THE UNIVERSITY OF CHICAGO

APPLICATIONS OF CONTINUOUS-TIME STOCHASTIC CONTROL IN PORTFOLIO OPTIMIZATION

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ABSTRACT

This dissertation applies stochastic control theory in portfolio optimization problems in two different scenarios.

In the first part, we consider a partially-informed trader who does not observe the true drift of a financial asset. Under price dynamics with stochastic unobserved drift, including cases of mean-reversion and momentum dynamics, we take a filtering approach to solve explicitly for trading strategies maximizing expected logarithmic, exponential, and power utility.

In the second part, we study the problem of dynamically trading multiple futures contracts subject to portfolio constraints under a stochastic basis model. The spreads between futures and spot prices are modeled by a multidimensional scaled Brownian bridge to account for their convergence at maturity. The optimal trading strategies are determined from a utility maximization problem with risk preferences of CRRA type.

CHAPTER 1

INTRODUCTION

This dissertation applies stochastic control theory in portfolio optimization problems in two different scenarios.

In the first scenario, we consider a partially-informed trader who does not observe the true drift of a financial asset. Under price dynamics with stochastic unobserved drift, including cases of mean-reversion and momentum dynamics, we take a filtering approach to solve explicitly for trading strategies which maximize expected logarithmic, exponential, and power utility. We prove that the optimal strategies depend on current price and an exponentially-weighted moving average (EMA) price, and in some cases current wealth−not on any other stochastic variables. We establish optimality over all price-history-dependent strategies satisfying integrability criteria, not just EMA-type strategies. We solve explicitly for the optimal parameters of the EMA-type strategies, and verify optimality rigorously.

In the second scenario, we study the problem of dynamically trading multiple futures contracts subject to portfolio constraints under a stochastic basis model. The spreads between futures and spot prices are modeled by a multidimensional scaled Brownian bridge to account for their convergence at maturity. The optimal trading strategies are determined from a utility maximization problem with risk preferences of CRRA type. This leads to the analysis of the associated system of Hamilton-Jacobi-Bellman (HJB) equations, which are reduced to a system of linear ODEs. A series of numerical examples are provided to illustrate the optimal strategies and examine the effects of model parameters.

1.1 Continuous Time Control

In general, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual condition and a n-dimensional Brownian motion $W = (W^1, \cdots, W^n)$ with respect to $\mathbb{F}$. We recall the basics from stochastic differential equations(SDEs) valued in $\mathbb{R}^m$:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,\ t \in [0, T], \tag{1.1.1}$$

where the drift vector $b(t, x)$ and dispersion matrix $\sigma(t, x)$ are deterministic functions, satisfying the (locally)Lipschitz continuous condition in the space variable x uniformly in t, i.e. there exists a constant $K > 0$ such that for every $t \geq 0$,

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|, \tag{1.1.2}$$

in order to ensure the existence and uniqueness of a strong Markovian solution to the SDE (1.1.1).

Control Process. Given a Borel subset A of $\mathbb{R}^d$, we denote $\mathcal{U}$ to be the set of progressively measurable processes $\alpha = (\alpha_t)_{t\leq T}$ valued in A. The elements of $\mathcal{U}$ are called control processes.

Controlled diffusion process. Let

$$\mathrm{d}X_t^\alpha = b(t, X_t^\alpha, \alpha_t)\mathrm{d}t + \sigma(t, X_t^\alpha, \alpha_t)\mathrm{d}W_t, \tag{1.1.3}$$

where the control $\alpha = (\alpha_t) \in \mathcal{U}$, and $b(t,x), \sigma(t,x)$ are continuous functions satisfying

$$\begin{aligned} \|b(t,x,a) - b(t,y,a)\| + \|\sigma(t,x,a) - \sigma(t,y,a)\| \leq K\|x-y\|, \\ \|b(t,x,a)\| + \|\sigma(t,x,a)\| \leq K(1 + \|x\| + \|a\|), \end{aligned} \tag{1.1.4}$$

for some constant K. We abuse the notation X_t^α with X_t if there is no confusion, and denote $X_s^{t,x}$ as the solution to (1.1.3) satisfying $X_t = x, a.s.$ Throughout the paper, we will focus on controlled Markov processes and admissible controls[1].

We denote $\mathcal{A}$ to be the set of admissible controls, i.e. $\alpha \in \mathcal{A}$ if it's progressively measurable and satisfies

$$\mathbb{E}\left(\int_0^T \|\alpha_s\|^2 ds\right) < \infty. \tag{1.1.5}$$

Furthermore, we define

$$\mathcal{A}(t,x) = \{\alpha \in \mathcal{A} | \alpha \text{ is independent with } \mathcal{F}_t\}. \tag{1.1.6}$$

A control process α which is adapted to the natural filtration generated by $\mathcal{F}^X$ is called a **feedback control**; in the form of $\alpha_s = h(s, X_s^{t,x})$, for some measurable function h, is called a **Markovian control**; deterministic is called an **open loop control**.

1. More general and rigorous conditions on controlled Markov processes and admissible controls are given in Appendix A.

1.2 Optimal Portfolio Selection

Let $U(x)$ be a non-decreasing and concave function used to describe an agent's degree of satisfaction with the outcome of wealth. This function is called utility function, and commonly used examples include:

$$
\begin{aligned}
U_{log}(x) &= \log(x). \\
U_{pow}(x) &= \frac{x^r - 1}{r}, r \neq 0. \\
U_{exp}(x) &= -\frac{e^{-px}}{p}, p \geq 0.
\end{aligned}
$$

In fact, since $\lim_{r \to 0} \frac{x^r - 1}{r} = \log(x)$, we will see log utility and power utility have quite lots of similarities.

The **risk tolerance** of a utility function is given by

$$T(U)(x) = -\frac{U'(x)}{U''(x)}.$$

It's easy to see that $T(U_{log})(x) = x,\ T(U_{pow})(x) = \frac{x}{1-p},\ T(U_{exp})(x) = \frac{1}{p}$.

Let X_t^α be some self-financed wealth process, and $U(x)$ be the utility function of an agent. We define an objective function as the expected utility function of the final wealth at terminal T at time t w.r.t control $\alpha \in \mathcal{A}(t, x)$ as:

$$J(t, x, \alpha) := \mathbb{E}\left(U(X_T^\alpha) | \mathcal{F}_t\right) = \mathbb{E}\left(U(X_T^\alpha) | X_t^\alpha = x\right), \tag{1.2.1}$$

The objective is to maximize the objective function over admissible control processes, and we introduce the associated value function:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha). \tag{1.2.2}$$

We say that $\hat{\alpha} \in \mathcal{A}(t,x)$ is optimal control if $v(t,x)^2 = J(t,x,\hat{\alpha})$.

In each scenario we discuss, we will clarify the specific admissible conditions for controls.

1.3 Thesis outline

In Chapter 2, we introduce dynamic programing principle and Hamilton-Jacobi-Bellman(HJB) equation following chapter 3 in Pham (2009) and Touzi (2013). This plays a role as bridges between the classical non-linear PDE approach to optimal portfolio selection problems. In Chapter 3, we set up a price dynamic of single asset with unobserved drift to describe partial information nature, and solve explicitly for the optimal parameters of the EMA-type strategies, and verify optimality rigorously. In Chapter 4, we investigate the problem of optimally trading futures portfolio dynamically with and without budget constraints, and illustrate our model and solutions through a series of numerical examples by simulation and solving a system of linear ODEs.

2. It's pointed out in Touzi (2013) Remark 3.2(iv), defining $\tilde{v}(t,x) := \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha)$, we have $\tilde{v}(t,x) = v(t,x)$.

CHAPTER 2

DYNAMIC PROGRAMMING

In this chapter, we introduce the dynamic programming method for solving stochastic control problems, following (Pham, 2009; Touzi, 2013). This approach yields a certain nonlinear second order partial differential equation(PDE), called Hamilton-Jacobi-Bellman(HJB) equation.

2.1 Dynamic Programming Principle

The dynamic programming principle (DPP) is a fundamental technique in the theory of stochastic control. For a controlled Markov process X_t given by (1.1.3), and the associated value function $v(t,x)$ to maximize the expected utility of $X_T^{t,x}$ among admissible controls.

Theorem 2.1.1 (Dynamic Programming Principle)**.** Let $(t,x) \in [0,T] \times \mathbb{R}^m$. Then we have

$$\begin{aligned} v(t,x) &= \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(v(\tau, X_\tau^{t,x}) \right), \\ &= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(v(\tau, X_\tau^{t,x}) \right), \end{aligned} \tag{2.1.1}$$

where $\mathcal{T}_{t,T}$ denote the set of stopping times taking values in $[t,T]$.

Proof. On one hand, by the strong Markov property of X and law of iterated conditional expectation, we have for any $\tau \in \mathcal{T}_{t,T}$

$$\begin{aligned} J(t,x,\alpha) &= \mathbb{E}\left(U(X_T^{t,x})\right) = \mathbb{E}\left(\mathbb{E}\left(U(X_T^{t,x})|\mathcal{F}_\tau\right)\right) \\ &= \mathbb{E}\left(U(X_T^{\tau,X_\tau^{t,x}})\right) = \mathbb{E}\left(J(\tau, X_\tau^{t,x}, \alpha)\right) \\ &\leq \mathbb{E}\left(v(\tau, X_\tau^{t,x})\right). \end{aligned} \tag{2.1.2}$$

Hence,

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha) \leq \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(v(\tau, X_\tau^{t,x})\right). \tag{2.1.3}$$

On the other hand, fix some arbitrary control $\alpha \in \mathcal{A}(t,x)$ and stopping time $\tau \in \mathcal{T}_{t,T}$. For any $\varepsilon > 0$, and probability measure μ on $[t,T] \times \mathbb{R}^m$ induced by map

$$\begin{aligned} \xi : \Omega &\to [t,T] \times \mathbb{R}^m \\ \omega &\mapsto \left(\tau(\omega),\ X_{\tau(\omega)}^{t,x}\right) \end{aligned} \tag{2.1.4}$$

should be $\mathcal{F}_\tau$-measurable, according to Lemma A.1.4 there exists[1] a Borel measurable map $\phi^\varepsilon : [t,T] \times \mathbb{R}^m \to \mathcal{A}(t,x)$ as ε-optimal control. Then

$$\begin{aligned} \zeta : (\Omega, \mathcal{F}_\tau) &\to (\mathcal{A}(t,x), \mathcal{B}_{\mathcal{A}(t,x)}) \\ \omega &\mapsto \phi^\varepsilon\left(\tau(\omega),\ X_{\tau(\omega)}^{t,x}(\omega)\right), \end{aligned} \tag{2.1.5}$$

is measurable. Then by condition **B** in Appendix A.1, there exists an admissible

