the Dodd-Frank Act in the US, the Capital Requirements Directive in the EU and Financial Services Authority prudential rules in the UK.

at a cost. With this slight modification, I show that mandatory risk retention can indeed improve the average quality of assets and obtain productive efficiency. However, the outcome is achieved with a considerable loss in allocative efficiency: mandatory risk retention prevents mutually beneficial transactions from happening. In the simple model, social welfare overall is worsened under the mandatory risk retention.

The literature on security design often finds it optimal for sellers to issue a single debt security after obtaining private information.[12] However, sellers in reality usually issue multiple debt securities rather than a single debt, and all of these securities are actively traded.[13] The counterfactual prediction of existing literature results from a full transparency assumption, that is, a buyer can observe how her counterparty trade with third parties. In practice, full transparency can be hard to achieve: sellers can provide trade with third parties privately, and the buyer cannot observe these transactions.[14]

In a novel contribution to the literature, my paper discusses the optimal security design problem when trading is not fully transparent. When trading is fully transparent, what matters is the integral incentive compatibility constraint over all securities and prices, and good sellers can signal their type by retaining a sufficiently large residual claim. When trading is not fully transparent, the incentive compatibility constraints have to be satisfied market-by-market, and good sellers can no longer use the retention of a residual claim to increase the selling probability of the debt security. Consequently, though selling a single debt security and retaining the residual claim further reduces the information sensitivity of the sold security, a good seller in my model cannot do so because buyers do not believe he will hold on to the residual claim. The lack of transparency gives rise to the issurance of multi-tiered debt securities.

Equilibrium models of adverse selection usually assume that sellers offer a single contract

12. See DeMarzo (2005) and DeMarzo, Frankel, and Jin (2015).

13. For example, even the unrated debt securities of a non-agency MBS deal are frequently sold to hedge funds or managers of collateral debt obligations (CDO).

14. For example, sellers can reduce their exposure through various off-balance sheet instruments.

and are matched with buyers one-to-one.[15] I extend existing models by allowing sellers to produce many goods from a production set and offer many contracts to many buyers.[16] This extension is crucial to the study of asset-backed security markets, both because we observe that sellers divide their assets into heterogeneous pieces and because a model without such an extension will underpredict aggregate trading volume and allocative efficiency. The application of this extension, however, is not limited to asset-backed security markets. In fact, the model can be readily used to explain the financing decisions of non-investment grade companies, among which a multi-tiered capital structure is common.[17]

In summary, my paper provides an alternative explanation for the use of multi-tiered debts in securitization activities, which helps us to understand why people tranche and how we should regulate securitization. The remainder of this paper proceeds as follows. Section 1.2 discusses the related literature. Section 2.1 models the asset-backed security markets with adverse selection. Section 2.2 demonstrates that the issuance of multi-tiered debt securities arises endogenously in equilibrium. Section 3.1 presents empirical evidence that supports the model's predictions. Section 3.2 evaluates the effects of two popular policies: mandatory risk retention and government security purchase. Section 4.1 extends the baseline model by assuming different degrees of transparency. Section 5.1 concludes.

1.2 Literature

This paper follows the literature that examines adverse selection in competitive search markets. Guerrieri, Shimer, and Wright (2010) set up the general method to analyze adverse selection in decentralized markets with search friction. Their analysis of asset markets illustrates that buyers will use rationing to screen good assets from bad ones because it's less costly for sellers with a good asset to be rationed. Moreover, when there is a positive search

15. See Gale (1996); Guerrieri, Shimer, and Wright (2010) and Williams (2016).

16. Lack of transparency makes the distinction between one contract and many contracts meaningful. If trading is fully transparent, selling multiple contracts is equivalent to selling one random contract.

17. See Rauh and Sufi (2010).

cost, sellers will not retain a positive share of the asset in equilibrium. Williams (2016) shows that under a different matching technology and positive search cost, retaining a share of the asset will never happen under uni-dimensional private information, but it can operate simultaneously with rationing when private information is multi-dimensional. In contrast to Guerrieri, Shimer, and Wright (2010) and Williams (2016), sellers and buyers in my model can trade not only shares of assets but also securities whose payouts are contingent on the asset-generated cash flow in nonlinear fashions. I find that expanding the security space beyond linear securities can increase aggregate trading volume and reduce inefficiency resulted from information frictions.

The model discussed in this paper is closely related to security design literature based on adverse selection. DeMarzo (2005) and the companion paper DeMarzo, Frankel, and Jin (2015) discuss which combination of securities a seller should issue if they can design securities after observing private information, and they find that selling a single debt security and retaining a residual claim is optimal for sellers.[18] My results are different. In my model, buyers cannot observe whether or not a seller trade with third parties. Consequently, good sellers have an incentive to offer the residual claim to another buyer rather than retaining it. On the other hand, anticipating that sellers will sell the residual claim with positive probability, buyers will start to ration the debt security. Thus, the equilibrium in DeMarzo (2005) falls apart when trading is not fully transparent. Another difference between DeMarzo (2005) and my paper regards the choice variable. In DeMarzo (2005), a monopolistic seller chooses the quantity, while the price is determined by a downward sloping market demand that results from buyers' concerns about adverse selection. In my paper, competitive sellers post prices while trading probability is determined endogenously through market tightness at each price. My approach delivers a similar result: instead of a downward sloping market demand, sellers in my model face a higher trading probability at a lower price.

18. Other papers in this area include Myers and Majluf (1984); Nachman and Noe (1994); Demarzo and Duffie (1999); DeMarzo (2005) and Biais and Mariotti (2005).

The models of Winton (1995) and Friewald, Hennessy, and Jankowitsch (2016) also predict the use of multi-tiered debt securities. Winton (1995) shows that multi-tiered debt securities can be optimal if investors must independently verify the output. This assumption is not suitable for asset-backed security markets in which ex-post verification is conducted by a master servicer communicating with all investors. Friewald, Hennessy, and Jankowitsch (2016) attribute the trading of multi-tiered debt securities to buyers' preference for variety. In their model, debt securities of different seniority face different degrees of price discounts in the secondary market. Investing in multi-tiered debt securities allows buyers to choose between retaining the subordinated debt or liquidating both debt securities, according to their holding cost shocks. In this paper, I show that multi-tiered debt securities will be actively traded in equilibrium even if buyers do not value variety. Moreover, my model predicts the observed patterns of illiquidity in primary markets,[19] which cannot be explained by preference-based models of tranching.

The concept of the information sensitivity of a security has been discussed in the literature. In models that focus on separating equilibrium, such as Demarzo and Duffie (1999), the information sensitivity of a security depends on the support of expected payouts from the security issued by different types of sellers. In models that focus on pooling equilibrium, such as Dang, Gorton, and Holmström (2013) and Dang, Gorton, and Holmström (2015), information sensitivity depends on the whole distribution of private valuation. Despite their differences, these three papers conclude that offering a single debt security and retaining the residual claim is optimal because it minimizes information sensitivity of the offered security. My paper focuses on separating equilibrium, so my notion of information sensitivity is closer to that of Demarzo and Duffie (1999). In my paper, the average information sensitivity of all securities offered equals the information sensitivity of the asset because sellers cannot credibly commit to retaining anything. However, by tranching the asset into multi-tiered debt contracts, buyers can disproportionately purchase the information insensitive compo-

19. See Section 3.1.

nents of the asset, and in this manner, the average information sensitivity of successfully sold securities is reduced.

My analysis is also related to the literature on multidimensional signaling and screening in financial markets. Chang (2017) and Guerrieri and Shimer (2016) allow for multidimensional types, but they focus only a unidimensional instrument. He (2009) and Williams (2016) consider multidimensional instruments, but the instruments they focus on are retaining or rationing shares of assets. In my model, sellers' types are unidimensional, but the space of instruments for them to signal their types is extremely large. A seller can go beyond retaining a share of his asset by designing various financial securities.

This paper is also relevant to the literature documenting the association between security "safeness" and trading speed or trading volume. Adelino, Gerardi, and Hartman-Glaser (2016) show that mortgages sold later are significantly less likely to default, conditioning on observable mortgage information at the time of sale. Begley and Purnanandam (2016) find that mortgages in MBS deals where a large fraction of the principal is not publicly offered have significantly lower delinquency rates, conditioning on observable mortgage characteristics in the deal prospectus. Friewald, Hennessy, and Jankowitsch (2016) show that, in the secondary market, senior asset-backed securities have a larger trading volume and a smaller round-trip cost. The empirical component of my paper contributes to this literature by looking into the fraction of securities sold during the initial placement of non-agency MBS. From the security-level holding data of a large MBS underwriter, I impute whether individual securities are sold and show that, other things being equal, liquidity increases with the seniority of debt securities, while it decreases with the ex-post performance of the security.

CHAPTER 2

THEORY

2.1 Model

2.1.1 Setup

In this section, I lay out a model of security design in decentralized asset-backed securities markets with adverse selection. The economy lasts for two periods. There are two types of risk-neutral agents in the economy.[1] There is a unit mass of sellers who discount consumption at date 2 with a factor β, $\beta < 1$, and there is a unit mass of buyers who are more patient than natural sellers. The discount factor of buyers is normalized to 1.[2] Each seller is endowed with 1 unit of asset that generates a stochastic amount of consumption good at date 2. The asset owned by the seller i yields a cash flow of $\tilde{a}^i$. The asset-generated cash flow has a support $A \subseteq \mathbb{R}_+$, with $|A| = N \geq 2$. Each buyer is endowed with b units of consumption good at date 1 (numeraire). For the sake of simplicity, I assume that buyers collectively have a deep pocket (a sufficiently large b); thus, competition will drive their equilibrium payoff down to zero.[3]

At the beginning of date 1, seller i privately observes a signal of $\tilde{a}^i$, which captures the information advantage of i on the quality of his asset. Agents other than i have inferior information: they do not observe this signal. Without a loss of generality, one can denote the type of seller i as $\tilde{z}^i$, where $\tilde{z}^i(a) = \text{Prob}\{\tilde{a}^i = a | \text{private signal of } i\}$. The distribution of $\tilde{z}^i$ in the population of sellers is π with a support $Z \subseteq \triangle(A)$.

1. One can relax risk neutrality by introducing stochastic discount factors. Since risk preference is not a focus of this paper, I choose not to do so here.

2. The preference of seller i, $i \in [0, 1]$, can be summarized by $u^i = c_1^i + \beta c_2^i$, where c_t^i is seller i's consumption at period t, $c_t^i \geq 0$. Similarly, the preference of buyer i, $i \in [0, 1]$, can be summarized by $u^i = c_1^i + c_2^i$.

3. The analysis in this paper follows through even when security prices are determined by cash in the market, so long as sellers get a positive share of gains from trade.

Since sellers are less patient than buyers, there will be gains from trade when future cash flow is transferred from sellers to buyers. In the model, this is done through the trading of asset-backed securities. Decentralized markets for asset-backed securities open up at date 1, after sellers observe their signals. Trading is non-exclusive and secret: a seller can trade with multiple buyers and a buyer does not observe how his counterparty trades with third parties.[4] An asset-backed security contract is the pair (s, p), where s represents the security and p the price. An asset-backed security is a promise of future payout which can depend on the realization of future cash flow. Upon issuing s, seller i promises to pay $s(a) \geq 0$ unit of consumption good at date 2 to any agent who holds 1 unit of security s when $\tilde{a}^i = a$. For example, a pass-through security whose payout is identical to the asset-generated cash flow can be represented by the identity function $s(a) = a$, while a standard debt with face value f would be $s(a) = \min\{f, a\}$. In exchange for such a promise, seller i charges a price p in consumption good at date 1. Each asset-backed security contract is traded in a separate market.[5] Each unit of future cash flow or numeraire can only be brought to one market.[6]

Denote S as the set of eligible asset-backed securities and P as the set of eligible prices. In this paper, I assume that agents can only trade securities whose payments are non-negative and weakly increasing in asset-generated cash flow, i.e., $S = \{s : A \to \mathbb{R}_+ | s \text{ weakly increasing}\}$.[7][8] Without loss of generality, let $P = \mathbb{R}_+$. For each type of eligible

4. At the other extreme, DeMarzo (2005) and DeMarzo, Frankel, and Jin (2015) assume that a seller must exclusively trade with a buyer. Their case of exclusive trading is isomorphic to the case that trading is non-exclusive but transparent, which is discussed in Subsection 4.1.2. Reality may be somewhat in the middle: the prospectus of a securitization deal announces how the asset is divided into securities, but it does not announce prices or how much of securities will be sold. Subsection 4.1.1 discusses this intermediate case.

5. From now on, I will use the terms "market" and "contract" interchangeably.

6. For example, even when a seller anticipates that he will be rationed for listing a unit security at a high price, he cannot secretly double list that unit of security.