1. A proof for weak dynamic programming principle without measurable selection theorem can be found in Touzi (2013).

control $\alpha^{\varepsilon} \in \mathcal{A}(t,x)$ such that $\zeta = \alpha^{\varepsilon}$, for any $\omega \in \Omega, s \geq \tau$, $Leb \times \mathbb{P}$-$a.s.$

$$\text{i.e, } v\left(\tau(\omega), X^{t,x}_{\tau(\omega)}(\omega)\right) - \varepsilon \leq J\left(\tau(\omega), X^{t,x}_{\tau(\omega)}(\omega), \alpha^{\varepsilon,\omega}\right), \ Leb \times \mathbb{P}\text{-}a.s.$$

We concat α and α^{ε} to construct a new control process as

$$\hat{\alpha}_s = \alpha_s \mathbf{1}_{s\in[t,\tau)} + \alpha^{\varepsilon}\mathbf{1}_{s\in[\tau,T]} \in \mathcal{A}(t,x), \tag{2.1.6}$$

by condition **A** of admissible controls, see Appendix A. Then,

$$v(t,x) \geq J(t,x,\hat{\alpha}) = \mathbb{E}\left(U(X_T^{\tau,X_\tau^{t,x}})\right) = J(\tau, X_\tau^{t,x}, \alpha^{\varepsilon}) \geq v(\tau, X_\tau^{t,x}) - \varepsilon, \tag{2.1.7}$$

let $\varepsilon \to 0$, we have $v(t,x) \geq v(\tau, X_\tau^{t,x})$, for arbitrary α and τ. Therefore,

$$v(t,x) \geq \sup_{\alpha\in\mathcal{A}(t,x)} \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(v(\tau, X_\tau^{t,x})\right). \tag{2.1.8}$$

In conclusion, Eq.(2.1.1) is verified by the two-side inequalities (2.1.3) and (2.1.8). $\square$

Corollary 2.1.1. For any stopping time $\tau \in \mathcal{T}_{t,T}$, we have

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}\left(v(\tau, X_\tau^{t,x})\right).$$

2.2 Hamilton-Jacobi-Bellman Equation(HJB)

The Hamilton-Jacobi-Bellman equation (HJB) is derived by applying Itô formula and dynamic programming principle, assuming the value function $v(t,x)$ given by

(1.2.2) is $C^{1,2}([0,T) \times \mathbb{R}^m, \mathbb{R})$.

The infinitesimal generator of the controlled diffusion process X_t with constant control $\alpha_t = a \in A$ in (1.1.3) is given by

$$\mathcal{L}^a u = b(t,x,a)\nabla_x u(t,x,a) + \frac{1}{2} tr\left(\sigma(t,x,a)\sigma(t,x,a)^\top \nabla_x^2 u(t,x,a)\right). \qquad (2.2.1)$$

Take $\tau^h = t + h,\ h > 0$ and

$$\theta^N = \inf\left\{s \geq t : \left|\left(\frac{\partial}{\partial t} + \mathcal{L}^a\right) v(s, X_s^{t,x})\right| + \|(\nabla_x v)^\top \sigma(s, X_s^{t,x})\| > N\right\}.$$

For sufficiently large N, we have $\mathbb{P}(\theta^N > t) > 0$, and in that case, $\theta^N \wedge \tau^h = \tau^h$ when $h \to 0^+$. Let $\tau = \theta^N \wedge \tau^h$, and plug into (2.1.1), we have

$$\begin{aligned} v(t,x) &\geq \mathbb{E}\left(v(\tau, X_\tau^{t,x})\right) \\ &\geq \mathbb{E}\left(v(t,x) + \int_t^\tau \left(\frac{\partial}{\partial t} + \mathcal{L}^a\right) v(s, X_s^{t,x}) \mathrm{d}s + \int_t^\tau (\nabla_x v)^\top \sigma(s, X_s^{t,x}) \mathrm{d}W_s\right) \\ &= v(t,x) + \mathbb{E}\left(\int_t^\tau \left(\frac{\partial}{\partial t} + \mathcal{L}^a\right) v(s, X_s^{t,x}) \mathrm{d}s\right), \end{aligned} \qquad (2.2.2)$$

thus, $\mathbb{E}\left(\frac{1}{h}\int_t^\tau \left(\frac{\partial}{\partial t} + \mathcal{L}^a\right) v(s, X_s^{t,x}) \mathrm{d}s\right) \leq 0$.

Let $h \to 0^+$, we have $\left(\frac{\partial}{\partial t} + \mathcal{L}^a\right) v(t,x) \leq 0$ by dominated convergence theorem and Lebegue differentiation theorem. Since a is arbitrary, we have

$$\sup_{a \in A} \left(\frac{\partial}{\partial t} + \mathcal{L}^a\right) v(t,x) \leq 0. \qquad (2.2.3)$$

On the other hand, suppose that $\alpha^* \in \mathcal{A}(t,x)$ is the optimal control[2], then all of the inequalities above are equalities, therefore, we have $\left(\frac{\partial}{\partial t} + \mathcal{L}^{\alpha_t^*}\right) v(t,x) = 0$.

In conclusion, we deduce the HJB for value function $v(t,x)$

$$\sup_{a \in A} \left(\frac{\partial}{\partial t} + \mathcal{L}^a \right) v(t,x) = 0. \tag{2.2.4}$$

with terminal condition $v(T,x) = U(x)$.

At this stage, we are equipped with the techniques, i.e. the HJB method for our purpose of solving portfolio optimization problems. As a standard step, we should verify the solution of (2.2.4) is exactly the value function, with some regular conditions assumed ahead. We will see how we complete the verification at the end in our next two chapters. As a matter of fact, the verification theorem can free us from the delicate measurability questions.

2. A proof without assuming existence of optimal control is given in Appendix A.3, Proposition A.3.1.

CHAPTER 3

EMA-TYPE TRADING STRATEGIES MAXIMIZE UTILITY UNDER PARTIAL INFORMATION

In this chapter[1], we consider a partially-informed trader who does not observe the drift of a financial asset driven by a hidden stochastic state variable. This framework includes cases where the drift pulls the market price to revert toward a hidden stochastic state variable that represents the "fair value" of the financial asset, as well as cases where the drift is a hidden "momentum" variable. The trader who dynamically trades this asset faces the question of what strategy to follow, given the price history.

On one hand, various popular "technical analysis" approaches include buy/sell signals that depend on comparing the current price against a *moving average* or exponentially-weighted moving average (EMA) of recent prices. It is common in practice to assess the empirical performance of such signals by backtesting them on historical data, but this does not address the foundational questions of *proving* rigorously that trading strategies should even use *moving average* signals in the first place, and justifying exactly what *explicit* functional form should map the moving averages into optimal trades.

On the other hand, the mainstream academic literature defines the objective of maximizing the expected utility of terminal wealth, and takes a stochastic optimal control approach to solving the maximization problem under various assumptions,

1. This chapter contains joint work with Roger Lee.

beginning with the Merton problem(1971) which assumed a known constant drift, more literatures can be found in (Mudchanatongsuk et al., 2008; Boguslavskaya and Boguslavsky, 2004). Extensions allowed a stochastic but observable drift. Then the filtering approach extended to optimization with hidden drift in (Lakner, 1995, 1998; Brendle, 2006, 2008), but still without settling the foundational questions listed above.

Building on the filtering approach, this paper's main contributions are threefold.

First, we prove that the optimal dynamic position (expressed in units of asset, or in the power utility case, expressed as a fraction of wealth) is indeed a function only of an EMA and the current price, and we express this function explicitly. It is crucial here to emphasize that we do not merely optimize *within* some class of EMA strategies, as discussed in Lorig et al. (2019); rather we optimize over *all* admissible functions of the *entire price history* and we *conclude* that an EMA-type strategy wins.

Second, in proving optimality relative to a specified universe of admissible strategies, we verify rigorously not only that the claimed optimal strategy is indeed admissible, but moreover that it satisfies a stronger regularity condition, namely a simple and natural sup-integrability condition purely on the induced value function, essentially establishing some control over the severity of the strategy's drawdowns.

Third, we obtain exact and explicit formulas for the optimal trading strategies given partial information in all cases: not only exponential and log utility, but also power utility; and for dynamics which include both mean-reversion and momentum cases. The explicit nature of the solutions not only simplifies implementation, but also provides intuition and insight, into which factors drive which features of the optimal trades in which directions.

3.1 Mean-reverting and Momentum Dynamics

On a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ supporting Brownian motions W and Z with correlation $\rho \in (-1, 1)$, let X and Θ be the unique strong solutions to the linear SDEs

$$\mathrm{d}X_t = (\kappa_{x\theta}\Theta_t - \kappa_x X_t)\mathrm{d}t + \sigma_x \mathrm{d}W_t, \tag{3.1.1}$$

where $\kappa_{x\theta} > 0$ and $\kappa_x \geq 0$ and $\sigma_x > 0$, and

$$\mathrm{d}\Theta_t = (\mu - \kappa_\theta \Theta_t)\mathrm{d}t + \sigma_\theta \mathrm{d}Z_t, \tag{3.1.2}$$

where $\kappa_\theta \geq 0$ and $\sigma_\theta > 0$.

Here X models the price of some underlying asset or combination of assets. For instance, X could be the spread between two futures contracts, or X could be the price of a long-short combination of two stocks, at some fixed ratio. The drift of X contains the unobserved factor Θ. The case $\kappa_{x\theta} = \kappa_x > 0$ produces mean-reversion dynamics where the level of X reverts toward Θ. The case $(\kappa_{x\theta}, \kappa_x) = (1, 0)$ produces "momentum" dynamics where the drift of X *is* Θ.

The hidden X-drift driver Θ itself varies stochastically. If $\kappa_\theta > 0$, then Θ reverts toward long-term level μ/κ_θ; on the other hand, if $\kappa_\theta = 0$ then Θ has constant drift μ.

Let us call $\mathcal{F}$ the *full-information* filtration, as both X and Θ are $\mathcal{F}$-adapted.

Denote by $\mathcal{F}^X$ the *partial-information* filtration generated by the observed price process X alone. With respect to $\mathcal{F}^X$, the drift factor Θ is unobserved, but does admit an estimate by Kalman-Bucy filter[2]

$$\hat{\Theta}_t := \mathbb{E}(\Theta_t|\mathcal{F}_t^X) \tag{3.1.3}$$

with an uncertainty quantified by the conditional variance

$$\gamma_t := \mathrm{Var}(\Theta_t|\mathcal{F}_t^X) = \mathbb{E}\Theta_t^2 - \mathbb{E}\hat{\Theta}_t^2, \tag{3.1.4}$$

which is deterministic and satisfies the ODE

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = -2\kappa_\theta\gamma_t + \sigma_\theta^2 - \Big(\frac{\kappa_{x\theta}\gamma_t}{\sigma_x} + \sigma_\theta\rho\Big)^2. \tag{3.1.5}$$

Hence

$$\gamma_t = \frac{\sigma_x^2}{\kappa_{x\theta}^2}\Big(R\frac{e^{2Rt}+g}{e^{2Rt}-g} - \kappa_\theta - \rho\frac{\sigma_\theta\kappa_{x\theta}}{\sigma_x}\Big), \tag{3.1.6}$$

where

$$R := \sqrt{\kappa_\theta^2 + 2\rho\kappa_\theta\frac{\sigma_\theta\kappa_{x\theta}}{\sigma_x} + \Big(\frac{\sigma_\theta\kappa_{x\theta}}{\sigma_x}\Big)^2} \tag{3.1.7}$$

and

$$g := \frac{\gamma_0\kappa_{x\theta}^2 + \sigma_\theta\sigma_x\kappa_{x\theta}\rho + \sigma_x^2\kappa_\theta - \sigma_x^2 R}{\gamma_0\kappa_{x\theta}^2 + \sigma_\theta\sigma_x\kappa_{x\theta}\rho + \sigma_x^2\kappa_\theta + \sigma_x^2 R} < 1. \tag{3.1.8}$$

2. The discussion on Kalman-Bucy filter can be found in Appendix B.

Thus, for all $\gamma_0 \geq 0$, we have $\gamma_t \to \bar{\gamma}$ as $t \to \infty$, where

$$\bar{\gamma} := \frac{\sigma_x^2}{\kappa_{x\theta}^2}\Big(R - \kappa_\theta - \rho\frac{\sigma_\theta \kappa_{x\theta}}{\sigma_x}\Big) \tag{3.1.9}$$

represents the long-term or steady-state level of uncertainty about Θ; moreover $\operatorname{sgn} g = \operatorname{sgn}(\gamma_0 - \bar{\gamma})$, and in particular $g = 0$ if and only if $\gamma_0 = \bar{\gamma}$.

Define x by

$$\mathrm{d}x_t = \frac{1}{\sigma_x}\mathrm{d}X_t + \frac{\kappa_x}{\sigma_x}X_t\mathrm{d}t = \frac{\kappa_{x\theta}}{\sigma_x}\Theta_t\mathrm{d}t + \mathrm{d}W_t. \tag{3.1.10}$$

Then

$$\mathrm{d}\hat{\Theta}_t = (\mu - \kappa_\theta\hat{\Theta}_t)\mathrm{d}t + \sigma_{\hat{\theta}}(t)\mathrm{d}\nu_t, \tag{3.1.11}$$

where

$$\sigma_{\hat{\theta}}(t) := \frac{\kappa_{x\theta}\gamma_t}{\sigma_x} + \sigma_\theta\rho = \frac{\sigma_x}{\kappa_{x\theta}}\Big(R\frac{e^{2Rt} + g}{e^{2Rt} - g} - \kappa_\theta\Big) \tag{3.1.12}$$

and the "innovation" process

$$\nu_t := x_t - \int_0^t \frac{\kappa_{x\theta}}{\sigma_x}\hat{\Theta}_s\mathrm{d}s \tag{3.1.13}$$

is an $\mathcal{F}^X$-Brownian motion. Therefore

$$\mathrm{d}X_t = \sigma_x\mathrm{d}x_t - \kappa_x X_t\mathrm{d}t = (\kappa_{x\theta}\hat{\Theta}_t - \kappa_x X_t)\mathrm{d}t + \sigma_x\mathrm{d}\nu_t. \tag{3.1.14}$$

3.1.1 The Filter Is a Moving Average

Proposition 3.1.1. We have

$$\hat{\Theta}_t = J(t)\frac{\mu}{\kappa_\theta} + \frac{1}{F(t)}\hat{\Theta}_0 - \frac{\sigma_{\hat{\theta}}(0)}{\sigma_x F(t)}X_0 + \frac{\sigma_{\hat{\theta}}(t)}{\sigma_x}X_t + \int_0^t K(s,t)X_s \mathrm{d}s, \tag{3.1.15}$$

where

$$\begin{aligned} F(t) &:= \exp\left(R\int_0^t \frac{e^{2Rs}+g}{e^{2Rs}-g}\mathrm{d}s\right) = \frac{e^{Rt}-ge^{-Rt}}{1-g} \\ J(t) &:= \frac{\kappa_\theta(e^{Rt}-1)(e^{Rt}-g)}{R(e^{2Rt}-g)} \\ K(s,t) &:= \frac{g(R+\kappa_x)(R+\kappa_\theta)e^{-R(t+s)} - (R-\kappa_x)(R-\kappa_\theta)e^{-R(t-s)}}{\kappa_{x\theta}\left(1-ge^{-2Rt}\right)} \end{aligned} \tag{3.1.16}$$

Proof. We have

$$\begin{aligned} \mathrm{d}\hat{\Theta}_t &= (\mu - \kappa_\theta\hat{\Theta}_t)\mathrm{d}t + \sigma_{\hat{\theta}}(t)\Big(\mathrm{d}x_t - \frac{\kappa_{x\theta}}{\sigma_x}\hat{\Theta}_t\mathrm{d}t\Big) \\ &= \Big(\mu - (\kappa_\theta + \frac{\kappa_{x\theta}}{\sigma_x}\sigma_{\hat{\theta}}(t))\hat{\Theta}_t\Big)\mathrm{d}t + \sigma_{\hat{\theta}}(t)\mathrm{d}x_t \\ &= \Big(\mu - R\frac{e^{2Rt}+g}{e^{2Rt}-g}\hat{\Theta}_t\Big)\mathrm{d}t + \sigma_{\hat{\theta}}(t)\mathrm{d}x_t \end{aligned} \tag{3.1.17}$$

and

$$\begin{aligned} \mathrm{d}(F(t)\hat{\Theta}_t) = \hat{\Theta}_t F'(t)\mathrm{d}t + F(t)\mathrm{d}\hat{\Theta}_t &= \mu F(t)\mathrm{d}t + F(t)\sigma_{\hat{\theta}}(t)\mathrm{d}x_t \\ &= \mu F(t)\mathrm{d}t + \frac{1}{\sigma_x}\Phi(t)\mathrm{d}X_t + \frac{\kappa_x}{\sigma_x}\Phi(t)X_t\mathrm{d}t, \end{aligned} \tag{3.1.18}$$

where $\Phi(t) := F(t)\sigma_{\hat{\theta}}(t)$. Then

$$F(t)\hat{\Theta}_t - F(0)\hat{\Theta}_0 = \mu \int_0^t F(s)\mathrm{d}s + \frac{\kappa_x}{\sigma_x}\int_0^t \Phi(s)X_s\mathrm{d}s + \frac{1}{\sigma_x}\Big(\Phi(t)X_t - \Phi(0)X_0 - \int_0^t X_s\mathrm{d}\Phi(s)\Big).$$

Hence,

$$\begin{aligned}\hat{\Theta}_t =& \frac{1}{F(t)}\hat{\Theta}_0 + \frac{\mu}{F(t)}\int_0^t F(s)\mathrm{d}s + \frac{\sigma_{\hat{\theta}}(t)}{\sigma_x}X_t - \frac{\sigma_{\hat{\theta}}(0)}{\sigma_x F(t)}X_0 \\ &+ \frac{1}{\sigma_x F(t)}\int_0^t (\kappa_x\Phi(s) - \Phi'(s))X_s\mathrm{d}s,\end{aligned} \tag{3.1.19}$$

which implies (3.1.15). □

Let the notation $f(t) \sim g(t)$ mean that $f/g \to 1$ as $t \to \infty$. Then the coefficients of μ/κ_θ, $\hat{\Theta}_0$, X_0 and X_t have, respectively, asymptotic behavior

$$\begin{aligned} J(t) &\sim \frac{\kappa_\theta}{R}(1 - e^{-Rt}) \\ \frac{1}{F(t)} &\sim e^{-Rt} \\ -\frac{\sigma_{\hat{\theta}}(0)}{\sigma_x F(t)} &\sim \frac{R-\kappa_\theta}{\kappa_{x\theta}}e^{-Rt} \qquad \text{if } R \neq \kappa_\theta \text{, otherwise} \sim -\frac{2gR}{\kappa_{x\theta}}e^{-Rt} \\ \frac{\sigma_{\hat{\theta}}(t)}{\sigma_x} &\sim \frac{R-\kappa_\theta}{\kappa_{x\theta}} \qquad \text{if } R \neq \kappa_\theta \text{, otherwise} \sim \frac{2gR}{\kappa_{x\theta}}e^{-2Rt} \end{aligned} \tag{3.1.20}$$

In the case $g = 0$, these asymptotic equivalences hold with *equality* for *all* $t \geq 0$, and

$$K(s,t) = -\frac{(R-\kappa_x)(R-\kappa_\theta)}{\kappa_{x\theta}}e^{-R(t-s)}$$

Remark 3.1.1. Let $g = 0$, let $\kappa_\theta > 0$, let t be large. Then the nonvanishing terms in (3.1.15) are

$$J(t)\frac{\mu}{\kappa_\theta} + \frac{\sigma_{\hat{\theta}}(t)}{\sigma_x}X_t + \frac{(\kappa_x - R)(R - \kappa_\theta)}{\kappa_{x\theta}} \int_0^t e^{-R(t-s)} X_s \mathrm{d}s$$
$$\approx \frac{\kappa_\theta}{R} \cdot \frac{\mu}{\kappa_\theta} + \left(1 - \frac{\kappa_\theta}{R}\right) \cdot \frac{\kappa_x}{\kappa_{x\theta}} \cdot \left(\frac{R}{\kappa_x} X_t + \left(1 - \frac{R}{\kappa_x}\right) \int_0^t R e^{-R(t-s)} X_s \mathrm{d}s\right) \quad (3.1.21)$$

which is a *weighted average* of

- ▷ the long-term Θ mean μ/κ_θ (with weight κ_θ/R) and
- ▷ an observed average X-level (with weight $1 - \kappa_\theta/R$), scaled by $\kappa_x/\kappa_{x\theta}$ to convert into a Θ level, because $\frac{\kappa_x}{\kappa_{x\theta}}X$ reverts toward Θ.

In turn, the average X-level is itself a weighted average (with weights R/κ_x and $1 - R/\kappa_x$) of

- ▷ the current X_t and
- ▷ an *exponentially weighted moving average* (EMA): $\int_0^t Re^{-R(t-s)} X_s \mathrm{d}s$.