7. One argument justifying the monotonicity restriction is that monotone securities eliminate the incentive to sabotage. When a seller sells securities with payouts that strictly decrease with total cash flow, she can increase her own gain by sabotaging the cash flow. Similar ideas have been proposed by Nachman and Noe (1994); Innes (1990) and Demarzo and Duffie (1999).

8. Readers may notice that I do not restrict the size of a security. Under risk neutrality, agents care only about expected payoffs. Trading a debt security of face value \$10 with probability ω is equivalent to trading 10 debt securities of face value \$1 independently, each with probability ω. Thus, I can let

security, a seller decides how much to sell at any given price; i.e., he chooses a Borel measure (say, φ) on $S \times P$.[9] Seller i cannot promise more payment than the cash flow he will get at date 2, for each possible realization of $\tilde{a}^i$.This is captured by the sellers' feasibility constraint:

$$\int_{S\times P} s(a)d\varphi(s,p) \leq a, \text{for all } a \in A. \tag{2.1}$$

At the same time, for each type of eligible security, buyers decide how much to buy at any given price, i.e., they also choose a Borel measure on $S \times P$.[10] Buyers cannot pay more than their endowments at date 1: they face feasibility constraint

$$\int_{S\times P} pd\varphi(s,p) \leq b. \tag{2.2}$$

I refer to Borel measures chosen by sellers and buyers as supply and demand schedules, respectively. [11]

In a market, when demand equals supply, both sides of the market can trade with certainty. Otherwise, the long side of a market will be rationed probabilistically. Moreover, different markets operate rationing independently: whether agent i is rationed in one market is independent from whether he is rationed in the other markets in which he participates. Define market tightness $\Theta(s,p)$ as the demand-supply ratio in market (s,p). For sellers who enter market (s,p), the probability of trading is $\min\{\Theta(s,p), 1\}$; while for buyers who enter

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$\mathbf{S} = \{(s_1, \cdots, s_J)|0 \leq s_1 \leq \cdots \leq s_J = 1\}$ without loss of generality.

9. In Section 2.2, I show equilibria in which $\sigma_{\mathbf{z}}$ is a discrete measure.

10. One interpretation of this formulation is that all buyers use the same demand schedule. However, I prefer to think that a buyer in equilibrium only posts one type of contract, and μ summarizes the contracts posted by all buyers. The second interpretation matches the secret trading assumption imposed in the baseline model: if a buyer only interacts with her counterparty in one specific market, she does not know how he trades with third parties.

11. Notice that the concepts of supply and demand schedules in my paper are different from that in a Walrasian auction. Fix a security. In a Walrasian auction, there is only one market clearing price. To determine the total supply of a seller, we only need to look at one point on his supply schedule, which is the point associated with market clearing price. In the competitive search setting, a seller in principle can supply at multiple prices. To determine the seller's total supply, we need to integrate his supply schedule over the range of prices he supplies.

market (s,p), the probability of trading is $\min\{\Theta(s,p)^{-1},1\}$.[12] In what follows, I interpret the selling probability as liquidity. At date 1, buyers who successfully meet their counterparties pay consumption good in exchange for securities, and agents consume their remaining consumption good. At the beginning of date 2, all sellers receive their cash flow, and those who traded at date 1 make promised payments, and agents consume their remaining consumption good. Conditional on trading, the payoff of seller z for selling contract (s,p) is $p-\beta\mathbb{E}_z[s]$, where $\mathbb{E}_z[s]$ represents the expected payout from contract (s,p) posted by type z seller.[13] Conditional on trading, the payoff for buying contract (s,p) is $\mathbb{E}_{\hat{z}(s,p)}[s]-p$, where $\mathbb{E}_{\hat{z}(s,p)}[s]$ represents the expected payout from a random sample of contract (s,p) in the market. Agents understand that they are infinitesimal: they take Θ and $\hat{z}$ as given when they make their optimal selling or buying decision.

2.1.2 Equilibrium

In this subsection, I develop the concept of the decentralized equilibrium in the setting of asset-backed securities market. Different from the work of Guerrieri, Shimer, and Wright (2010), my definition captures the scenario where an agent with private information can trade with multiple principles and a principle does not observe how the agent trades with other principles., i.e., trading is non-exclusive and secret. This is a meaningful departure. Exclusivity or non-exclusivity with full transparency has very different implication compared to non-exclusivity together with secrecy. In the former case, only one incentive compatibility constraint has to be enforced per type of agents. In the latter case, incentive compatibility constraints must be enforced contract-by-contract for a given type of agents.

Adding to the complexity, markets for different securities are interconnected through the

12. Notice that there is no search friction in the current model. In particular, if supply equals demand, both sides trade with certainty. The analysis of this paper can generalize to other matching functions with embedded search friction.

13. All payoffs are gross payoffs net of agents' utilities in autarky.

feasibility constraints of sellers.[14] When a seller creates one security with asset-generated cash flow, he forgoes the opportunity to create other securities with the same cash flow. I introduce functional u to facilitate analysis. Here $u(e,z)$ is the maximum payoff seller z can get if, counterfactually, he has $e(a) \geq 0$ unit of date-2 consumption good to sell when his asset-generated cash flow is a and if he chooses his supply schedule optimally. The maximum payoff seller z can get is $u(e,z)$ evaluated at e equals to the identity function $l(a) = a$. When the seller creates h unit of security s, he can sell the rest of his asset for $u(l - hs, z)$. Then the marginal opportunity cost of posting contract (s,p) for seller z is $\lim_{h\downarrow 0} \frac{u(l,z)-u(l-hs,z)}{h}$.[15]

With these notations, we can proceed to define the equilibrium.

Definition 1. An *equilibrium* is $\mu : \mathcal{B}(S \times P) \to \mathbb{R}_+$, $\sigma_z : \mathcal{B}(S \times P) \to \mathbb{R}_+$, $\Theta : S \times P \to [0,\infty]$, $\hat{z} : S \times P \to \Delta(A)$ that satisfies the following conditions: [16]

D1. Optimal Selling Decision: given Θ, for all $z \in Z$,

$$\sigma_z \in \text{argmax}_\varphi \int \underbrace{\min\{\Theta(s,p), 1\}}_{\text{selling prob.}} \underbrace{(p - \beta \mathbb{E}_z[s])}_{\text{profit} \mid \text{selling}} d\varphi(s,p) \quad \text{s.t. } (2.1)$$

D2. Optimal Buying Decision: given Θ and $\hat{z}$, for all $z \in Z$,

$$\mu \in \text{argmax}_\varphi \int \underbrace{\min\{\Theta(s,p)^{-1}, 1\}}_{\text{buying prob.}} \underbrace{(\mathbb{E}_{\hat{z}(s,p)}[s] - p)}_{\text{profit} \mid \text{buying}} d\varphi(s,p) \quad \text{s.t. } (2.2)$$

D3. Consistency: Let $\sigma = \mathbb{E}_\pi[\sigma_z]$. For $\sigma-$almost all (s,p), (a)

$$\Theta(s,p) = \frac{d\mu(s,p)}{d\sigma(s,p)}$$

14. Because I assume buyers collectively have a deep pocket, $v(e) = 0$. In the end, a buyer's decision to enter one market is independent of her decision to enter other markets because she faces a slack feasibility constraint.

15. One can show u is concave in its first argument. Consequently, $\lim_{h\downarrow 0} \frac{u(l,z)-u(l-hs,z)}{h} = \inf_{h>0} \frac{u(l,z)-u(l-hs,z)}{h}$ is always well defined.

16. Θ and $\hat{z}$ are Borel functions.

and (b)

$$\hat{z}(s,p) = \mathbb{E}_\pi[\frac{d\sigma_z(s,p)}{d\sigma(s,p)}z].$$

R. Refinement: for all (s,p), either (a)

$$\hat{z}(s,p) \in \text{conv}(Z^*(s,p)),$$

where

$$Z^*(s,p) = \{z \in Z | \underbrace{\min\{\Theta(s,p),1\}(p-\beta\mathbb{E}_z[s])}_{\text{marginal benefit}} = \underbrace{\lim_{h\downarrow 0}\frac{u(l,z)-u(l-hs,z)}{h}}_{\text{marginal cost}}\},$$

and

$$u(e,z) = \max_\varphi \int \min\{\Theta(s,p),1\}(p-\beta\mathbb{E}_z[s])d\varphi(s,p)$$
$$\int s(a)d\varphi(s,p) \leq e(a), \text{for all } a \in A$$

or (b) $\Theta(s,p) = \infty$.

The equilibrium definition above is very natural. D1 requires that sellers maximize their payoff by optimally supplying asset-backed contracts subject to their feasibility constraints, given the difficulty of selling in each market. D2 requires that buyers maximize their payoff by optimally demanding asset-backed security contracts subject to their feasibility constraint, given the difficulty of buying and the conjectured seller composition in each market. D3 requires that if a market has positive supply, market tightness conjectured by agents must coincide with the actual demand-supply ratio derived from the optimal strategies of both sides, and a buyer's belief about the average seller type must be Bayesian consistent with sellers' optimal strategies.

The concept of equilibrium restricts an agent's beliefs about inactive markets in a manner analogous to Banks and Sobel (1987). Buyers believe that if they were to buy in an inactive market, they would be randomly matched with a mixture of sellers who are indifferent between deviating or not. Why is this plausible? Guerrieri, Shimer, and Wright (2010) provide a useful thought experiment. When a buyer considers deviating to a market lacking active sellers, she initially imagines an infinite demand-supply ratio. If the market is unable to attract any sellers even at that ratio, she stops imagining because her belief on seller composition in that market does not matter. Otherwise, she thinks that some sellers would go to the market, which would pull down the demand-supply ratio. As the demand-supply ratio goes down, some sellers may no longer find the market attractive and leave. When the adjustment process stops, the value of the demand-supply ratio will make some sellers indifferent between deviating or not, and only these sellers can possibly stay until then end. Therefore, it's reasonable for buyers to believe they will meet these indifferent sellers.

Lemma 1. *Any equilibrium is payoff-equivalent to a seperating equilibrium where sellers price their securities fairly conditional on their private information:* $p = \mathbb{E}_z[s]$ *holds* σ_z *-almost surely.*

As in Guerrieri, Shimer, and Wright (2010), sellers in the equilibrium cannot strictly benefit from over price their securities. In this paper, I focus on equilibria where sellers post securities at prices that, conditional on their private information, are actuarially fair. I call these equilibria as separating equilibria because despite prices may not perfectly reveal sellers' types,[17] they are informative enough in the sense that they summarize sellers' information that's payoff relevant to buyers. Pooling equilibria that satisfy D1-D3 can exist in this model. For example, trading assets at the pooling price can be an equilibrium that satisfies D1-D3, given appropriate distributional assumptions. However, this type of equilibrium does not survive the equilibrium refinement, as indicated in Lemma 1.

17. It's because different types of sellers can issue securities with the same conditional expected payout (e.g., a safe debt).

2.2 Multi-tiered Debt Securities

In this section, I discuss the condition under which tranching will endogenously arise as an equilibrium outcome. To provide some intuitions, I will start with an example where are only two possible cash flow realizations and two types of sellers. Then I will discuss a more general case with multiple cash flow realizations and continuous types of sellers.

2.2.1 A Binary Example

Fix $G = (0, 1)$, $B = (1-q, q)$ with $q \in [0, 1)$ and $H > L$; i.e., a good asset (type G) generates high cash flow H for sure, while a bad asset (type B) generates high cash flow with a less than 1 probability. Proposition 1 characterizes the equilibrium in this setting.

Proposition 1. *Assume $A = \{H, L\}$ and $Z = \{G, B\}$. There is an equilibrium in which sellers divide their assets into L unit of senior and $(H - L)$ unit of subordinated debts. A senior debt has the form $s_2(a) = 1$ and it always perform. Both types of sellers sell senior debts at $p = 1$ with certainty. A subordinated debt has the form $s_1(a) = 1(a = H)$ and it performs only in the high cash flow state. Bad sellers sell subordinated debts at $p = q$ with certainty, while good sellers sell subordinated debts at $p = 1$ with probability $\frac{(1-\beta)q}{1-\beta q} < 1$.*

Table 2.1 shows the probability that seller z will trade debt security s in equilibrium. In equilibrium, illiquidity acts as a costly signal of a seller's type. Senior debts are liquid because markets for senior debts do not suffer from informational friction: senior debts issued by either type have an identical payout. Subordinated debts, on the other hand, are subject to adverse selection and illiquidity because buyers value a subordinated debt issued by bad sellers less than a subordinated debt issued by good sellers. In equilibrium, good sellers accept illiquidity in exchange for selling at a high price, while bad sellers enjoy perfect liquidity by selling at a low price. Therefore, the liquidity of a security weakly increases with its observable seniority, but it decreases with the unobservable quality of the underlying asset.