The decay rate of the EMA is R from (3.1.7), where

$$R^2 = \frac{\kappa_\theta^2 \sigma_x^2 + 2\rho\kappa_\theta\sigma_\theta\kappa_{x\theta}\sigma_x + \sigma_\theta^2\kappa_{x\theta}^2}{\sigma_x^2}. \quad (3.1.22)$$

Intuitively, this ratio balances the trade-off between two factors: On one hand, the greater the variation of the observed X levels – as measured by the σ_x^2 in the denominator – the longer the period over which the X averaging should be done, and therefore the smaller the decay rate R. On the other hand, the greater the variation of the X-undetermined part of the Θ dynamics, the less informative the older X data as an indicator of the current Θ level, and therefore the larger the R; this variation of the X-undetermined part of Θ can be expressed by rewriting (3.1.2):

$$\mathrm{d}\kappa_{x\theta}\Theta_t = -(\kappa_{x\theta}\mu - \kappa_x\kappa_\theta X_t)\mathrm{d}t - \kappa_\theta \mathrm{d}X_t + \kappa_\theta\sigma_x \mathrm{d}W_t + \kappa_{x\theta}\sigma_\theta \mathrm{d}Z_t, \tag{3.1.23}$$

which has eliminated Θ from the dt term, making both the dt and dX terms $\mathcal{F}^X$-observable; the remaining variation, taking into account the correlation between dW and dZ, is given by the numerator of (3.1.22).

3.1.2 $\mathcal{F}^X$-dynamics of price, drift, and wealth

Denote the estimated instantaneous drift of X by

$$Y_t := \kappa_{x\theta}\hat{\Theta}_t - \kappa_x X_t, \tag{3.1.24}$$

which has dynamics

$$\begin{aligned} \mathrm{d}Y_t &= \kappa_{x\theta}[(\mu - \kappa_\theta\hat{\Theta}_t)\mathrm{d}t + \sigma_{\hat{\theta}}(t)\mathrm{d}\nu_t] - \kappa_x(Y_t\mathrm{d}t + \sigma_x\mathrm{d}\nu_t) \\ &= (\kappa_{x\theta}\mu - \kappa_\theta\kappa_{x\theta}\hat{\Theta}_t - \kappa_x Y_t)\mathrm{d}t + (\kappa_{x\theta}\sigma_{\hat{\theta}}(t) - \kappa_x\sigma_x)\mathrm{d}\nu_t \\ &= (\kappa_{x\theta}\mu - \kappa_\theta(Y_t + \kappa_x X_t) - \kappa_x Y_t)\mathrm{d}t + (\kappa_{x\theta}\sigma_{\hat{\theta}}(t) - \kappa_x\sigma_x)\mathrm{d}\nu_t. \end{aligned} \tag{3.1.25}$$

Suppose that some $\mathcal{F}^X$-adapted processes H and V satisfy

$$\mathrm{d}V_t = H_t \mathrm{d}X_t, \tag{3.1.26}$$

(for which sufficient conditions are given in Section 3.3.1). Refer to H as the *control* process or the *trading strategy*, and V as the *wealth* process. Implicit in the specification (3.1.26) are zero interest rates and frictionless markets.

Then (X, Y, V) have $\mathcal{F}^X$-dynamics

$$\begin{aligned} \mathrm{d}X_t &= Y_t \mathrm{d}t + \sigma_x \mathrm{d}\nu_t \\ \mathrm{d}Y_t &= (\kappa_{x\theta}\mu - (\kappa_\theta + \kappa_x)Y_t - \kappa_\theta \kappa_x X_t)\mathrm{d}t + \sigma_x \bar{\sigma}(t)\mathrm{d}\nu_t \\ \mathrm{d}V_t &= Y_t H_t \mathrm{d}t + \sigma_x H_t \mathrm{d}\nu_t \end{aligned} \tag{3.1.27}$$

where

$$\bar{\sigma}(t) := \frac{\kappa_{x\theta}\sigma_{\hat{\theta}}(t)}{\sigma_x} - \kappa_x = R\frac{e^{2Rt} + g}{e^{2Rt} - g} - \kappa_\theta - \kappa_x \tag{3.1.28}$$

is the ratio of the signed $\mathcal{F}^X$-volatilities of Y and X.

3.2 Optimal Strategies

For fixed $T > 0$ we will solve for H to maximize expected utility of terminal wealth

$$\mathbb{E}(U(V_T)) \tag{3.2.1}$$

for a utility function $U \in \{U_{\exp}, U_{\text{pow}}, U_{\log}\}$ where

$$\begin{aligned} U_{\log}(v) &:= \log v \\ U_{\text{pow}}(v) &:= \frac{v^{1-q}}{1-q} \qquad\qquad q > 1 \\ U_{\exp}(v) &:= -\exp(-pv) \qquad p > 0, \end{aligned} \tag{3.2.2}$$

where q and p are risk-aversion parameters in the cases of power and exponential utility, respectively.

For any continuous function $G : [0,T] \times O \to \mathbb{R}$ that is $C^{1,2}$ on $(0,T) \times O$, where O is an open subset of $\mathbb{R}^3$, let $\mathcal{L}G$ denote the function $(0,T) \times O \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{L}G(t,x,y,v,\mathsf{H}) &:= G_t + yG_x + (\mu\kappa_{x\theta} - \kappa_\theta\kappa_x x - (\kappa_\theta + \kappa_x)y)G_y + y\mathsf{H}G_v + \frac{\sigma_x^2}{2}G_{xx} \\ &\quad + \frac{\sigma_x^2\bar{\sigma}(t)^2}{2}G_{yy} + \frac{\mathsf{H}^2\sigma_x^2}{2}G_{vv} + \sigma_x^2\bar{\sigma}(t)G_{xy} + \mathsf{H}\sigma_x^2\bar{\sigma}(t)G_{vy} + \mathsf{H}\sigma_x^2 G_{xv}. \end{aligned} \tag{3.2.3}$$

The partial derivatives of G are denoted by subscripts, and evaluated at (t,x,y,v,H).

Proposition 3.2.1. If $G_{vv} < 0$ on $(0,T) \times O$, then for each $(t,x,y,v,\mathsf{H}) \in (0,T) \times O \times \mathbb{R}$,

$$\begin{aligned} \mathcal{L}\mathcal{G} &\le G_t + yG_x + (\mu\kappa_{x\theta} - \kappa_\theta\kappa_x x - (\kappa_\theta + \kappa_x)y)G_y + \frac{\sigma_x^2}{2}G_{xx} + \sigma_x^2\bar{\sigma}(t)G_{xy} \\ &\quad + \frac{\sigma_x^2\bar{\sigma}(t)^2}{2}G_{yy} - \frac{(yG_v + \sigma_x^2 G_{xv} + \sigma_x^2\bar{\sigma}(t)G_{yv})^2}{2\sigma_x^2 G_{vv}}. \end{aligned} \tag{3.2.4}$$

Equality holds if

$$\mathsf{H} = \frac{yG_v + \sigma_x^2 G_{xv} + \sigma_x^2 \bar{\sigma}(t) G_{yv}}{-\sigma_x^2 G_{vv}}. \tag{3.2.5}$$

Proof. This follows from regarding $\mathcal{L}G$ as a quadratic in H with negative coefficient on H^2. □

Given processes (X, Y, V, H) as introduced in the previous section, define the process $\mathcal{L}^H G$ by

$$(\mathcal{L}^H G)_t := \mathcal{L}G(t, X_t, Y_t, V_t, H_t).$$

Remark 3.2.1. Intuitively, the operator $\mathcal{L}^H$ is the infinitesimal generator of the H-controlled process (t, X, Y, V). By finding G such that the right-hand side of (3.2.4) vanishes, with terminal condition given by the utility of terminal wealth, we will solve the Hamilton-Jacobi-Bellman equation

$$\sup_H \mathcal{L}^H G = 0, \qquad G(T, x, y, v) = U(v). \tag{3.2.6}$$

for G and H, which will be rigorously verified to be the value function and optimal trading strategy respectively.

3.2.1 Log Utility

In the case of log utility ($U = U_{\log}$), let

$$G(t, x, y, v) = A(t)x^2 + B(t)xy + \alpha(t)y^2 + C(t)x + \beta(t)y + D(t) + U_{\log}(v) \tag{3.2.7}$$

where $A, B, C, D, \alpha, \beta$ are given below.

Proposition 3.2.2. In the case $U = U_{\log}$ define

$$
\begin{aligned}
A(t) &= -\kappa_x\kappa_\theta\bar{A}(t), \qquad B(t) = \kappa_x\kappa_\theta\bar{B}(t), \\
C(t) &= 2\kappa_{x\theta}\mu\bar{A}(t), \qquad \beta(t) = -\kappa_{x\theta}\mu\bar{B}(t)
\end{aligned}
\tag{3.2.8}
$$

and $D(t) = \int_t^T \mu\kappa_{x\theta}\beta(s) + \sigma_x^2 A(s) + \sigma_x^2\bar{\sigma}(s)^2\alpha(s) + \sigma_x^2\bar{\sigma}(s)B(s)\mathrm{d}s$, where, with $\tau := T - t$,

$$
\begin{aligned}
\bar{A}(t) &= -\frac{(\kappa_x-\kappa_\theta)^2 + 4e^{-(\kappa_x+\kappa_\theta)\tau}\kappa_x\kappa_\theta - e^{2\kappa_\theta t}\kappa_x(\kappa_x+\kappa_\theta) - e^{2\kappa_x t}\kappa_\theta(\kappa_x+\kappa_\theta)}{4\sigma_x^2(\kappa_x-\kappa_\theta)^2(\kappa_x+\kappa_\theta)} \\
\bar{B}(t) &= -\frac{(e^{-\kappa_\theta\tau} - e^{-\kappa_x\tau})^2}{2\sigma_x^2(\kappa_x-\kappa_\theta)^2} \\
\alpha(t) &= \frac{(\kappa_x-\kappa_\theta)^2 + 4e^{-(\kappa_x+\kappa_\theta)\tau}\kappa_x\kappa_\theta - e^{-2\kappa_x\tau}\kappa_x(\kappa_x+\kappa_\theta) - e^{-2\kappa_\theta\tau}\kappa_\theta(\kappa_x+\kappa_\theta)}{4\sigma_x^2(\kappa_x-\kappa_\theta)^2(\kappa_x+\kappa_\theta)}
\end{aligned}
\tag{3.2.9}
$$

for $\kappa_x \neq \kappa_\theta$, otherwise $\bar{A}(t) = \frac{(1+2\kappa_x\tau+2\kappa_x^2\tau^2)e^{-2\kappa_x\tau}-1}{8\sigma_x^2\kappa_x}$, $\bar{B}(t) = -\frac{\tau^2 e^{-2\kappa_x\tau}}{2\sigma_x^2}$, and $\alpha(t) = \frac{\tau e^{-2\kappa_x\tau}}{2\sigma_x^2} - \bar{A}(t)$.

Then G defined by (3.2.7) satisfies $G(T, x, y, v) = U_{\log}(v)$ and $\mathcal{L}^H G \leq 0$ for arbitrary H. The trading strategy defined by

$$
H_t^* = h_t^* V_t, \quad h_t^* = \frac{Y_t}{\sigma_x^2},
\tag{3.2.10}
$$

attains equality $\mathcal{L}^{H^*}G = 0$.

Proof. Direct substitution shows that, for this (G, H^*), the right-hand side of (3.2.4) vanishes, the right-hand side of (3.2.5) evaluated at (t, X_t, Y_t, V_t) produces H_t^*, and

$G(T, x, y, v) = U_{\log}(v)$. $\square$

3.2.2 Exponential and Power Utility

In the cases of exponential and power utility, define

$$G(t, x, y, v) = e^{A(t)x^2+B(t)xy+\alpha(t)y^2+C(t)x+\beta(t)y+D(t)}U(v) \tag{3.2.11}$$

where $A, B, C, D, \alpha, \beta$ will be defined below (and will depend on whether $U = U_{\exp}$ or $U = U_{\text{pow}}$).

3.2.3 Exponential Utility

In the case of exponential utility, the interdependencies of the ODEs for $A, B, C, D, \alpha, \beta$ may be resolved sequentially, by solving a sequence of scalar ODEs. The result is as follows:

Proposition 3.2.3. In the case $U = U_{\exp}$ define

$$\begin{aligned} A(t) &= -\frac{\kappa_x^2\kappa_\theta^2}{2\sigma_x^2}\bar{A}(t), \qquad B(t) = \frac{\kappa_x\kappa_\theta}{\sigma_x^2}\bar{B}(t), \qquad \alpha(t) = -\frac{1}{2\sigma_x^2}\bar{\alpha}(t) \\ C(t) &= \frac{\kappa_{x\theta}\mu\kappa_x\kappa_\theta}{\sigma_x^2}\bar{A}(t), \qquad \beta(t) = -\frac{\kappa_{x\theta}\mu}{\sigma_x^2}\bar{B}(t) \end{aligned} \tag{3.2.12}$$

and

$$
\begin{aligned}
\bar{A}(t) &= \frac{2R\tau - 3 + 4e^{-R\tau} - e^{-2R\tau} - ge^{-2RT}(2R\tau - 4e^{R\tau} + e^{2R\tau} + 3)}{2R^3\left(1 - ge^{-2RT}\right)}, \\
\bar{B}(t) &= \frac{(e^{R\tau} - 1)^2}{2R^2} \times \frac{e^{2Rt} - g}{e^{2RT} - g} \\
\bar{\alpha}(t) &= \frac{e^{2R\tau} - 1}{2R} \times \frac{e^{2Rt} - g}{e^{2RT} - g} \\
D(t) &= \int_t^T \mu\kappa_{x\theta}\beta(s) + \sigma_x^2 A(s) + \sigma_x^2 \bar{\sigma}(s)^2 \alpha(s) + \sigma_x^2 \bar{\sigma}(s) B(s) \mathrm{d}s.
\end{aligned}
\tag{3.2.13}
$$

where $\tau := T - t$.

Then G defined by (3.2.11) satisfies $G(T, x, y, v) = U_{\exp}(v)$ and $\mathcal{L}^H G \leq 0$ for arbitrary H. The trading strategy defined by

$$
H_t^* = \frac{C(t) + \bar{\sigma}(t)\beta(t)}{p} + \frac{1 + \sigma_x^2 B(t) + 2\sigma_x^2 \bar{\sigma}(t)\alpha(t)}{p\sigma_x^2} Y_t + \frac{2A(t) + \bar{\sigma}(t)B(t)}{p} X_t
\tag{3.2.14}
$$

attains equality $\mathcal{L}^{H^*} G = 0$.

Proof. Direct substitution shows that, for this (G, H^*), the right-hand side of (3.2.4) vanishes, the right-hand side of (3.2.5) evaluated at (t, X_t, Y_t, V_t) produces H_t^*, and $G(T, x, y, v) = U_{\exp}(v)$. □

Remark 3.2.2. The prospective optimal trading strategy (3.2.14) or equivalently

$$
H_t^* = \frac{1 + \kappa_\theta \kappa_x \bar{B}(t) - \bar{\sigma}(t)\bar{\alpha}(t)}{p\sigma_x^2} (\kappa_{x\theta}\hat{\Theta}_t - \kappa_x X_t) + \frac{\kappa_\theta \kappa_x \bar{A}(t) - \bar{\sigma}(t)\bar{B}(t)}{p\sigma_x^2} (\mu\kappa_{x\theta} - \kappa_\theta \kappa_x X_t)
\tag{3.2.15}
$$

is a linear combination of two "drifts" of X:

- ▷ First, $Y_t = \kappa_{x\theta}\hat{\Theta}_t - \kappa_x X_t$ is the "instantaneous" drift of X. If $\kappa_x \neq 0$, then this is the drift of X toward its estimated "instantaneous" estimated reversion level $\hat{\Theta}_t \kappa_{x\theta}/\kappa_x$, and

- ▷ Second, $\mu\kappa_{x\theta} - \kappa_x\kappa_\theta X_t$ is the "forward-looking" drift of X. If $\kappa_x\kappa_\theta \neq 0$, then this is the drift of X toward its long-term reversion level $\mu\kappa_{x\theta}/(\kappa_\theta\kappa_x)$.

The coefficients on these two drifts have the effect of reducing the bet sizes when some notion of overall "risk" (or aversion to risk) is high:

- ▷ The denominator $p\sigma_x^2$ is higher (and therefore scales the position sizes toward 0) when risk aversion p is higher, or when the X volatility σ_x is higher.

- ▷ The 1 in the first numerator multiplies $Y_t/(p\sigma_x^2)$, producing the purely *myopic* component that depends only on the instantaneous drift Y_t, instantaneous variance σ_x, and risk aversion p, in a ratio analogous to the Merton solution, but here with a filtered drift.

- ▷ As $t \to T$, the myopic direction dominates the long-term prospects: the coefficient on the instantaneous drift approaches the Merton weight $1/(p\sigma_x^2)$, while the coefficient on the long-term drift approaches 0.

- ▷ Both numerators have a positive contribution from the product $\kappa_\theta\kappa_x$ of the mean reversion rates of X and Θ. Intuitively, the stronger this is, the smaller the variability of the expected profits, and the bigger the bet sizes that the trader should make.

- ▷ Both numerators include $-\bar{\sigma}(t)$, the negative of the ratio of the signed $\mathcal{F}^X$-volatilities of Y and X. Intuitively, in the case that $\bar{\sigma} < 0$, the random fluctuations of Y and X have negative correlation, so when the price X moves against the trader, the drift Y moves in favor of the trader, mitigating the losses. Of course, prices X that move in favor of the trader are also mitigated, by drifts Y that move against the trader; this two-sided mitigation of risk causes risk-averse utility maximizers to increase position sizing, hence the negative sign on $\bar{\sigma}$.

- ▷ The coefficient on the forward-looking drift can be negative. For instance, in the case that $g = \kappa_x = \kappa_\theta = 0$, we have $\bar{\sigma}(t) = R$ and $H_t^* = \frac{(1-R\bar{\alpha}(t))}{p\sigma_x^2}Y_t - \frac{R\bar{B}(t)}{p\sigma_x^2}\mu\kappa_{x\theta}$, which places a negative coefficient on the forward-looking drift $\mu\kappa_{x\theta}$, in contrast to the positive coefficient on the instantaneous drift Y_t. The intuition is that increasing a positive forward-looking drift causes the optimal current position to become *less* positive, because it becomes more advantageous to *delay* up-sizing that bet until a later date, when the drift is expected to be more favorable.

3.2.4 Power Utility

In the case of power utility, the interdependencies of the ODEs for $A, B, C, D, \alpha, \beta$ cannot be resolved sequentially, but rather require the solution of a matrix Riccati equation. In the case where the initial uncertainty is at the equilibrium level $\gamma_0 = \bar{\gamma}$, the $\bar{\sigma}(t)$ and thus the Riccati coefficients are constant, leading to an explicit solution. In the case of general initial level of uncertainty γ_0, this matrix Riccati equation has time-dependent coefficients involving $\bar{\sigma}(t)$, which still admits an explicit solution provided that $\kappa_x\kappa_\theta = 0$.

Lemma 3.2.1. In the case $\gamma_0 = \bar{\gamma}$ and $\kappa_x\kappa_\theta \neq 0$ let $\mathbf{I} \in \mathbb{R}^{2\times 2}$ be the identity matrix and define

$$S := \begin{pmatrix} \frac{2\sigma_x^2}{q} & \frac{2\sigma_x^2\bar{\sigma}}{q} \\ \frac{2\sigma_x^2\bar{\sigma}}{q} & \frac{2\sigma_x^2\bar{\sigma}^2}{q} \end{pmatrix}, \qquad P := \begin{pmatrix} 0 & \frac{1}{q} \\ -\kappa_x\kappa_\theta & \frac{(1-q)\bar{\sigma}}{q} - \kappa_\theta - \kappa_x \end{pmatrix}, \qquad Q := \begin{pmatrix} 0 & 0 \\ 0 & \frac{q-1}{2q\sigma_x^2} \end{pmatrix}. \tag{3.2.16}$$

For $\varepsilon \geq 0$ define $\Phi_\varepsilon, \Psi_\varepsilon : [0,T] \to \mathbb{R}^{2\times 2}$ by $\Phi_\varepsilon(T) = \mathbf{I}$, $\Psi_\varepsilon(T) = -\varepsilon\mathbf{I}$, and, for all $t < T$, by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \Phi_\varepsilon \\ \Psi_\varepsilon \end{pmatrix} = \begin{pmatrix} P & S \\ Q & -P^* \end{pmatrix}\begin{pmatrix} \Phi_\varepsilon \\ \Psi_\varepsilon \end{pmatrix}. \tag{3.2.17}$$

Then $M_\varepsilon(t) := \Phi_\varepsilon^{-1}(t)\Psi_\varepsilon(t)$ is well-defined and negative semi-definite for all $t \in [0,T]$, all $\varepsilon \geq 0$. In particular this conclusion holds for M_0, equivalently M.

Proof. The ODE (3.2.17) satisfies a Lipschitz condition, which implies the existence, uniqueness, and continuous dependence on boundary conditions (and in particular, on ε) of $\Phi_\varepsilon(t)$ and $\Psi_\varepsilon(t)$.