Table 2.1: Liquidity in Active Markets

	Security Type	
Seller Type	s_1	s_2
B	1	1
G	$\frac{(1-\beta)q}{1-\beta q}$	1

Good sellers in the model strictly prefer to issue multi-tiered debt securities. In equilibrium, in order for buyers to enter market $(s, \mathbb{E}_G[s])$, it has to be that $\hat{z}(s, \mathbb{E}_G[s]) = G$. To screen out bad sellers, the incentive compatibility constraint has to hold:

$$\underbrace{\min\{\Theta(s, \mathbb{E}_G[s]), 1\}(\mathbb{E}_G[s] - \beta \mathbb{E}_B[s])}_{\text{marginal benefit of deviation}} \leq \underbrace{(1-\beta)\mathbb{E}_B[s]}_{\text{marginal cost}}. \tag{2.3}$$

Here the right-hand side is the bad sellers' payoff if they post a fair price for security s, in which case they will not be rationed. The left-hand side is their payoff if they deviate by posting a price of $\mathbb{E}_G[s]$. In equilibrium, competition among buyers will push up the liquidity for contract $(s, \mathbb{E}_G[s])$ so that (2.3) holds as equality and we have

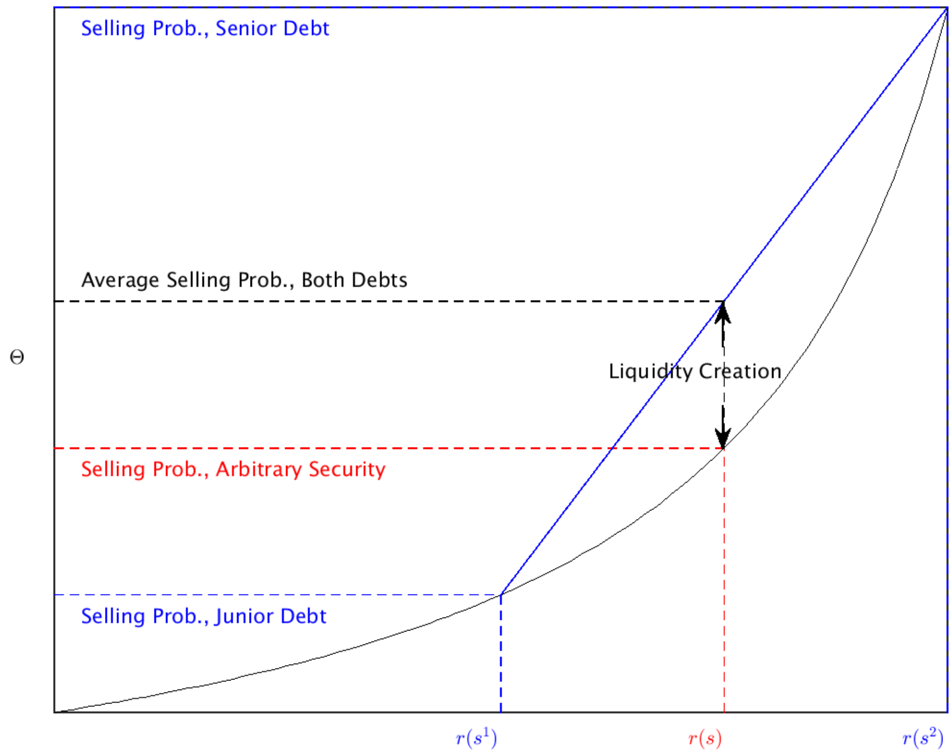
$$\Theta(s, \mathbb{E}_G[s]) = \frac{(1-\beta)r(s)}{1-\beta r(s)},$$

where $r(s) = \mathbb{E}_B[s]/\mathbb{E}_G[s]$.[18] The fair value ratio $r(s)$ captures the sensitivity of the payout of security s to the private information of sellers. When $r(s)$ is close to 1, i.e., s is information insensitive, the return from deviating is low for bad sellers, conditional on trading. Thus, buyers can provide more liquidity in market $(s, \mathbb{E}_G[s])$ without worrying about attracting bad sellers.

The convexity of $\Theta(s, \mathbb{E}_G[s])$ in $r(s)$ implies that good sellers can benefit from decomposing an intermediate security s into extreme securities s_1 and s_2, because the average liquidity

18. If 2.3 is slack, buyers can earn a positive profit by entering market $(s, \mathbb{E}_G[s] - \epsilon)$, so it cannot be an equilibrium.

Figure 2.1: Liquidity Enhancement through Tranching



of the two tranched debt securities is higher than the liquidity of the intermediate security. This is shown in Figure 2.1.

Intuitively, issuing an intermediate security forces buyers to purchase the information insensitive and sensitive components of the cash flow at a fixed proportion; while tranching allows buyers to disproportionately purchase the information insensitive component of the asset. Through tranching, good sellers reduce the average information sensitivity of successfully sold securities and make imitation less desirable for bad sellers. Tranching allows good sellers to sell a larger fraction of their cash flow compared to issuing an intermediate security. Meanwhile, anticipating that good sellers will not sell any intermediate securities, buyers will not try to buy them at high prices.

The reasoning above heavily relies on the equilibrium refinement. Given arbitrary beliefs about inactive markets, many trading patterns can arise as an equilibrium outcome. Say, buyers believe that $\hat{z}(s, \mathbb{E}_G[s]) = B$, for any s that is not identity function. Under this unnaturally belief, good sellers will not divide their assets. Instead, they will attempt to sell pass-through securities whose payouts are identical to assets. In fact, one can construct many separating equilibria without tranching that satisfies D1-D3, but such equilibria will violate R. Thus, the Banks and Sobel (1987) type of refinement generates strong predictions regarding what type of securities will be actively traded in equilibrium.

When $N = 2$, the information insensitive debt security is a safe debt in the sense that its payout is flat over the support of the total cash flow. However, in the more general case, a risky debt security can also be information insensitive. Consider the case of $N = 3$. Assume that the complementary CDF's conditional on private information are as displayed in Figure 2.2. The debt security $s_2(a) = 1(a \geq a_2)$ is risky because its payout increases from 0 to 1 when the total cash flow goes from a_3 to a_2. But since $\Pr_G(\tilde{a} \geq a_2) = \Pr_B(\tilde{a} \geq a_2)$, $r(s_2) = 1$; i.e., s_2 is information insensitive. This example shows that information insensitivity differs conceptually from safeness.[19]

2.2.2 The General Case

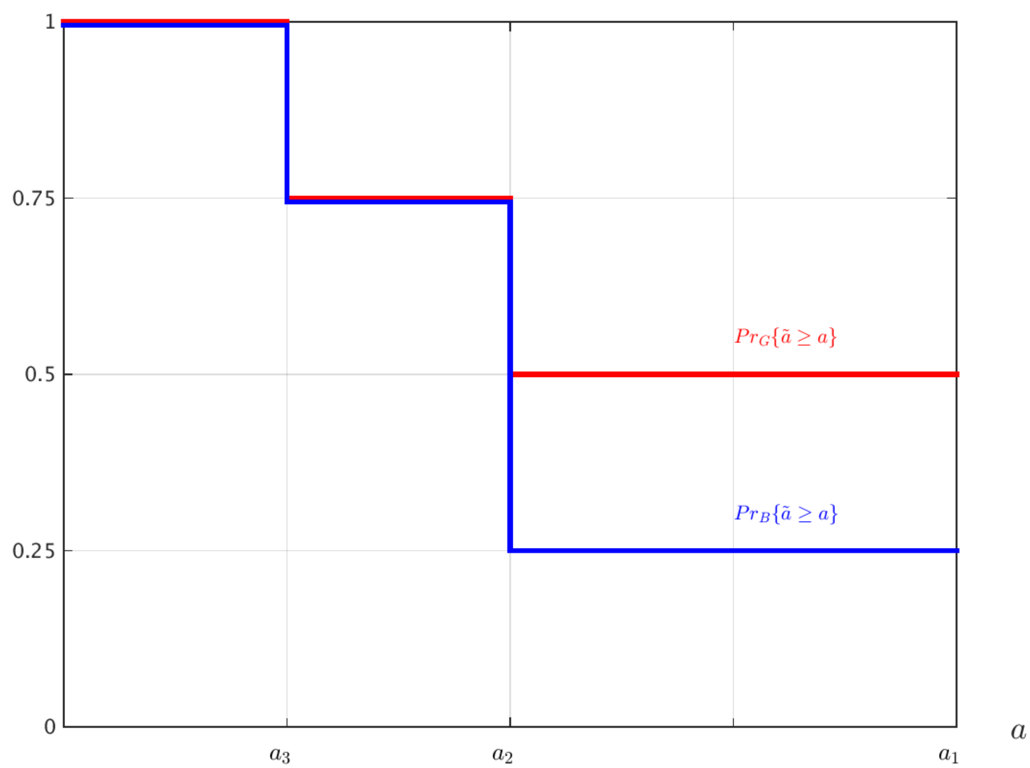
In this section, I assume that Z follows the following restrictions:

Assumption 1. *Let $\preceq$ be the hazard rate order, we have $z \preceq z'$ if $\frac{Pr_z(\tilde{a} \geq a)}{Pr_{z'}(\tilde{a} \geq a)}$ weakly decreases in a.[20] For all z, z' in Z, either $z \preceq z'$ or $z' \preceq z$. Also, Z has the least element $\underline{z}$.*

19. In this particular example, since both s_2 and s_3, with the form $s_3(a) = 1$, are information insensitive, good sellers can combine $(a_2 - a_3)$ unit of s_2 and a_3 unit of s_3 and into a_2 unit of security $(\frac{a_3}{a_2}, 1, 1)$ and obtain the same payoff.

20. $z \preceq z'$ if and only if the left truncation of z at a is weakly first-order stochastic dominated by the left truncation of z' at a, for all a. Thus, some authors also refer to Assumption 1 as conditional stochastic dominance.

Figure 2.2: Information Insensitivity v.s. Safeness



Assumption 2. *Z is the image of some continuously differentiable bijection $f : [0,1] \to \triangle(A)$ that satisfies monotonicity: $f(x) \preceq f(x')$ whenever $x \leq x'$.*

Assumption 1 requires that the conditional distributions of total cash flow are hazard rate ordered. This is a standard assumption in the security design literature.[21] The assumption says that one can order the private signal by its "goodness" (in the sense of first-order stochastic dominance) and, additionally, the difference in conditional distributions is more prominent at the right tail. When $N = 2$, any pair of distributions are hazard rate ordered. This is no longer true when $N \geq 3$. Assumption 2 is assumed to derive a close form solution. Based on numerical solutions, I believe that the results in this paper hold qualitatively for any finite set $\mathbf{Z}$ that satisfies Assumption 1.

Under the assumptions above, an equilibrium in which agents trade multi-tiered debt securities exists and is payoff-unique.[22]

Proposition 2. *Assume Assumption 1 and Assumption 2. There is an equilibrium in which sellers divide their assets into $s_1, \cdots, s_N$, where*

$$s_n(a) = 1\{a \geq a_n\}$$

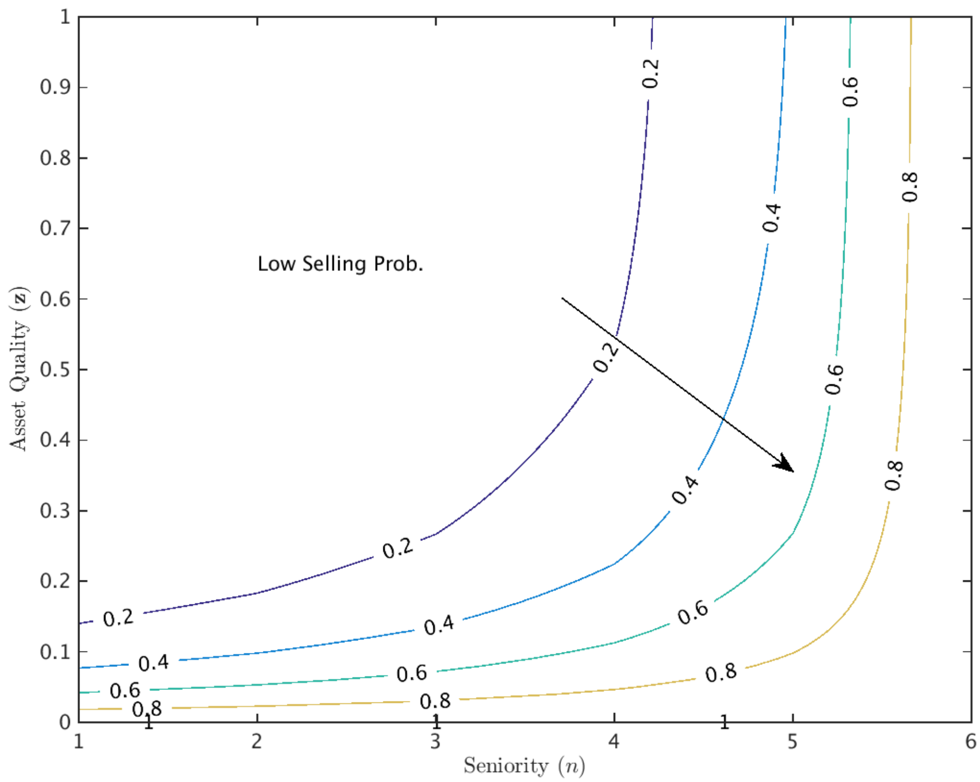
is nth subordinated debt and a_n is n^{th}-largest element in A. s_n performs in n highest cash flow states and defaults in the rest of states. The quantity of the most senior debt corresponds to the smallest cash flow, and the quantities of other debts correspond to the differences of two consecutive cash flows. Sellers of type z sell security s_n at price $\mathbb{E}_z[s_n]$ with probability $(\frac{\mathbb{E}_{\underline{z}}[s_n]}{\mathbb{E}_z[s_n]})^{\frac{1}{1-\beta}}$. The equilibrium is payoff-unique.