To show that $\Phi_\varepsilon(t)$ and $\Psi_\varepsilon(t)$ are invertible on $[0,T)$, follow the proof of lemma 4.4.1 in W.T.Reid (1972). Suppose to the contrary that there exists some $t_0 \in [0,T)$ and some nonzero $\pi \in \mathbb{R}^2$ such that $\Phi_\varepsilon(t_0)\pi = \mathbf{0}$ or $\Psi_\varepsilon(t_0)\pi = \mathbf{0}$. Let $u(t) := \Phi_\varepsilon(t)\pi$

and $v(t) := \Psi_\varepsilon(t)\pi$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u^*(t)v(t)) &= (u^*(t)P^* + v^*(t)S)v(t) + u^*(t)(Qu(t) - P^*v(t)) \\ &= u^*(t)Qu(t) + v^*(t)Sv(t),\end{aligned} \tag{3.2.18}$$

hence

$$\int_{t_0}^{T} u^*(t)Qu(t) + v^*(t)Sv(t)\mathrm{d}t = u^*(T)v(T) - u^*(t_0)v(t_0) = -\varepsilon\pi^*\pi. \tag{3.2.19}$$

Because Q and S are positive semi-definite, we have $\varepsilon = 0$. So $\Phi_\varepsilon(t)$ and $\Psi_\varepsilon(t)$ are nonsingular for all $\varepsilon > 0$ and $t < T$. Let's now focus on $\varepsilon = 0$, and denote $\Phi_0(t) = (\phi_{i,j}(t))_{1\le i,j\le 2}, \Psi_0(t) = (\psi_{i,j}(t))_{1\le i,j\le 2}$. Because $t_0 < T$, we have

$$u^*(t)Qu(t) = v^*(t)Sv(t) = 0, \qquad t \in [t_0, T]. \tag{3.2.20}$$

In particular $0 = u^*(T)Qu(T) = \pi^*Q\pi$, so $\pi = c\mathbf{e}_1$ for some scalar $c \neq 0$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbb{R}^2$. Hence $u(t) = \Phi_0(t)\pi = c\phi_{11}(t)\mathbf{e}_1 + c\phi_{21}(t)\mathbf{e}_2$. Substituting into (3.2.20), we have $\phi_{21}(t) = 0$ for all $t \in [t_0, T]$. By (3.2.17), we have $\dot{\Phi} = P\Phi + S\Psi$. Comparing the $(2,1)$ entry of both sides, at all $t \in (t_0, T)$,

$$-\kappa_\theta\kappa_x\phi_{11}(t) + \frac{2\sigma_x^2\bar{\sigma}}{q}\psi_{11}(t) + \frac{2\sigma_x^2\bar{\sigma}^2}{q}\psi_{21}(t) = 0,$$

but taking $t \to T$ contradicts $\kappa_x\kappa_\theta \neq 0$, hence $\Phi_0(t)$ is nonsingular. Then, $M_\varepsilon(t) := \Phi_\varepsilon^{-1}(t)\Psi_\varepsilon(t)$ exists for all $\varepsilon \ge 0, t \le T$, and is nonsingular when $\varepsilon > 0$. For $\varepsilon > 0$, therefore, $M_\varepsilon(T)$ negative definite implies $M_\varepsilon(t)$ negative definite for all $t \in [0, T]$.

For each $t \in [0,T]$, by continuous dependence, $M_0(t) = \lim_{\varepsilon \to 0} M_\varepsilon(t)$ is negative semi-definite. $\square$

Proposition 3.2.4. In the case $\gamma_0 = \bar{\gamma}$ let

$$\begin{aligned} \mathcal{H}_s^{\pm}(t) &:= \sinh\left(\frac{z_{\pm}}{\sqrt{q}}(t-T)\right) \\ \mathcal{H}_c^{\pm}(t) &:= \cosh\left(\frac{z_{\pm}}{\sqrt{q}}(t-T)\right) \end{aligned} \tag{3.2.21}$$

where

$$\begin{aligned} a &:= (q-1)(\bar{\sigma} + \kappa_x + \kappa_\theta)^2 + \kappa_x^2 + \kappa_\theta^2 \\ b &:= \sqrt{a^2 - 4q\kappa_x^2\kappa_\theta^2} \\ z_{\pm} &:= \sqrt{\frac{a \pm b}{2}} \end{aligned} \tag{3.2.22}$$

(A) In case $\kappa_x \kappa_\theta \neq 0$, define matrix-valued functions $\Phi, \Psi : [0,T] \to \mathbb{R}^{2\times 2}$ by

$\Phi = (\phi_{ij})_{1\leq i,j\leq 2}$ and $\Psi = (\psi_{ij})_{1\leq i,j\leq 2}$ where[3]

$$\begin{aligned}
\phi_{11} &:= \sum_{+,-}\Bigg(\pm(\kappa_x+\kappa_\theta)\frac{z_\mp}{b}\mathcal{H}_s^\pm(t) \mp \frac{z_\mp^2+\kappa_x\kappa_\theta}{b}\mathcal{H}_c^\pm(t)\Bigg)\\
\phi_{12} &:= \sum_{+,-}\Bigg(\pm\Big(\frac{z_\pm}{b\sqrt{q}} + \frac{\sqrt{q}}{bz_\pm}\kappa_x\kappa_\theta\Big)\mathcal{H}_s^\pm(t) \mp \frac{\kappa_\theta+\kappa_x}{b}\mathcal{H}_c^\pm(t)\Bigg)\\
\phi_{21} &:= \sum_{+,-}\Bigg(\mp(z_\mp^2+q\kappa_x\kappa_\theta)\frac{z_\pm}{b\sqrt{q}}\mathcal{H}_s^\pm(t) \pm \kappa_x\kappa_\theta\frac{(q-1)\bar{\sigma}+q(\kappa_x+\kappa_\theta)}{b}\mathcal{H}_c^\pm(t)\Bigg)\\
\phi_{22} &:= \sum_{+,-}\Bigg(\mp((q-1)\bar{\sigma}+q(\kappa_x+\kappa_\theta))\frac{z_\pm}{b\sqrt{q}}\mathcal{H}_s^\pm(t) \pm \frac{z_\pm^2+\kappa_x\kappa_\theta}{b}\mathcal{H}_c^\pm(t)\Bigg)
\end{aligned} \tag{3.2.23}$$

and

$$\begin{aligned}
\psi_{11} &:= \sum_{+,-}\mp\frac{(q-1)\sqrt{q}\kappa_x^2\kappa_\theta^2}{2\sigma_x^2 bz_\pm}\mathcal{H}_s^\pm(t) & \psi_{12} &:= \sum_{+,-}\pm\frac{(q-1)\kappa_x\kappa_\theta}{2b\sigma_x^2}\mathcal{H}_c^\pm(t)\\
\psi_{21} &:= -\psi_{12} & \psi_{22} &:= \sum_{+,-}\pm\frac{q-1}{2\sigma_x^2}\frac{z_\pm}{b\sqrt{q}}\mathcal{H}_s^\pm(t)
\end{aligned} \tag{3.2.24}$$

Then Φ is invertible for all $t \in [0,T]$. Define A, B, C, α, β, and M by

$$\begin{pmatrix} A(t) & \frac{B(t)}{2}\\ \frac{B(t)}{2} & \alpha(t)\end{pmatrix} := M(t) := \Psi(t)\Phi^{-1}(t) \qquad \textit{and} \qquad \begin{aligned} C(t) &:= -\frac{2\mu\kappa_{x\theta}}{\kappa_\theta\kappa_x}A(t)\\ \beta(t) &:= -\frac{\mu\kappa_{x\theta}}{\kappa_\theta\kappa_x}B(t)\end{aligned} \tag{3.2.25}$$

3. If $b \notin \mathbb{R}$, it does not matter which branch to choose for the complex square root because (3.2.23,3.2.24) are invariant under the transformation $(b, z_\pm, \mathcal{H}_s^\pm, \mathcal{H}_c^\pm) \mapsto (-b, z_\mp, \mathcal{H}_s^\mp, \mathcal{H}_c^\mp)$, and likewise the branch of z does not matter due to the symmetry $(z_\pm, \mathcal{H}_s^\pm, \mathcal{H}_c^\pm) \mapsto (-z_\pm, -\mathcal{H}_s^\pm, , \mathcal{H}_c^\pm)$.

and

$$D(t) := \int_t^T \left(\frac{2q\sigma_x^2 A(s) + 2q\sigma_x^2 \bar{\sigma}(s) B(s) + 2q\sigma_x^2 \bar{\sigma}(s)^2 \alpha(s)}{2q} + \frac{2q\mu\kappa_{x\theta}\beta(s) + 2\sigma_x^2 \bar{\sigma}(s) C(s)\beta(s) + \sigma_x^2 C(s)^2 + \sigma_x^2 \bar{\sigma}(s)^2 \beta(s)^2}{2q} \right) \mathrm{d}s. \tag{3.2.26}$$

(B) In the case $\kappa_x \kappa_\theta = 0$, let $A = B = C = 0$ and let

$$\begin{aligned} \alpha(t) &= \frac{q-1}{2\sigma_x^2} \times \frac{\mathcal{H}_s^+(t)}{(\bar{\sigma} - q(\kappa_x + \kappa_\theta + \bar{\sigma}))\mathcal{H}_s^+(t) + \sqrt{aq}\mathcal{H}_c^+(t)} \\ \beta(t) &= \frac{(1-q)\sqrt{q}\kappa_{x\theta}\mu}{\sigma_x^2 \sqrt{a}} \frac{\mathcal{H}_c^+(t) - 1}{\sqrt{aq}\mathcal{H}_c^+(t) + (\bar{\sigma} - q(\kappa_x + \kappa_\theta + \bar{\sigma}))\mathcal{H}_s^+(t)} \end{aligned} \tag{3.2.27}$$

Then, in both (A,B) cases, G defined by (3.2.11) satisfies $G(T, x, y, v) = U_{\mathrm{pow}}(v)$ and $\mathcal{L}^H G \leq 0$ for arbitrary H. The trading strategy defined by

$$h_t^* = \frac{C(t) + \bar{\sigma}(t)\beta(t)}{q} + \frac{1 + \sigma_x^2 B(t) + 2\sigma_x^2 \bar{\sigma}(t)\alpha(t)}{q\sigma_x^2} Y_t + \frac{2A(t) + \bar{\sigma}(t)B(t)}{q} X_t \tag{3.2.28}$$

$$H_t^* = h_t^* V_t \tag{3.2.29}$$

attains equality $\mathcal{L}^{H^*} G = 0$.