In Figure 2.3, I plot the equilibrium selling probabilities in active markets. Fixing debt seniority, one can see that the probability of selling a security at the fair price decreases with

21. See Nachman and Noe (1994); DeMarzo, Frankel, and Jin (2015) and Friewald, Hennessy, and Jankowitsch (2016).

22. For completeness, Section A.3 discusses what the equilibrium will look like when Assumption 1 fails.

Figure 2.3: Liquidity in Multi-tiered Debt Equilibrium



Note: In this numerical example, I set $\beta = 0.8$, $N = 6$, $a_n = N - n$ and

$$f(a_n; x) = \frac{\exp(a_n x)}{\sum_{n=1}^{N} \exp(a_n x)}.$$

asset quality: a bad seller is more likely to sell her security, while a good seller is less likely to do so. Fixing an asset quality, the liquidity of a security increases with debt seniority: the senior debt is liquid, while the subordinated debts are less and less so. Moreover, one can clearly see the interaction effect of seniority and asset quality: liquidity decreases with asset quality more slowly for more senior securities.

The predictions regarding the association between liquidity and private information cannot be tested directly because private information itself is not observed by econometricians.

However, for security s_n, we are able to observe the unconditional selling probability,

$$L(n) = \mathbb{E}_\pi[\Theta(s_n, \mathbb{E}_z[s_n])],$$

and the selling probability conditional on performing

$$L(n|\text{performing}) = \mathbb{E}_\pi[\frac{\mathbb{E}_z[s_n]}{\mathbb{E}_\pi[\mathbb{E}_z[s_n]]}\Theta(s_n, \mathbb{E}_z[s_n])],$$

where$\mathbb{E}_z[s_n]$ is the performing probability of s_n backed by asset z.

Proposition 3. *In the equilibrium of Proposition 2,*

(a) both $L(n|performing)$ and $L(n)$ increases in n;

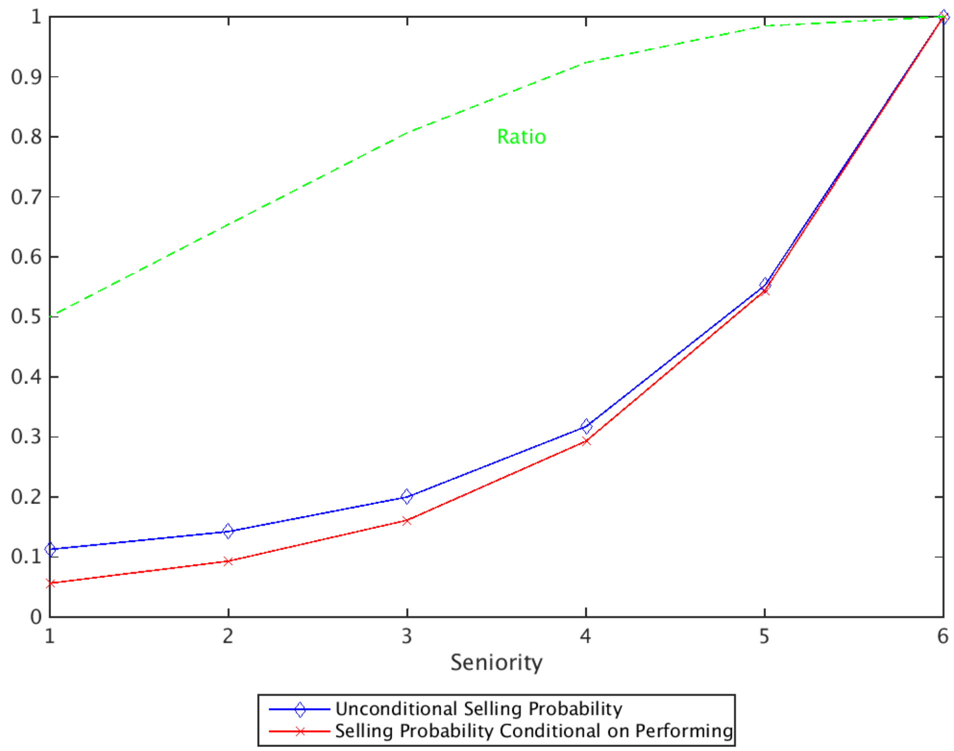
(b) for a fix n, $L(n|performing) \leq L(n)$;

(c) the ratio $L(n|performing)/L(n)$ increases in n.

Proposition 3 formalizes the testable implications of the model. Part (a) in Proposition 3 says that the probability of selling a debt security increases in its seniority unconditionally and conditional on performing. Part (b) says that, fixing seniority, the conditional selling probability is lower than the unconditional selling probability.[23] Part (c) goes a step further than (b): the ratio of conditional and unconditional selling probabilities increases with seniority. Figure 2.4 shows these results graphically. In Section 3.1, I provide evidence that supports these predictions.

23. This is a classic symptom of adverse selection. Since securities issued by good sellers are more likely to perform and good sellers sell their security with a lower probability, securities that perform well ex-post should be less liquid ex-ante. Many empirical papers implicitly use this argument to identify adverse selection in the markets.

Figure 2.4: Unconditional and Conditional Selling Probabilities: Model



Note: In this numerical example, I set β, N, a_n and f as in Figure 2.3. Furthermore, I assume π is such that x is uniformly distributed on $[0, 1]$.

CHAPTER 3

EMPIRICAL EVIDENCE AND POLICY DISCUSSION

3.1 Empirical Evidence

An asset-backed security can be "safe" either because it is backed by an asset that is more likely to generate a higher total cash flow or because its holder has the priority to receive payments when the total cash flow is low. My model predicts that the liquidity of an asset-backed security decreases with the former "safeness" measure, which is unobservable to buyers, while it increases in the latter "safeness" measure, which is observable to buyers. But because I do not observe seller types, I focus on verifying the testable predictions in Proposition 3.

3.1.1 Data

To test the empirical predictions of my model, I use data from the non-agency MBS market, in which buyers have legitimate concerns about adverse selection.[1] In particular, I focus on MBS underwritten by Bear Stearns between 2000 and 2007 under the shelf name Bear Stearns Asset Backed Securities (BSABS) since security-level holding data from other major MBS underwriters is lacking.[2] My sample includes 528 debt securities from 63 MBS deals.

The non-agency MBS held by Bear Stearns before its collapse can be inferred from the holding reports of Maiden Lane LLC in 2010. Maiden Lane was a special purpose vehicle formed in 2008 to facilitate the merger between J.P. Morgan and Bear Stearns. Maiden Lane inherited a portfolio of assets that originated from the mortgage desk of Bear Stearns, including 5.1 billion dollars worth of investment-grade non-agency MBS securities. Assets held in the Maiden Lane portfolio were managed by BlackRock Financial Management with

1. Agency MBS, on the other hand, are backed by government guarantees either explicitly or implicitly. Thus, credit risk is not a concern for buyers.

2. Bear Stearns Asset Backed Securities Inc., a wholly owned subsidiary of Bear Steans, specialized in issuing non-agency Alt-A and subprime MBS.

the objective to repay New York Fed's senior loan. Per the New York Fed's mandate, Maiden Lane cannot purchase non-agency MBS after the initial transaction. Therefore, any the holding of Maiden Lane LLC in 2010 contains MBS that were held by Bear Stearns in March 2008 and not sold during the 2-year period.

I obtained the total outstanding volume in 2010 for each security from Wells Fargo CTSLink. Then I calculated two measures of liquidity at the security level. First, I compared the balance held by Maiden Lane with the total balance to determine whether a positive sale was associated with a given security.[3] Then, to compute the fraction sold, I divided the balance that was not held by Maiden Lane by the total balance.

I complement the liquidity data with security-level seniority and ex-post performance data. In securitization deals, Class A securities are subordinated by both Class M and Class B securities, and Class M securities are subordinated by Class B securities. Within each broad class, there can be multiple subclasses, such as Class A-2 and Class B-2. To obtain the seniority for each debt security, I read through deal prospectuses and record seniority at the finer level as one plus the number of subclasses that are senior to the current security.[4] I also refer to a security as a senior security in the broader sense if it is a Class A security. To determine if a debt security is performing, I use information from remittance reports. I define a security as performing if it did not incur any loss by August 2017.

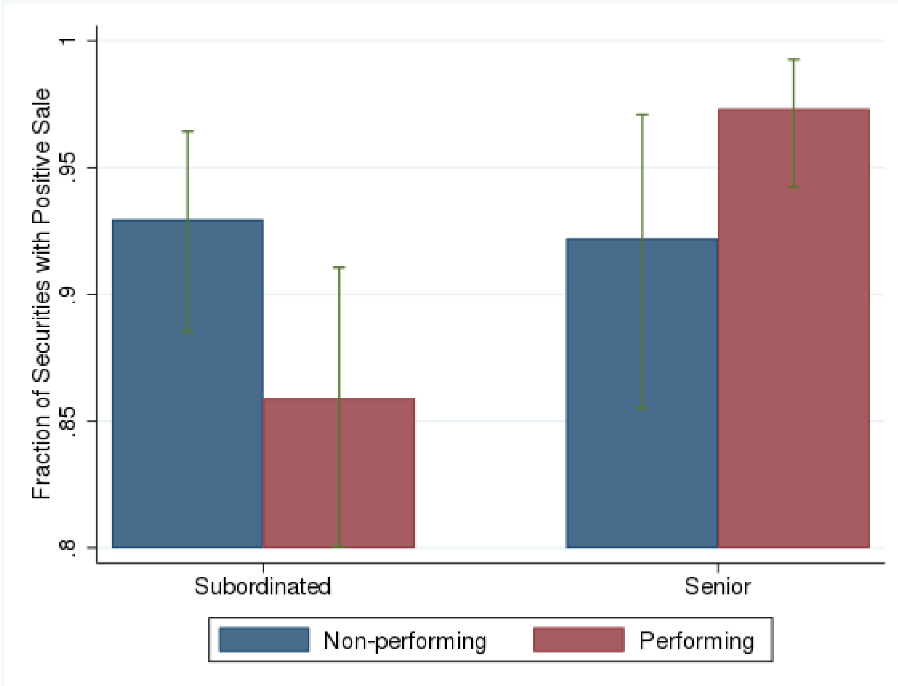
3.1.2 Findings

Figure 3.1 presents the raw data on fractions of securities that have a positive sale for different subgroups. Although subordinated MBS are generally less likely to be sold than senior MBS, it is the subgroup that has a good ex-post performance that drives down the average selling probability. In fact, subordinated securities that default have a reasonable

3. As noted above, I use the security-level holding data of Maiden Lane to proxy Bear Stearns's holding as of March 2008.

4. Say a deal consists of Class A-1, Class A-2, Class M-1, Class M-2, Class B-1, and Class B-2. I define the seniority for the series of securities as 6, 5, 4, 3, 2 and 1, respectively. When a deal offers more than 6 tranches, I still code seniority as 6 for the most senior debt and code seniority as 1 for least senior debts.

Figure 3.1: Liquidity, Seniority, and Performance



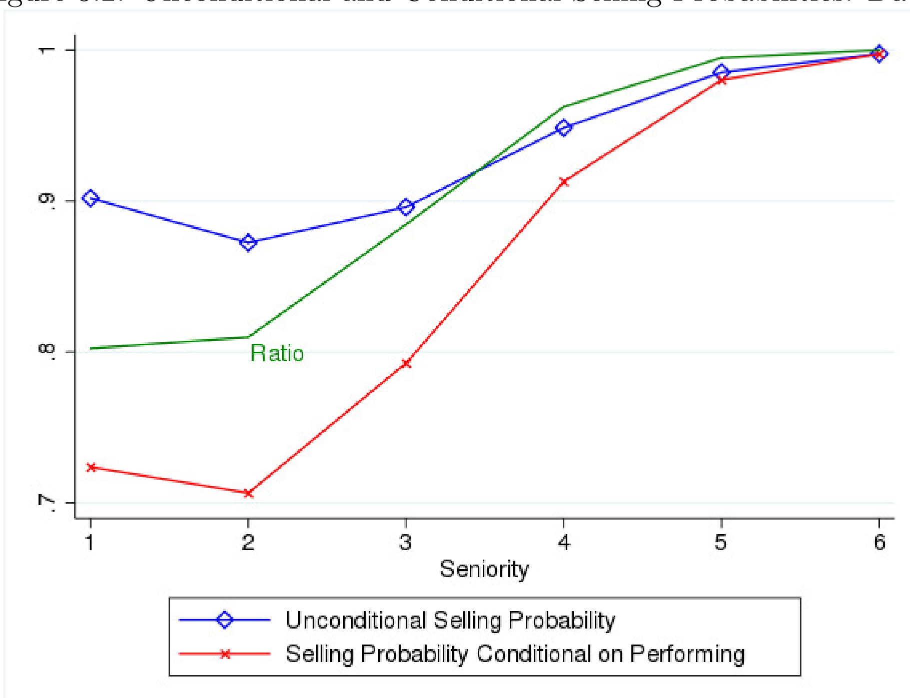
Note: Senior is an indicator for Class A securities, which can include multiple subclasses. Performing is an indicator for securities that incur no losses by Aug 2017.

chance of being sold ex-ante.