Proof. In the case $\kappa_x \kappa_\theta \neq 0$, the definitions (3.2.23,3.2.24) of Φ and Ψ satisfy the $\varepsilon = 0$ case of (3.2.17), hence $\Phi = \Phi_0$ and $\Psi = \Psi_0$. By Lemma 3.2.1, therefore, Φ is invertible and M exists. We have

$$\begin{aligned} M'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(\Psi(t)\Phi^{-1}(t)) = \Psi'(t)\Phi^{-1}(t) - \Psi(t)\Phi^{-1}(t)\Phi'(t)\Phi^{-1}(t) \\ &= (Q\Phi(t) - P^*\Psi(t))\Phi^{-1}(t) - \Psi(t)\Phi^{-1}(t)(P\Phi(t) + S\Psi(t))\Phi^{-1}(t) \\ &= -M(t)SM(t) - P^\top M(t) - M(t)P + Q. \end{aligned} \tag{3.2.30}$$

Calculating the derivatives of (3.2.11) and simplifying using the A', B', and α' formulas contained in the M' expression (3.2.30), shows that, for this (G, H^*), the right-hand side of (3.2.4) vanishes, and the right-hand side of (3.2.5) evaluated at (t, X_t, Y_t, V_t) produces H_t^*. Moreover G satisfies the U_{pow} terminal condition. □

Remark 3.2.3. Thus the prospective optimal trading strategy in the power utility $\kappa_x\kappa_\theta \neq 0$ case is

$$\begin{aligned} H_t^* =& \frac{1 + \kappa_x\kappa_\theta\bar{B}(t) - \bar{\sigma}(t)\bar{\alpha}(t)}{q\sigma_x^2}(\kappa_{x\theta}\hat{\Theta}_t - \kappa_x X_t)V_t \\ &+ \frac{\kappa_x\kappa_\theta\bar{A}(t) - \bar{\sigma}(t)\bar{B}(t)}{q\sigma_x^2}(\mu\kappa_{x\theta} - \kappa_x\kappa_\theta X_t)V_t. \end{aligned} \tag{3.2.31}$$

where $\bar{A}(t) := -\frac{2\sigma_x^2}{\kappa_x^2\kappa_\theta^2}A(t)$, $\bar{B}(t) := \frac{\sigma_x^2}{\kappa_x\kappa_\theta}B(t)$, and $\bar{\alpha}(t) := -2\sigma_x^2\alpha(t)$.

As in the case of exponential utility, the optimal control is proportional to a linear combination of the "instantaneous" drift $\kappa_{x\theta}\hat{\Theta}_t - \kappa_x X_t$ and the "forward-looking" drift $\kappa_{x\theta}\mu - \kappa_x\kappa_\theta X_t$, and the coefficients on these drifts have similar interpretations to the exponential case.

Unlike the case of exponential utility, the power-utility maximizer takes position sizes that are proportional also to current wealth V_t.

The results (3.2.14) and (3.2.28) show that the only stochastic variables on which the optimal strategy depends are: X_t, V_t, and an exponentially weighted moving average (EMA) of X.

3.3 Verification

Let us rewrite the definitions (3.2.7) and (3.2.11)

$$G(t,x,y,v) := e^{f(t,x,y)}U(v), \qquad U \in \{U_{\mathrm{exp}}, U_{\mathrm{pow}}\} \tag{3.3.1}$$

$$G(t,x,y,v) := f(t,x,y) + U(v), \qquad U = U_{\mathrm{log}} \tag{3.3.2}$$

where

$$f(t,x,y) := A(t)x^2 + B(t)xy + \alpha(t)y^2 + C(t)x + \beta(t)y + D(t).$$

Here $A, B, C, D, \alpha, \beta$ are defined by (3.2.12,3.2.13) in case $U = U_{\mathrm{exp}}$, by (3.2.25) in case $U = U_{\mathrm{pow}}$, and by (3.2.8,3.2.9) in case $U = U_{\mathrm{log}}$. Any of $G, f, A, B, C, D, \alpha, \beta$ may be written with a subscript U if we wish to emphasize their dependence on U.

Lemma 3.3.1. In the cases $U = U_{\mathrm{exp}}$ and $U = U_{\mathrm{pow}}$ we have

$$\sup_{(t,\omega)\in[0,T]\times\Omega} f(t, X_t, Y_t) < \infty \tag{3.3.3}$$

Proof. If $\kappa_x\kappa_\theta \neq 0$, then for all $t \in [0,T)$ we have

$$\begin{aligned} f(t,x,y) &= A(t)(x-x_0)^2 + B(t)(x-x_0)y + \alpha(t)y^2 + D(t) - x_0^2A(t) \quad \text{where } x_0 := \frac{\mu\kappa_{x\theta}}{\kappa_x\kappa_\theta} \\ &\leq D(t) - x_0^2A(t) \end{aligned} \tag{3.3.4}$$

because $\begin{pmatrix} A(t) & B(t)/2 \\ B(t)/2 & \alpha(t) \end{pmatrix}$ is negative semi-definite, which is verified in the case $U = U_{\text{exp}}$ by

$$4A(t)\alpha(t) - B^2(t) = \frac{\kappa_x^2\kappa_\theta^2\left(e^{R\tau}-1\right)\left(e^{2Rt}-g\right)}{2\sigma_x^4R^4\left(e^{2RT}-g\right)}\left(R\tau + e^{R\tau}(R\tau-2)+2\right) > 0, \tag{3.3.5}$$

and $A(t) < 0$; and is verified in the case $U = U_{\text{pow}}$ by Lemma 3.2.1.

If $\kappa_x\kappa_\theta = 0$, then in both cases $U = \{U_{\text{exp}}, U_{\text{pow}}\}$, we have $\alpha(t) < 0$ hence

$$f(t,x,y) = \alpha(t)y^2 + \beta(t)y + D(t) \leq D(t) - \beta(t)^2/(4\alpha(t)). \tag{3.3.6}$$

From either (3.3.4) or (3.3.6), the result follows because D, A, and β^2/α are all bounded. □

3.3.1 Admissible Controls

Given (X_0, Y_0), let (X,Y) be the unique strong solution to the linear SDE (from (3.1.27))

$$\begin{aligned} \mathrm{d}X_t &= Y_t\mathrm{d}t + \sigma_x\mathrm{d}\nu_t \\ \mathrm{d}Y_t &= (\kappa_{x\theta}\mu - (\kappa_\theta+\kappa_x)Y_t - \kappa_\theta\kappa_xX_t)\mathrm{d}t + \sigma_x\bar{\sigma}(t)\mathrm{d}\nu_t \end{aligned} \tag{3.3.7}$$

Let $\mathbb{H}$ denote the set of progressively measurable h such that

$$\mathbb{E}\int_0^T h_t^2 \mathrm{d}t < \infty. \tag{3.3.8}$$

Moreover Y is Gaussian hence $\int_0^T Y_s^2 \mathrm{d}s < \infty$ which allows us to define for $h \in \mathbb{H}$

$$X_t^h := \int_0^t Y_s h_s \mathrm{d}s + \int_0^t \sigma_x h_s \mathrm{d}\nu_s. \tag{3.3.9}$$

Given initial wealth $V_0 > 0$ in case $U \in \{U_{\log}, U_{\text{pow}}\}$, or $V_0 \in \mathbb{R}$ in case $U = U_{\exp}$, define

$$\begin{aligned} V_t^H &:= V_0 + X_t^h && \text{if } U = U_{\exp} \\ V_t^H &:= V_0\ \mathcal{E}(X^h)_t && \text{if } U \in \{U_{\log}, U_{\text{pow}}\} \end{aligned} \tag{3.3.10}$$

where $\mathcal{E}$ denotes the stochastic (Doleans-Dade) exponential. Thus we have wealth dynamics

$$\mathrm{d}V_t^H = H_t \mathrm{d}X_t, \qquad \text{where } H_t := \begin{cases} h_t & \text{if } U = U_{\exp} \\ h_t V_t^H & \text{if } U \in \{U_{\text{pow}}, U_{\log}\} \end{cases} \tag{3.3.11}$$

In cases $U = U_{\exp}$ or $U = U_{\text{pow}}$, let $r := p$ or $r := q - 1$ respectively, and let

$$\begin{aligned} \xi_t^h &:= f_x(t, X_t, Y_t)\sigma_x + f_y(t, X_t, Y_t)\sigma_x \bar{\sigma}(t) - r\sigma_x h_t \\ \Xi_t^h &:= \exp\Big(\int_0^T \xi_s^h \mathrm{d}\nu_s - \frac{1}{2}\int_0^T (\xi_s^h)^2 \mathrm{d}t\Big) \end{aligned} \tag{3.3.12}$$

Definition 3.3.1 (Admissible trading strategies)**.** Let $\mathcal{A}_U$ (abbreviated as $\mathcal{A}$) denote the set of admissible controls: all $h \in \mathbb{H}$ such that H satisfies (i) and (ii):

(i) In the cases $U \in \{U_{\text{pow}}, U_{\log}\}$, the wealth process is a.s. positive: $V_t^H > 0$.

(ii) In the cases $U \in \{U_{\exp}, U_{\text{pow}}\}$, the strategy h satisfies

$$\mathbb{E}(\Xi_T^h) = 1. \tag{3.3.13}$$

This condition is implied by a sufficient condition on the wealth process alone:

Proposition 3.3.1. In the cases $U \in \{U_{\exp}, U_{\text{pow}}\}$, for $h \in \mathbb{H}$ satisfying (i), a sufficient condition for (3.3.13) is

$$\mathbb{E} \sup_{t\in[0,T]} |G(t, X_t, Y_t, V_t^H)| < \infty \tag{3.3.14}$$

In turn, a sufficient condition for (3.3.14) is the following sup-integrability condition, involving no other stochastic variables aside from wealth:

$$\mathbb{E} \sup_{t\in[0,T]} |U(V_t^H)| < \infty, \tag{3.3.15}$$

For instance, a sufficient condition for (3.3.15) is that wealth be bounded below: $V_t^H > v^H$ for some nonrandom $v^H \in \mathbb{R}$ in the exponential case or nonrandom $v^H > 0$ in the power case.

Proof. By Lemma 3.3.1, condition (3.3.15) implies (3.3.14).

To obtain (3.3.13) let $G_t := G(t, X_t, Y_t, V_t^h)$. Then

$$\mathrm{d}G(t, X_t, Y_t, V_t^h) = (\mathcal{L}^H G)_t \mathrm{d}t + \xi_t^h G_t \mathrm{d}\nu_t \tag{3.3.16}$$

and

$$
\begin{aligned}
G_t &= G_0\ \mathcal{E}\left(\int_0^{\cdot} \frac{(\mathcal{L}^h G)_s}{G_s} ds + \int_0^{\cdot} \xi^h d\nu_s\right)_t \\
&= G_0 \exp\left(\int_0^t \frac{(\mathcal{L}^h G)_s}{G_s} \mathrm{d}s + \int_0^t \xi^h \mathrm{d}\nu_s - \frac{1}{2}\int_0^t (\xi^h_s)^2 \mathrm{d}s\right) \le G_0 \Xi^h_t
\end{aligned}
\tag{3.3.17}
$$

because $G < 0$, and Propositions 3.2.1 and 3.2.3/3.2.4 imply $\mathcal{L}^H G \le 0$. So if (3.3.14) holds, then

$$
\mathbb{E} \sup_{t\in[0,T]} \Xi^h_t \le \mathbb{E} \sup_{t\in[0,T]} \frac{G_t}{G_0} < \infty
$$

hence Ξ^h is a true martingale and (3.3.13) follows. □

This section verifies that G is the value function, and that the strategy h^* is optimal, where h^* is defined by (3.2.14) in case $U = U_{\text{exp}}$, by (3.2.28) in case $U = U_{\text{pow}}$, and by (3.2.10) in case $U = U_{\log}$.