Figure 3.2 plots the estimated unconditional and conditional selling probabilities by debt seniority at the finer level. We can see that selling probabilities in the data have a pattern similar to the selling probabilities in the model (Figure 2.4). The unconditional and conditional selling probabilities both increase with seniority. Fixing seniority, the unconditional selling probability is larger than the conditional selling probability. And finally, the ratio of unconditional and conditional selling probabilities increases with seniority.

Table 3.1 formally tests the theoretical predictions in Proposition 3. The results in column (1) to (4) are based on linear probability models that control for the vintage of securities. Column (1) shows that selling probability increases in seniority unconditionally,

Figure 3.2: Unconditional and Conditional Selling Probabilities: Data



Note: Due to the small sample size, selling probabilities are estimated using logistic models with second-order polynomials of seniority and vintage. In this figure, selling probabilities are computed for debt securities originated in 2005.

Table 3.1: Liquidity, Seniority, and Performance

	(1) Positive Sale	(2) Positive Sale	(3) Positive Sale	(4) Positive Sale
Seniority	0.0205***	0.0358***	0.0326***	0.0336***
	(0.00635)	(0.0122)	(0.00877)	(0.0116)
Performing			-0.134***	-0.140**
			(0.0475)	(0.0653)
Seniority x Performing				0.00128
				(0.0113)
N	528	297	528	528
Mean	0.920	0.916	0.920	0.920

Standard errors in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Note: "Performing" is an indicator for securities that incur no losses by Aug 2017. "Positive sale" is an indicator variable for non-zero sale. The second regression is restricted to the subsample of performing securities. Vintage is controlled in all specifications. Standard errors are robust and clustered at the deal level (63 clusters).

and column (2) shows that this is also true conditional on performing. Column (3) shows that, controlling for seniority, the selling probability is lower for a performing security than for a non-performing security. In column (4), I test whether the absolute difference in selling probability between a performing security and a non-performing security is attenuated by seniority. Although the result in column (4) is not statistically significant, the signs of the point estimates are consistent with Proposition 3.[5] I also run the same set of regressions using "fraction sold" as the dependent variable. The results are quantitatively similar.

3.2 Policy Discussion

3.2.1 Mandatory Risk Retention

The regulation of securitization is an ongoing policy debate. The Dodd-Frank Act (implemented in Dec. 2015 for MBS and Dec. 2016 for other ABS) requires that a sponsor of a securitization deal must retain at least 5% of the aggregate credit risk of the assets that

5. Notice that the theoretical model is nonlinear. A more careful exercise would test monotonicity in a nonlinear regression model. This is left for the future.

collateralize an issuance. Proposers argue that mandatory risk retention will mitigate adverse selection and moral hazard problems related to securitization activities. In an effort to repeal Dodd-Frank, the Financial CHOICE Act, which passed in the House in June 2017, exempts asset-backed securities other than MBS from risk retention. In this subsection, I extend Subsection 2.2.1 to evaluate the merit of risk retention policy.

To add a flavor of the agency problem into the baseline model, I assume that some sellers are able to choose their types: they can become good sellers by exerting some effort, or they can become bad ones.[6] The overall payoff of not exerting effort is

$$v_0 = \beta \mathbb{E}_B[l] + (1-\beta)\mathbb{E}_B[l].$$

Here the first term represents a seller's valuation of a bad asset, while the second term represents a bad seller's gain from trading in the decentralized market. The overall payoff of exerting effort is

$$v_1 = \beta \mathbb{E}_G[l] + (1-\beta)\overline{\theta}_S \mathbb{E}_G[l] - c$$

where $\overline{\theta}_S < 1$ is the average liquidity of both tranched debts issued by good sellers. Here the first term represents a seller's valuation of a good asset, the second term represents the gain from trade for a good seller, and the third term represents the utility cost of exerting effort. Thus, the benefit of exerting effort depends on the difference in the seller's valuation of different assets and the difference in the gain from trade. The first term is positive because sellers value a good asset over a bad asset. The second term, however, is negative because bad sellers have a lower opportunity cost and, consequently, enjoy a higher gain from trade

6. One interpretation of this assumption is that a seller in the model is a vertically integrated unit comprising of mortgage lenders and the MBS issuer. One can always do so if there is no friction between the two parties.

in equilibrium.

$$\underbrace{\beta(\mathbb{E}_G[l]-\mathbb{E}_B[l])}_{\text{difference in sellers' valuation }(+)} + \underbrace{(1-\beta)(\bar{\theta}_S\mathbb{E}_G[l]-\mathbb{E}_B[l])}_{\text{difference in gain from trade }(-)}$$

Suppose the government requires all sellers to retain γ fraction of all issued tranches.[7] Under mandatory risk retention, the benefit of exerting effort becomes

$$\underbrace{\beta(\mathbb{E}_G[l]-\mathbb{E}_B[l])}_{\text{difference in sellers' valuation }(+)} + \underbrace{(1-\gamma)(1-\beta)(\bar{\theta}_S\mathbb{E}_G[l]-\mathbb{E}_B[l])}_{\text{difference in gain from trade }(-)}$$

From the equation above, we can see that γ decreases the absolute value of the difference in the gain from trade between the high type and the low type, which provides more incentive for sellers to exercise effort. One can choose γ such that the benefit of exerting effort equals the cost c, as long as $c \leq \beta(\mathbb{E}_G[l]-\mathbb{E}_B[l])$. In this case, mandatory risk retention can promote productive efficiency and increase the average asset quality in the market by encouraging sellers to exert effort. However, the risk retention rule hinders allocative efficiency because it reduces trading volume and the gain from trade. In fact, in this simple model, the social cost of mandatory risk retention outweighs the social benefit.

Another issue associated with risk retention rule is the unequal treatment of securitization and whole loan sale. In the analysis above I assume that retention is required even if sellers trade the whole asset. In practice, the risk retention rule introduced by most regulatory authorities applies to securitization activities but not to whole loan sale. Theoretically, if

7. The risk retention rule in Dodd-Frank allows two types of retention: sellers can either retain a fraction of all issued tranches (vertical retention) or retain a first loss tranche (horizontal retention). Here I assume the former.

regulators care only about productive efficiency, a more stringent risk retention rule should be applied to loan sale. Under direct asset sale, the benefit of exerting effort is

$$\underbrace{\beta(\mathbb{E}_G[l] - \mathbb{E}_B[l])}_{\text{difference in sellers' valuation } (+)} + \underbrace{(1-\beta)(\overline{\theta}_A \mathbb{E}_G[l] - \mathbb{E}_B[l])}_{\text{difference in gain from trade } (-)}$$

where $\overline{\theta}_A < \overline{\theta}_S$ is the liquidity of the whole asset for good sellers. Direct asset sale reduces the gain from trade for good sellers and makes the agency problem even more severe.

To summarize, mandatory risk retention can indeed improve average asset quality and promote productive efficiency. But the welfare cost of mandatory risk retention is more prominent because it prevents the gain from trade from being realized. Moreover, securitization provides better incentives to sellers than direct loan sale. I can find no theoretical justification for regulatory authority to target securitization rather than whole loan sale.

3.2.2 Government Security Purchase

During the crisis, many financial assets were purchased by government programs in order to promote financial market stability. In the U.S., examples include the 2008 Troubled Asset Relief Program and the 2009 Public-Private Investment Program. In this subsection, I discuss the effect of government security purchases in my model.

As noted in Subsection 2.2.1, the private market for senior debts is perfectly liquid. Any government purchase of senior debts will only crowd out private funding.[8] Here I focus on government purchases of subordinated debts. Suppose that before the markets for asset-backed security contracts open up, the government announces it will buy any amount of subordinated tranche at price $p > q$. When $p \geq 1$, both bad and good sellers will

8. The purchase of senior, liquid securities, such as the execution of quantitative easing, can improve the liquidity of other fixed income instruments through the portfolio rebalancing channel. This is not modeled in this paper.

sell their senior tranche in the private market and sell the subordinated tranche to the government with probability 1. When $p \in (q, 1)$, both types still sell the senior tranche in the private market. Now that the price offered by the government is less attractive, only bad sellers sell the subordinated tranche to the government with probability 1, while good sellers will sell the subordinated tranche to private investors at fair price q. Because the government security purchase increases bad sellers' equilibrium payoff, the incentive compatibility constraint (2.3) is relaxed. Consequently, good sellers can sell the subordinated debt with a higher probability. Therefore, the government program not only increases the price of the asset purchased by the government, but also has a positive spillover effect on the liquidity of securities traded in the private market.

CHAPTER 4

EXTENSION

4.1 Extension

4.1.1 Partially Transparent Trading

In this subsection, I discuss the intermediate case of partial transparency: a buyer observes how his counterparty trade with third parties in the quantity dimension. An asset-backed security contract is the triplet (s, p, m), where m is a message that sets the maximum quantities the seller can offer for each type of securities, and it is ex-ante verifiable for buyers.[1] An eligible message is an exhaustive partition of an asset.[2] Denote M as the set of eligible messages, then $M = \{m : \mathcal{B}(S) \to \mathbb{R}_+ | \int s dm = l\}$.

Definition 2. An equilibrium with partial transparency is $\mu : \mathcal{B}(S \times P \times M) \to \mathbb{R}_+$, $m_z : \mathcal{B}(S) \to \mathbb{R}_+$ and $\sigma_z : \mathcal{B}(S \times P \times M) \to \mathbb{R}_+$ for all $z \in Z$, $\Theta : S \times P \times M \to [0, \infty]$, $\hat{z} : S \times P \times M \to \Delta(A)$ that satisfies the following conditions:

D1. Selling Decision: given Θ, for all $z \in Z$,[3]

$$
\begin{aligned}
(m_z, \sigma_z) \in & \arg\max_{\omega,\varphi} \int \min\{\Theta(s, p, m), 1\}(p - \beta \mathbb{E}_z[s]) d\varphi(s, p, m) \\
& \text{s.t. } \varphi^P \leq \omega \times \delta_\omega \\
& \quad\;\; \omega \in M
\end{aligned}
$$

1. In reality, the message resembles the information content of a prospectus.

2. This requirement captures the fact that sellers cannot credibly commit to retain a piece of their asset. I regard this as a plausible assumption. For example, even the equity tranche of non-agency MBS was often sold to hedge funds and CMO managers.

3. δ_ω denotes the Dirac measure. φ^P is supply schedule φ integrated over all possible prices: $\varphi^P(C) = \varphi(\{(s, p, m)|(s, m) \in C\})$.

D2. Buying Decision: given Θ and $\hat{z}$,

$$\mu \in \arg\max_{\varphi} \int \min\{\Theta(s,p,m)^{-1},1\}(\mathbb{E}_{\hat{z}(s,p,m)}[s]-p)d\varphi(s,p,m)$$
$$\text{s.t.} \int p d\varphi(s,p,m) \leq b$$

D3. Consistency: Let $\sigma = \mathbb{E}_{\pi}[\sigma_z]$. For $\sigma-$almost all (s,p,m), (a)

$$\Theta(s,p,m) = \frac{d\mu(s,p,m)}{d\sigma(s,p,m)}$$

and (b)

$$\hat{z}(s,p,m) = \mathbb{E}_{\pi}[\frac{d\sigma_z(s,p,m)}{d\sigma(s,p,m)}z].$$

R. Refinement: for all (s,p,m), either (a)

$$\hat{z}(s,p,m) \in \text{conv}(Z^*(s,p,m)),$$

where

$$Z^*(s,p,m) = \{z \in Z | \min\{\Theta(s,p,m),1\}(p-\beta\mathbb{E}_z[s]) = \lim_{h\downarrow 0}\frac{u(l,z;m_z)-u(l-hs,z;m)}{h}\},$$

and

$$u(e,z;\omega) = \max_{\varphi} \int \min\{\Theta(s,p,m),1\}(p-\beta\mathbb{E}_z[s])d\varphi(s,p,m) \tag{4.1}$$
$$\text{s.t. } \varphi^P \leq \omega \times \delta_\omega$$
$$\int s(a)d\varphi(s,p,m) \leq e(a), \text{for all } a \in A$$

or (b) $\Theta(s,p,m) = \infty$.

Compared to the equilibrium concept in Subsection 2.1.2, the new concept is different in two ways. First, contracts or markets now are indexed by (s,p,m), meaning buyers can observe one additional message from a seller. Secondly, the supply schedule chosen by a seller must be consistent with the message he sent in the quantity dimension: a seller who sends a certain message can only supply in market indexed by such message, and the quantities of securities he supplies integrated over all possible prices cannot exceeds the quantities listed in the message.

Proposition 4 shows that the trading of multi-tiered debt securities is robust even if trading is partially transparent.

Proposition 4. *Assume Assumption 1 and Assumption 2. There is an equilibrium with partial transparency in which sellers divide their assets into* $s_1, \cdots, s_N$*, where*

$$s_n(a) = 1\{a \geq a_n\}$$

is nth subordinated debt and a_n *is* n^{th}*-largest element in A.* s_n *performs in n best states and defaults in the rest of states. The quantity of the most senior debt corresponds to the smallest cash flow, and the quantities of other debts correspond to the differences of two consecutive cash flows. Sellers of type z sell security* s_n *at price* $\mathbb{E}_z[s_n]$ *with probability* $(\frac{\mathbb{E}_{\underline{z}}[s_n]}{\mathbb{E}_z[s_n]})^{\frac{1}{1-\beta}}$.

4.1.2 Fully Transparent Trading

In this subsection, I show that when trading is fully transparent, each seller effectively sells one debt security. Full transparency here means a buyer observes how his counterparty trade with third parties in both quantity and price dimensions. An asset-backed security contract is still the triplet (s, p, m), but m now sets the maximum quantities the seller can offer for each type of security at each possible price. Denote M as the set of eligible messages, $M = \{m : \mathcal{B}(S \times P) \to \mathbb{R}_+ | \int s dm = l\}$.

Definition 3. An equilibrium with full transparency is $\mu : \mathcal{B}(S \times P \times M) \to \mathbb{R}_+$, $m_z : \mathcal{B}(S \times P) \to \mathbb{R}_+$ and $\sigma_z : \mathcal{B}(S \times P \times M) \to \mathbb{R}_+$ for all $z \in Z$, $\Theta : S \times P \times M \to [0, \infty]$, $\hat{z} : S \times P \times M \to \Delta(A)$ that satisfies the following conditions:

D1. Selling Decision: given Θ, for all $z \in Z$,

$$(m_z, \sigma_z) \in \arg\max_{\omega,\varphi} \int \min\{\Theta(s,p,m),1\}(p - \beta \mathbb{E}_z[s]) d\varphi(s,p,m)$$
$$\text{s.t. } \varphi \leq \omega \times \delta_\omega, \omega \in M$$

D2. Buying Decision: given Θ and $\hat{z}$,

$$\mu \in \arg\max_{\varphi} \int \min\{\Theta(s,p,m)^{-1},1\}(\mathbb{E}_{\hat{z}(s,p,m)}[s] - p) d\varphi(s,p,m)$$
$$\text{s.t. } \int p d\varphi(s,p,m) \leq b$$

D3. Consistency: Let $\sigma = \mathbb{E}_\pi[\sigma_z]$. For $\sigma-$almost all (s,p,m), (a)

$$\Theta(s,p,m) = \frac{d\mu(s,p,m)}{d\sigma(s,p,m)}$$

and (b)

$$\hat{z}(s,p,m) = \mathbb{E}_\pi[\frac{d\sigma_z(s,p,m)}{d\sigma(s,p,m)} z].$$

R. Refinement: for all (s,p,m), either (a)

$$\hat{z}(s,p,m) \in \text{conv}(Z^*(s,p,m)),$$

where

$$Z^*(s,p,m) = \{z \in Z | \min\{\Theta(s,p,m),1\}(p - \beta \mathbb{E}_z[s]) = \lim_{h \downarrow 0} \frac{u(l,z;m_z) - u(l-hs,z;m)}{h}\},$$

and

$$u(e,z;\omega) = \max_{\varphi} \int \min\{\Theta(s,p,m),1\}(p-\beta\mathbb{E}_z[s])d\varphi(s,p,m)$$

$$\text{s.t. } \varphi \le \omega \times \delta_\omega$$

$$\int s(a)d\varphi(s,p,m) \le e(a), \text{for all } a \in A$$

or (b) $\Theta(s,p,m) = \infty$.

Lemma 2. *In an equilibrium with full transparency, for any $m \in M$, we have $\Theta(s^m, p^m, \delta_{(s^m,p^m)}) \ge 1$, where $s^m = \int \Theta(s,p,m)sdm$ and $p^m = \int \Theta(s,p,m)pdm$.*

Lemma 2 shows that in an equilibrium with full transparency, sellers are indifferent between m and $\delta_{(s^m,p^m)}$. This means that sellers in equilibrium cannot benefit from issuing multiple securities.

Proposition 5. *Assume Assumption 1 and Assumption 2. There is an equilibrium with full transparency in which sellers divide their assets into $s_1, \cdots, s_N$, where*

$$s_n(a) = 1\{a \ge a_n\}$$

is nth subordinated debt and a_n is n^{th}-largest element in A. s_n performs in n best states and defaults in the rest of states. The quantity of the most senior debt corresponds to the smallest cash flow, and the quantities of other debts correspond to the differences of two consecutive cash flows. Sellers of type z attemp to sell security s_n at price $\mathbb{E}_z[s_n]$. For all type z, there exists a hurdle class $\hat{n}$ such that contract $(s_n, \mathbb{E}_z[s_n])$ with $n > \hat{n}$ is sold with certainty, while contract $(s_n, \mathbb{E}_z[s_n])$ with $n < \hat{n}$ is never sold, i.e., each seller effectively sells one debt security.

Proposition 5 shows that the trading of multi-tiered debt securities is no longer robust if buyers also observe the prices of other securities offered by a seller. In contrary to the previous subsection, the message sent by sellers contains pricing information in addition to quantities. The finer contract space here leads to the issuance of a single debt security, while the coarser contract space in the previous subsection generates the trading of multiple securities.

CHAPTER 5

CONCLUSION

5.1 Conclusion

Existing literature on security design often finds that a single debt security is optimal under adverse selection. Behind this result is a key assumption: a buyer can observe how his counterparty trade with third parties. In a novel contribution to the literature, my paper discusses the optimal security design problem without such transparency. I show that the trading of multi-tiered debt securities is a robust equilibrium outcome. The result is consistent with the fact that multi-tiered debt securities are prevalent in many securitization activities. Moreover, I show that tranching improves social welfare by mitigating information frictions in the markets.

An asset-backed security can be "safe" either because it is backed by an asset that is more likely to generate a high total cash flow or because its holder has the priority to receive payments when the total cash flow is low. My model predicts that the liquidity of an asset-backed security decreases with the former "safeness", which is ex-ante unobservable to buyers, while it increases in the latter "safeness", which is observable to buyers. My empirical findings from the non-agency MBS market support these predictions.

This paper has implications for the regulations of securitization. As part of the Dodd-Frank Act, the risk retention rule that came into effect in 2015 required the sponsor of a securitization deal to retain at least 5 percent of the securitized exposure. Through the lens of my model, mandatory risk retention causes welfare loss because the benefit in productive efficiency is outweighed by the cost in allocative efficiency. Therefore, I do not find a theoretical justification for mandatory risk retention.

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APPENDIX A

APPENDIX

A.1 Proof of Lemmas

Lemma 1:

Proof. In the first step, I want to show that if seller z_1 find market (s, p) optimal, seller z_0 with $\mathbb{E}_{z_0}[s] < \mathbb{E}_{z_1}[s]$ will not find $(s, p+\epsilon)$ optimal for any $\epsilon > 0$. Since seller z_1 find market (s, p) optimal, it must be that $\min\{\Theta(s, p), 1\} > \min\{\Theta(s, p+\epsilon), 1\}$. Notice that sellers' payoffs satisfy strict decreasing difference property in $\min\{\Theta(s, p), 1\}$ and $\mathbb{E}_z[s]$. Thus, seller z_0 will strictly prefer (s, p) over $(s, p+\epsilon)$. In the second step, I want to show that if $(s, p) \in \text{supp}(\mu)$, then $\mathbb{E}_z[s] - p \leq 0$ for all $z \in Z^*(s, p)$. By way of contradiction, suppose $\mathbb{E}_z[s] > p$. From the first step, we know $\mathbb{E}_{\hat{z}(s,p+\epsilon)}[s] \geq \mathbb{E}_z[s]$. Then buyers will earn a strictly positive profit in market $(s, p+\epsilon)$ for small enough ϵ, a contradiction. Now we can turn to the main argument.

$$
\begin{aligned}
0 &= \int \min\{\Theta(s, p), 1\}(\mathbb{E}_{\hat{z}(s,p)}[s] - p)d\sigma(s, p) \\
&= \mathbb{E}_\pi[\int \min\{\Theta(s, p), 1\}(\mathbb{E}_z[s] - p)d\sigma_z(s, p)],
\end{aligned}
$$

where the first equality comes from the deep pocket assumption and D(3)(a), the second equality come from D(3)(b). Combined with the result from the second step, we have

$$
\int \min\{\Theta(s, p), 1\}(\mathbb{E}_z[s] - p)d\sigma_z(s, p) = 0,
$$

π-almost surely. Since value functions are continuous in z, the equation above holds for all $z \in Z$.

Set

$$
\tilde{\sigma}_z(C) = \sigma_z(\{(s, p)|(s, \mathbb{E}_z[s]) \in C\})
$$

and $\tilde{\mu}$ according to D3. Then $(\tilde{\mu}, \{\tilde{\sigma}_z\}_{z\in Z}, \Theta, \hat{z})$ is a separating equilibrium. □

Lemma 2:

Proof. Suppose not, i.e., $\Theta(s^m, p^m, \delta_{(s^m,p^m)}) < 1$. Then all seller will strictly prefer m over $\delta_{(s^m,p^m)}$. But then $\Theta(s^m, p^m, \delta_{(s^m,p^m)}) = \infty$ according to the refinement, a contradiction. □

A.2 Proof of Propositions

Proposition 1:

Proof. Let φ be the solution of $\sum_{n=1}^{N} s_n \varphi(s_n; e) = e$, or in matrix form

$$\begin{bmatrix} s_1(a_1) & s_1(a_1) & \cdots & s_n(a_1) \\ s_1(a_2) & s_1(a_2) & \cdots & s_n(a_2) \\ \vdots & \vdots & & \vdots \\ s_1(a_n) & s_1(a_n) & \cdots & s_n(a_n) \end{bmatrix} \begin{bmatrix} \varphi(s_1; e) \\ \varphi(s_2; e) \\ \vdots \\ \varphi(s_n; e) \end{bmatrix} = \begin{bmatrix} e(a_1) \\ e(a_2) \\ \vdots \\ e(a_n) \end{bmatrix}.$$

We have $\varphi(s_n; e) = e(a_n) - e(a_{n+1})$, with $e(a_{N+1}) = 0$. Set

$$\sigma_z(C) = \sum_{n=1}^{N} 1[(s_n, \mathbb{E}_z[s_n]) \in C]\varphi(s_n, l).$$

For active markets $(s_n, \mathbb{E}_z[s_n])$, set

$$\Theta(s_n, \mathbb{E}_z[s_n]) = \begin{cases} \frac{(1-\beta)r(s_n)}{1-\beta r(s_n)} & \text{if } z = G \\ 1 & \text{if } z = B \end{cases},$$

and $\hat{z}(s_n, \mathbb{E}_z[s_n]) = z$.

Let

$$u(e, z) = \sum_{n=1}^{N} \Theta(s_n, \mathbb{E}_z[s_n])(1-\beta)\mathbb{E}_z[s_n]\varphi(s_n; e),$$

then

$$D_s u(l,z) = \sum_{n=1}^{N} \Theta(s_n, \mathbb{E}_z[s_n])(1-\beta)\mathbb{E}_z[s_n]\varphi(s_n, s).$$

For inactive markets $(s,p) \notin \{(s_n, \mathbb{E}_z[s_n]) | 1 \leq n \leq N, z \in Z\}$, set

$$\Theta(s,p) = \begin{cases} \frac{D_s u(l,z)}{p-\beta\mathbb{E}_z[s_n]} & \text{if } \frac{D_s u(l,z)}{p-\beta\mathbb{E}_z[s_n]} \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

$$\hat{z}(s,p) = z$$

where z is a minimizer of $\min_{\mathbb{E}_z[s]\leq p} \frac{D_s u(l,z)}{p-\beta\mathbb{E}_z[s]}$. Finally, set μ according to D3.

First, I want to show for active markets $(s_n, \mathbb{E}_{z'}[s_n])$, we have

$$D_{s_n} u(l,z) \geq \min\{\Theta(s_n, \mathbb{E}_{z'}[s_n]), 1\}(\mathbb{E}_{z'}[s_n] - \beta\mathbb{E}_z[s_n]). \tag{A.1}$$

Notice that LHS $= \min\{\Theta(s_n, \mathbb{E}_z[s_n]), 1\}(1-\beta)\mathbb{E}_z[s]$. If $z = B$, LHS = RHS by construction. If $z = G$, since sellers' payoffs satisfy strict decreasing difference property in $\min\{\Theta(s,p), 1\}$ and $\mathbb{E}_z[s]$, so LHS > RHS.

Then, I want to show for inactive market (s,p), we have

$$D_s u(l,z) \geq \min\{\Theta(s,p), 1\}(p - \beta\mathbb{E}_z[s]). \tag{A.2}$$

I only need to consider the case that $p \in (\beta\mathbb{E}_z[s], \mathbb{E}_z[s])$. If $z = B$,

$$\text{LHS} = (1-\beta)\mathbb{E}_z[s] \geq \text{RHS}$$

Notice

$$\begin{aligned}\frac{D_s u(l,G)}{(1-\beta)\mathbb{E}_G[s]} &= w^1 g(r(s_1)) + w^2 g(r(s_2)) \\ &\geq g(w^1 r(s_1) + w^2 r(s_2)) \\ &= g(r(s)) \\ &= \frac{D_s u(l,B)}{\mathbb{E}_G[s] - \beta\mathbb{E}_B[s]}\end{aligned}$$

where $w^n = \frac{\varphi(s_n;s)\mathbb{E}_G[s_n]}{\mathbb{E}_G[s]}$ and $g(x) = \frac{(1-\beta)x}{1-\beta x}$. If $z = G$, we have

$$\begin{aligned}\text{LHS} &\geq \frac{D_s u(l,B)}{\mathbb{E}_G[s] - \beta\mathbb{E}_B[s]}(1-\beta)\mathbb{E}_G[s] \\ &\geq \frac{D_s u(l,B)}{p - \beta\mathbb{E}_B[s]}(p - \beta\mathbb{E}_G[s]) \\ &\geq \text{RHS}\end{aligned}$$

which finishes the proof of (A.2).

Now we can verify $(\mu, \{\sigma_z\}_{z\in Z}, \Theta, \hat{z})$ conjectured above is indeed an equilibrium. For D1, sellers of type z have no incentive to deviate because the marginal benefit of going to market (s,p) is at most as large as the marginal opportunity cost $D_s u(l,z)$. Also, sellers cannot post additional contracts because the feasibility constraints hold tight with respect to any eligible security. For D2(a), buyers have no incentive to deviate because the marginal benefit of going to any market is at most 0. Also, buyers feasibility constraint holds since b is large enough. D3 and R is ensured by the construction of the equilibrium. □

Proposition 2:

Proof. Set $(\mu, \{\sigma_z\}_{z\in Z}, \Theta, \hat{z})$ as in the proof of Proposition 1, with the exception that

$$\Theta(s_n, \mathbb{E}_z[s_n]) = \left(\frac{\mathbb{E}_{\underline{z}}[s_n]}{\mathbb{E}_z[s_n]}\right)^{\frac{1}{1-\beta}}$$

and $\hat{z}(s_n, \mathbb{E}_z[s_n]) = z$.

First, I want to show (A.1) holds for active markets. One can verify that for a fixed s_n, the local IC constraint

$$\partial_p \min\{\Theta(s_n, p), 1\}(1-\beta)p + \min\{\Theta(s_n, p), 1\} = 0$$

holds on the set of active prices $\{p | p = \mathbb{E}_{z'}[s_n], z' \in Z\}$, and the value function is convex in $\mathbb{E}_z[s_n]$. Thus,

$$\mathbb{E}_z[s_n] \in \max_p \min\{\Theta(s_n, p), 1\}(p - \beta \mathbb{E}_z[s_n]), \tag{A.3}$$

which finishes the proof of (A.1).

Then, I want to show (A.2) holds for inactive markets. Let $g(x; s, p) = \frac{D_s u(l, f(x))}{p - \beta \mathbb{E}_{f(x)}[s]}$. We have

$$g'(x; s, p) = \frac{D_x D_s u(l, f(x))}{(p - \beta \mathbb{E}_{f(x)}[s])^2}(p - p(x, s)) \tag{A.4}$$

where

$$\begin{aligned} p(x; s) &= \beta \mathbb{E}_{f(x)}[s] - \frac{\beta D_x \mathbb{E}_{f(x)}[s] D_s u(l, f(x))}{D_x D_s u(l, f(x))} \\ &= \mathbb{E}_{f(x)}[s][\beta + (1-\beta)\frac{\sum \varphi^n D_x \mathbb{E}_{f(x)}[s_n] \sum \varphi^n \theta^n \mathbb{E}_{f(x)}[s_n]}{\sum \varphi^n \mathbb{E}_{f(x)}[s_n] \sum \varphi^n \theta^n D_x \mathbb{E}_{f(x)}[s_n]}] \\ &\geq \mathbb{E}_{f(x)}[s]. \end{aligned}$$

The inequality comes from the fact that under Assumption 1, θ^n increases in n while

$\frac{D_x\mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]}$ decreases in n. Also notice

$$
\begin{aligned}
p'(x;s) &= -\beta D_s u(l,f(x)) D_x\left(\frac{D_x\mathbb{E}_{f(x)}[s]}{D_x D_s u(l,f(x))}\right) \\
&= -\beta \frac{D_s u(l,f(x)) D_x\mathbb{E}_{f(x)}[s]}{D_x D_s u(l,f(x))}\left(\frac{D_x^2\mathbb{E}_{f(x)}[s]}{D_x\mathbb{E}_{f(x)}[s]} - \frac{D_x^2 D_s u(l,f(x))}{D_x D_s u(l,f(x))}\right) \\
&> -\beta \frac{D_s u(l,f(x)) D_x\mathbb{E}_{f(x)}[s]}{D_x D_s u(l,f(x))}\left(\frac{\sum \varphi^n D_x^2\mathbb{E}_{f(x)}[s_n]}{\sum \varphi^n D_x\mathbb{E}_{f(x)}[s_n]} - \frac{\sum \varphi^n \theta^n D_x^2\mathbb{E}_{f(x)}[s_n]}{\sum \varphi^n \theta^n D_x\mathbb{E}_{f(x)}[s_n]}\right) \\
&\geq 0
\end{aligned}
$$

with $\varphi^n = \varphi(s_n;s)$, $\theta^n = \Theta(s_n, \mathbb{E}_{f(x)}[s_n])$. The first inequality comes from $D_x^2 D_s u(l,f(x)) > -\beta \sum \varphi^n \theta^n D_x^2 \mathbb{E}_{f(x)}[s_n]$, and the second inequality comes from the fact that under Assumption 1, θ^n increases in n while $\frac{D_x^2\mathbb{E}_{f(x)}[s_n]}{D_x\mathbb{E}_{f(x)}[s_n]}$ decreases in n. Thus, we have $\frac{D_s u(l,f(x))}{p-\beta\mathbb{E}_{f(x)}[s]} \geq \min_{\mathbb{E}_{f(x)}[s]\leq p} \frac{D_s u(l,f(x))}{p-\beta\mathbb{E}_{f(x)}[s]} \geq \min\{\Theta(s,p),1\}$, which finishes the proof of (A.2).

As in the proof of Proposition 1, one can verify $(\mu, \{\sigma_z\}_{z\in Z}, \Theta, \hat{z})$ conjectured above is indeed an equilibrium.

For uniqueness, it suffices to show that for a given type of sellers, his payoff in multi-tiered debt equilibrium equals to his payoff in any other refined equilibrium. Let $(\tilde{\mu}, \{\tilde{\sigma}_z\}_{z\in Z}, \tilde{\Theta}, \tilde{z})$ be the multi-tiered debt equilibrium, and $(\mu, \{\sigma_z\}_{z\in Z}, \Theta, \hat{z})$ be any equilibrium.

First, I want to show $u(l,z) \geq \tilde{u}(l,z)$. I claim that for a fixed s_n,

$$\min\{\Theta(s_n,p),1\} \geq \min\{\tilde{\Theta}(s_n,p),1\} \tag{A.5}$$

holds on the set of active prices. If this is true then we have $u(l,z) \geq \tilde{u}(l,z)$ from D1. In any equilibrium,

$$\min\{\Theta(s_n, \mathbb{E}_{\underline{z}}[s_n]),1\} = 1; \tag{A.6}$$

otherwise buyers have incentive to deviate to $(s_n, \mathbb{E}_{\underline{z}}[s_n]-\epsilon)$ and earn a positive profit. From

R we have some sellers find (s_n, p) weakly optimal, thus

$$D_p \min\{\Theta(s_n, p), 1\}(p - \beta\mathbb{E}_{\hat{z}}(s_n, p)) + \min\{\Theta(s_n, p), 1\} = 0$$

and $-\beta \min\{\Theta(s_n, p), 1\}$, as the sub-gradient of a convex function, is weakly increasing. Since buyers earn zero payoff in equilibrium, we have $\mathbb{E}_{\hat{z}}(s_n, p) \le p$. Then $D_p \min\{\Theta(s_n, p), 1\} \in [-\frac{1}{1-\beta}\frac{\min\{\Theta(s_n,p),1\}}{p}, 0]$, which finishes the proof of (A.5).

Then, I want to show $u(l, z) \le \tilde{u}(l, z)$. From Lemma 1, we have $u(l, f(0)) \le \tilde{u}(l, f(0))$. Thus, it is sufficient to show

$$D_x u(l, f(x)) \le D_x \tilde{u}(l, f(x)).$$

Notice

$$\begin{aligned} -D_x u(l, f(x)) &= \beta \int \min\{\Theta(s, p), 1\} D_x \mathbb{E}_{f(x)}[s] d\sigma_{f(x)}(s, p) \\ &\ge \int \frac{\beta D_s \tilde{u}(l, f(x))}{p - \beta\mathbb{E}_{f(x)}[s]} D_x \mathbb{E}_{f(x)}[s] d\sigma_{f(x)}(s, p) \\ &\ge -\int D_x(D_s \tilde{u}(l, f(x))) d\sigma_{f(x)}(s, p) \\ &= -D_x \tilde{u}(l, f(x)) \end{aligned} \tag{A.7}$$

Here the first equality comes from applying envelop theorem to D1; the first inequality comes from the optimality of σ_z and (A.5); the second inequality comes from (A.4). □

Proposition 3:

Proof. Since $\Theta(s_n, \mathbb{E}_z[s_n])$ increases in n, $\int \Theta(s_n, \mathbb{E}_z[s_n]) d\pi(z)$ increases in n. Now I want to show $\frac{\int \mathbb{E}_z[s_n]\Theta(s_n,\mathbb{E}_z[s_n])\mathbb{E}_z[s_n]d\pi(z)}{\int \mathbb{E}_z[s_n]d\pi(z)} / \int \Theta(s_n, \mathbb{E}_z[s_n])d\pi(z)$ increase in n.

$$\begin{aligned}
&\frac{\int \mathbb{E}_z[s_{n+1}]\Theta(s_{n+1}, \mathbb{E}_z[s_{n+1}])d\pi(z)}{\int \mathbb{E}_z[s_{n+1}]d\pi(z)} / \int \Theta(s_{n+1}, \mathbb{E}_z[s_{n+1}])d\pi(z) \\
\geq &\frac{\int \mathbb{E}_z[s_n]\Theta(s_{n+1}, \mathbb{E}_z[s_{n+1}])d\pi(z)}{\int \mathbb{E}_z[s_n]d\pi(z)} / \int \Theta(s_{n+1}, \mathbb{E}_z[s_{n+1}])d\pi(\mathbf{z}) \\
\geq &\frac{\int \mathbb{E}_z[s_n]\Theta(s_n, \mathbb{E}_z[s_n])d\pi(z)}{\int \mathbb{E}_z[s_n]d\pi(z)} / \int \Theta(s_n, \mathbb{E}_z[s_n])d\pi(z).
\end{aligned}$$

For the first inequality, I notice the fact that $\Theta(s_{n+1}, \mathbb{E}_{f(x)}[s_{n+1}])$ decreases in x while $\frac{\mathbb{E}_{f(x)}[s_{n+1}]}{\mathbb{E}_{f(x)}[s_n]}$ decreases in x, and use Chebyshev's algebraic inequality. For the second inequality, I notice the fact that $\frac{\Theta(s_{n+1},\mathbb{E}_z[s_{n+1}])}{\Theta(s_n,\mathbb{E}_z[s_n])}$ increases in x while $\mathbb{E}_{f(x)}[s_n]$ increases in x, and apply Chebyshev's algebraic inequality again. Thus we have the ratio of selling probability conditional on performing and unconditional selling probability increases in n. But since the ratio equals 1 when $n = N$, so we have the conditional probability is always smaller than the unconditional probability. □

Proposition 4:

Proof. Set $m_z = m$ such that

$$m(S^0) = \sum_{n=1}^{N} 1[s_n \in S^0]\varphi(s_n, l).$$

Let $(\tilde{\mu}, \{\tilde{\sigma}_z\}_{z \in Z}, \tilde{\Theta}, \tilde{z})$ be the equilibrium in Proposition 2. Set

$$u(e, z; m) = \tilde{u}(e, z)$$

$$\Theta(s, p; m) = \tilde{\Theta}(s, p)$$

$$\hat{z}(s, p; m) = \hat{z}(s, p)$$

$$\sigma_z = \tilde{\sigma}_z \times \delta_m$$

$$\mu = \tilde{\mu} \times \delta_m$$

I only need to show that u conjectured above is achieved in (4.1). It is sufficient to show for any $z \in Z$ and $s \in S$, there is $p \in P$ so that (A.2) holds as equality. Solving for p by setting the RHS of (A.4) to zero, we have

$$p(s, f(x)) = \mathbb{E}_{f(x)}[s][\beta + (1-\beta)\frac{\sum w^n \theta^n \sum w^n \frac{D_x \mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]}}{\sum w^n \theta^n \frac{D_x \mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]}}].$$

Then

$$x \in \text{argmin} g(x; s, p(s, f(x))),$$

and (A.2) holds as equality. □

Proposition 5:

Proof. Consider the linear programming problem:

$$F(x, U) = \max_{\{\theta^n\}} -\beta \sum \theta^n \varphi(s_n, l) D_x \mathbb{E}_{f(x)}[s_n]$$

$$\text{s.t. } 0 \leq \theta^n \leq 1$$

$$(1-\beta) \sum \theta^n \varphi(s_n, l) \mathbb{E}_{f(x)}[s_n] = U$$

Notice the Lagrangian is linear in θ^n with partial derivative

$$\varphi(s_n, l)\mathbb{E}_{f(x)}[s_n](-\beta \frac{D_x \mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]} + (1-\beta)\lambda),$$

where $\lambda \geq 0$ is the Lagrangian multiplier. Thus, the optimal solution has the form

$$\theta^n = \begin{cases} 0 & -\beta \frac{D_x \mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]} + (1-\beta)\lambda \leq 0 \\ 1 & -\beta \frac{D_x \mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]} + (1-\beta)\lambda \geq 0 \end{cases}$$

From Assumption 1, we have $\frac{D_x \mathbb{E}_{f(x)}[s_n]}{\mathbb{E}_{f(x)}[s_n]}$ decreases with n, so θ^n has to increase in n.

Now consider the ODE problem $U'(x) = F(x, U(x))$ with initial condition $U(0) = (1 - \beta)\sum \varphi(s_n, l) f(0; a_n)$, and denote the solution of the ODE as $U(x)$.

Set m_z such that

$$m_z(S^0 \times P^0) = \sum_{n=1}^{N} 1[(s_n, \mathbb{E}_{f(x)}[s_n]) \in S^0 \times P^0]\varphi(s_n, l).$$

For active markets $(s_n, \mathbb{E}_z[s_n], m_z)$, set

$$\Theta(s_n, \mathbb{E}_z[s_n], m_z) = \theta^n(f^{-1}(x))$$

and $\hat{z}(s_n, \mathbb{E}_z[s_n], m_z) = z$. For inactive markets $(s, p, m) \notin \{(s_n, \mathbb{E}_z[s_n], m_z) | 1 \leq n \leq N, z \in Z\}$, set

$$\Theta(s, p, m) = \begin{cases} \frac{U(x^*)}{p^m - \beta \mathbb{E}_{f(x^*)}[s^m]} & \text{if } \frac{U(x^*)}{p^m - \beta \mathbb{E}_{f(x^*)}[s^m]} \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

$$\hat{z}(s, p, m) = f(x^*)$$

where x^* is a minimizer of $\min_{\mathbb{E}_{f(x)}[s^m] \leq p^m} \frac{U(x)}{p^m - \beta \mathbb{E}_{f(x)}[s^m]}$. Finally, set μ according to D3.

First, I want to show for any active message $m_{f(x')}$,

$$U(x) \geq \int \min\{\Theta(s, p, m), 1\}(p - \beta \mathbb{E}_{f(x)}[s]) dm_{f(x')}$$

From the construction of m_z, we have LHS $= \sum \theta^n(x')\varphi(s_n, l)[\mathbb{E}_{f(x')}[s_n] - \beta \mathbb{E}_{f(x)}[s_n]]$. From the linear programming problem, we have

$$U'(x) = -\beta \sum \theta^n \varphi(s_n, l) D_x \mathbb{E}_{f(x)}[s_n] = D_x \text{LHS},$$

and

$$
\begin{aligned}
U''(x) =& -\beta \sum \theta^n \varphi(s_n, l) D_x^2 \mathbb{E}_{f(x)}[s_n] \\
&+ \lambda \sum \theta^n \varphi(s_n, l) D_x \mathbb{E}_{f(x)}[s_n] \\
\geq& D_x^2 \text{LHS},
\end{aligned}
$$

which ensure the inequality holds.

Secondly, I want to show or any inactive message m,

$$
U(x) \geq \int \min\{\Theta(s, p, m), 1\}(p - \beta \mathbb{E}_{f(x)}[s])dm
$$

From construction, we have RHS $= \frac{U(x^*)}{p^m - \beta \mathbb{E}_{f(x^*)}[s^m]}(p^m - \beta \mathbb{E}_{f(x)}[s^m])$. Only need to consider the case that $\mathbb{E}_{f(x)}[s^m] \in (p^m, p^m/\beta)$. Let $g(x; s, p) = \frac{U(x)}{p - \beta \mathbb{E}_{f(x)}[s]}$. When $p < \mathbb{E}_{f(x)}[s]$, we have

$$
\begin{aligned}
g'(x; s, p) \geq& \frac{U(x)(1-\beta)\mathbb{E}_{f(x)}[s_n]}{(p - \beta \mathbb{E}_{f(x)}[s])^2} \\
&[\frac{U'(x)}{U(x)} + \frac{\beta D_x \mathbb{E}_{f(x)}[s]}{(1-\beta)\mathbb{E}_{f(x)}[s]}] \\
\geq& 0
\end{aligned}
$$

where the final inequality comes from the linear programming problem above.

One can verify $(\mu, \{\sigma_z\}_{z \in Z}, \Theta, \hat{z})$ conjectured above is indeed an equilibrium. □

A.3 Secret Trading without Hazard Rate Ordering

For general type space and security space, one can follow the algorithm below and numerically solve for an approximate equilibrium. The fixed-point algorithm is more general than the sequential maximization algorithm provided by Guerrieri, Shimer, and Wright (2010) because it can deal with cases in which types are not first-order stochastic ordered or securities are non-monotone.[1]

Algorithm. *Discretize set S and Z into $\{s_1, \cdots, s_J\}$ and $\{z_1, \cdots, z_K\}$. Pick small positive h and ϵ. Initialize $u^{(0)}(l-hs_j, z_k) = (1-\beta)\mathbb{E}_{z_k}[l-hs_j]$ for all j and k. Initialize $u^{(0)}(l, z_k) = (1-\beta)\mathbb{E}_{z_k}[l]$.*

1. Update directional derivative: compute

$$D_{s_j}u(l, z_k)^{(m)} = \frac{u(l, z_k) - u(l - hs_j, z_k)}{h}$$

   for all j and k.

2. Update market tightness: compute

$$\Theta^{(m)}(s_j, p) = \min_{\mathbb{E}_{z_k}[s_j] \leq p} \frac{D_{s_j}u(l, z_k)^{(m)}}{p - \beta\mathbb{E}_{z_k}[s_j]}$$

   for all j and k.

3. Update payoff: compute

$$\begin{aligned} u^{(m)}(e, z_k) = \min \quad & \sum_{a \in A} e(a)q(a) \\ \text{s.t.} \quad & s_j(a)q(a) \geq \max_{p \geq \mathbb{E}_{z_k}[s_j]} \min\{\Theta^{(m)}(s_j, p), 1\}(p - \beta\mathbb{E}_{z_k}[s_j]) \qquad \text{(A.8)} \\ & \text{for all } j \end{aligned}$$

   for $e \in \{l\} \cup \{l - hs_j | 1 \leq j \leq J\}$.

1. When types are first order stochastic ordered and securities are monotone, the fixed-point algorithm reduces to a sequential maximization algorithm similar to Guerrieri, Shimer, and Wright (2010).

4. Check for convergence: stop if

$$\| u^{(m)}(l - hs_j, z_k) - u^{(m-1)}(l - hs_j, z_k) \| < \epsilon$$

for all j and k, and

$$\| u^{(k)}(l, z_k) - u^{(k-1)}(l, z_k) \| < \epsilon$$

for all k; otherwise, add 1 to counter m and go to step 1.

The approximate equilibrium always exists due to Kakutani fixed-point theorem. I find the algorithm above typically converges. When the algorithm converges, one can find

$$\sigma_{z_k}(\{(s_j, p)\}) = \begin{cases} \lambda(j,k) & \text{if } p = \mathbb{E}_{z_k}[s_j] \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

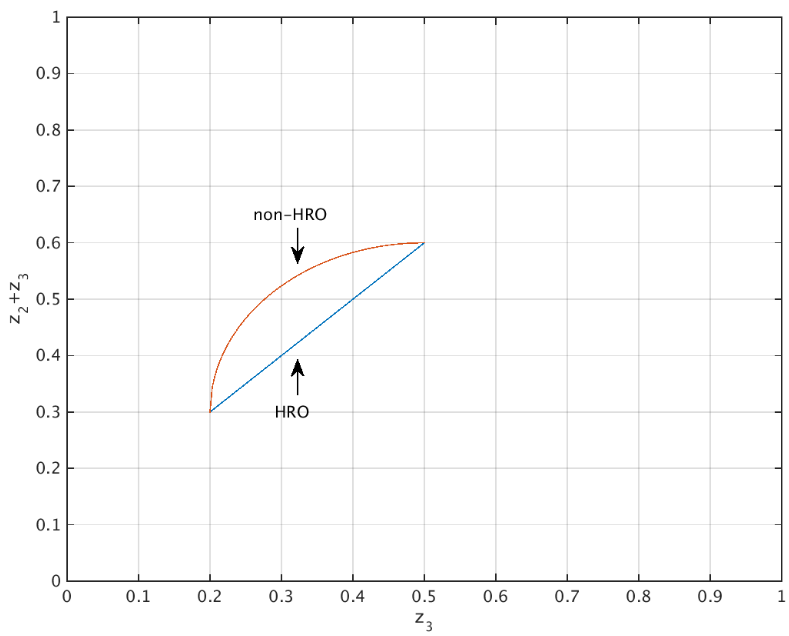
where $\lambda(j,k) \geq 0$ is a KKT multiplier of (A.8);

$$\hat{z}(s_j, p) = \begin{cases} z_k & \text{if } \sigma_{z_k}(\{(s_j, p)\})) > 0 \text{ for some } k \\ z^* & \text{otherwise} \end{cases} \tag{A.10}$$

where $z^* \in \arg\min_{\mathbb{E}_{z_k}[s_j] \leq p} \frac{D_{s_j} u(l, z_k)}{p - \beta \mathbb{E}_{z_k}[s_j]}$.

Figure A.1 shows two possible Z's when $N = 3$. The first one satisfies Assumption 1, while the second one does not. In the non-HRO example, I find that sellers with $x \in [0.13, 0.56]$ will issue interior securities rather than debt securities in the equilibrium.

Figure A.1: First Order Stochastic Order



Note: In this example, the hazard-rate-ordered type set (lower) is

$$\{z|\Pr_z(\tilde{a} \geq a_2) = 0.3(x+1), \Pr_z(\tilde{a} \geq a_3) = 0.3x + 0.2, x \in [0,1]\}$$

the non-HRO type set (upper) is

$$\{z|\Pr_z(\tilde{a} \geq a_2) = 0.3(1+\sqrt{1-(x-1)^2}), \Pr_z(\tilde{a} \geq a_3) = 0.3x + 0.2, x \in [0,1]\}$$

The latter set is non-HRO: pick $z = (1, 0.1, 0.2)$ and $z' = (1, 0.22, 0.3)$, we have $\frac{\sum_{j\geq 1} z_j}{\sum_{j\geq 1} z'_j} \geq \frac{\sum_{j\geq 3} z_j}{\sum_{j\geq 3} z_{j'}} \geq \frac{\sum_{j\geq 2} z_j}{\sum_{j\geq 2} z'_j}$. $A = \{0, 0.28, 0.96\}$.