Proposition 3.3.2 (Admissiblity)**.** We have $h^* \in \mathcal{A}$. Indeed it satisfies the stronger condition (3.3.14).

Proof. We have h^* Gaussian, therefore in $\mathbb{H}$.

In the log and power cases, $V_0^{H^*} > 0$, and V^{H^*} is a stochastic exponential, hence $V_t^{H^*} > 0$.

In the power and exponential cases, to show H^* satisfies (3.3.14), let $G^*_t := G(t, X_t, Y_t, V_t^{H^*})$. Then

$$
\mathrm{d}G^*_t = \big(\lambda_0(t) + \lambda_1(t)X_t + \lambda_2(t)Y_t\big)G^*_t \mathrm{d}\nu_t \tag{3.3.18}
$$

where the $\lambda_i(t)$ are deterministic and bounded on $[0,T]$ for $i = 0, 1, 2$. (Indeed, in the exponential case, they are constant: $\lambda_0 = \lambda_1 = 0$ and $\lambda_2 = -1/\sigma_x$.) In both power and exponential cases, then, G^* is a stochastic exponential. Lemma 3.3.2 shows that G^* satisfies the Beneš condition; therefore F. Klebaner and R. Liptser (2011) Theorem 2.2 concludes that G^* is a martingale. Applying Doob's $L \log L$ maximal inequality to $-G^* = |G^*|$, we have

$$\mathbb{E} \sup_{t\in[0,T]} |G_t^*| \leq \frac{e}{e-1}(1 + \mathbb{E}(|G_T^*| \log^+ |G_T^*|)),$$

so it suffices to show that $\mathbb{E}(|G_T^*| \log |G_T^*|) < \infty$.

The proof of Theorem 2.2 in F. Klebaner and R. Liptser (2011) defines a sequence of stopping times $\tau_n \uparrow \infty$ and establishes that $\sup_n \mathbb{E}(|G_{T\wedge\tau_n}^*| \log |G_{T\wedge\tau_n}^*|) < \infty$. Therefore

$$\begin{aligned} \mathbb{E}(|G_T^*| \log |G_T^*|) &= \mathbb{E}(\liminf_{n\to\infty} |G_{T\wedge\tau_n}^*| \log |G_{T\wedge\tau_n}^*|) \\ &\leq \liminf_{n\to\infty} \mathbb{E}(|G_{T\wedge\tau_n}^*| \log |G_{T\wedge\tau_n}^*|) < \infty \end{aligned} \tag{3.3.19}$$

as claimed. □

Lemma 3.3.2. Given $T > 0$, there exists some constant C such that for all $t \in [0,T]$

$$(1 + |X_t| + |Y_t|)^2 \leq C\Big(1 + \sup_{s\leq t} \nu_s^2\Big). \tag{3.3.20}$$

Proof. We have

$$\begin{aligned}\hat{\Theta}_t &= e^{-\kappa_\theta t}\hat{\Theta}_0 + \sigma_{\hat{\theta}}(t)\nu_t + \int_0^t (\mu - (\kappa_\theta \sigma_{\hat{\theta}}(t) + \sigma'_{\hat{\theta}}(t))\nu_s)e^{\kappa_\theta(s-t)}\mathrm{d}s \\ X_t &= e^{-\kappa_x t}X_0 + \sigma_x \nu_t + \int_0^t (\kappa_\theta \hat{\Theta}_s - \kappa_x \sigma_x \nu_s)e^{\kappa_x(s-t)}\mathrm{d}s\end{aligned} \tag{3.3.21}$$

Therefore there exists C such that

$$|\hat{\Theta}_t| + |X_t| \leq C\big(1 + \sup_{s\leq t}|\nu_s|\big). \tag{3.3.22}$$

which implies (3.3.20). □

3.3.2 Verification Theorem

To verify optimality of h^*, some key steps, that h^* satisfies (3.3.14) and hence the true martingale condition (3.3.13), were already verified in Propositions 3.3.1 and 3.3.2. The conclusion then follows from:

Proposition 3.3.3 (Verification Theorem)**.** In all cases $U \in \{U_{\text{exp}}, U_{\text{pow}}, U_{\text{log}}\}$, we have

$$\sup_{h\in\mathcal{A}} \mathbb{E}U(V_T^H) = \mathbb{E}U(V_T^{H^*}) = G(0, X_0, Y_0, V_0) \tag{3.3.23}$$

where H is driven by h via (3.3.11).

Proof. For any $h \in \mathcal{A}$ we have $\mathcal{L}^H G \leq 0$ by Propositions 3.2.2/3.2.3/3.2.4. Let $G_t^H := G(t, X_t, Y_t, V_t^H)$.

For $0 \le r \le t \le T$, by Ito's rule,

$$G_t^H = G_r^H + \int_r^t \mathcal{L}^H G(s, X_s, Y_s, V_s^H)\mathrm{d}s + M_t - M_r \le G_r^H + M_t - M_r, \quad (3.3.24)$$

where in the U_{exp} and U_{pow} cases, $M_t = \Xi_t^h$ is a true martingale by (3.3.13); and in the $U_{\log}$ case

$$M_t = \int_0^t \Big(\sigma_x(2A(t)X_s+B(s)Y_s+C(s))+\sigma_x\bar{\sigma}(t)(B(t)X_s+2\alpha(s)Y_s+\beta(s))+\sigma_x h_t\Big)\mathrm{d}\nu_s \quad (3.3.25)$$

from (3.3.2,3.3.7,3.3.11), which is a true martingale because X and Y are Gaussian, and h satisfies (3.3.8).

So in all cases of U we have

$$G(0, X_0, Y_0, V_0) \ge \mathbb{E}U(V_T^H). \quad (3.3.26)$$

In particular if $h = h^*$, then by combining $h \in \mathcal{A}$ by Proposition 3.3.2 and $\mathcal{L}^{H^*}G = 0$ by Propositions 3.2.2/3.2.3/3.2.4, the inequalities (3.3.24,3.3.26) then hold with equality, proving (3.3.23). □

Now, let's move back and suppose the trader has access to complete market information.

3.4 Full Information Case

The spread price X and mean reversion factor Θ satisfy the same SDEs (3.1.1, 3.1.2).

$$\begin{aligned} \mathrm{d}X_t &= (\kappa_{x\theta}\Theta_t - \kappa_x X_t)\mathrm{d}t + \sigma_x \mathrm{d}W_t, \\ \mathrm{d}\Theta_t &= (\mu - \kappa_\theta \Theta_t)\mathrm{d}t + \sigma_\theta \mathrm{d}Z_t. \end{aligned} \tag{3.4.1}$$

We are considering the *full-information* filtration $\mathcal{F}$, where both X and Θ are $\mathcal{F}$-adapted, and Θ_0 is a constant. The control process H now is also $\mathcal{F}$-adapted, and the wealth process V satisfies

$$\mathrm{d}V_t = H_t \mathrm{d}X_t = H_t Y_t \mathrm{d}t + \sigma_x H_t \mathrm{d}W_t, \tag{3.4.2}$$

where $Y_t = \kappa_{x\theta}\Theta_t - \kappa_x X_t$, and then satisfies

$$\mathrm{d}Y_t = (\kappa_{x\theta}\mu - (\kappa_x + \kappa_\theta)Y_t - \kappa_x \kappa_\theta X_t)\mathrm{d}t + \kappa_{x\theta}\sigma_\theta \mathrm{d}Z_t - \kappa_x \sigma_x \mathrm{d}W_t. \tag{3.4.3}$$

3.4.1 Log Utility Revisited

Let's begin our comparison between the trader accessing complete information and those only being exposed to partial information in the logarithmic utility case. Here we would apply a direct approach without 'ansatz'. And let's follow (3.3.11), where $H_t = h_t V_t$, then we can rewrite the wealth process from SDE(3.4.2) to integral as

$$V_T = V_0 \exp\left(\int_0^T \left(h_s Y_s - \frac{h_s^2 \sigma_x^2}{2} \right) \mathrm{d}s + \int_0^T \sigma_x h_s \mathrm{d}W_s \right). \tag{3.4.4}$$

Then take the expected log utility, we have

$$
\begin{aligned}
\mathbb{E}\left(U_{\log}(V_T)\right) &= U_{\log}(V_0) + \mathbb{E}\left(\int_0^T \left(h_s Y_s - \frac{h_s^2\sigma_x^2}{2}\right) \mathrm{d}s + \int_0^T \sigma_x h_s \mathrm{d}W_s\right) \\
&= U_{\log}(V_0) + \mathbb{E}\left(\int_0^T \left(h_s Y_s - \frac{h_s^2\sigma_x^2}{2}\right) \mathrm{d}s\right) \\
&\leq U_{\log}(V_0) + \mathbb{E}\left(\int_0^T \frac{Y_s^2}{2\sigma_x^2} \mathrm{d}s\right) \\
&= U_{\log}(V_0) + \int_0^T \frac{\mathbb{E}(Y_s^2)}{2\sigma_x^2} \mathrm{d}s,
\end{aligned}
\tag{3.4.5}
$$

where the equality holds when $h_t = \frac{Y_t}{\sigma_x}$.

Proposition 3.4.1. The second moment of the Gaussian process Y_t is given by

$$
\mathbb{E}(Y_t^2) = \mu^2 F_1(t) + \mu F_2(t, X_0, Y_0) + F_3(t, X_0, Y_0), \tag{3.4.6}
$$

where

$$
\begin{aligned}
F_1(t) = -\kappa_{x\theta}^2 \Bigg(& \frac{4e^{-\kappa_\theta t}}{\kappa_\theta(2\kappa_x - \kappa_\theta)} - \frac{e^{-2\kappa_\theta t}}{(\kappa_x - \kappa_\theta)^2} \\
& + \frac{e^{-2\kappa_x t}(4\kappa_x^2 - 6\kappa_x\kappa_\theta + \kappa_\theta^2)}{\kappa_\theta(2\kappa_x - \kappa_\theta)(\kappa_x - \kappa_\theta)^2} + \frac{2e^{-(\kappa_x+\kappa_\theta)t}(3\kappa_\theta - 2\kappa_x)}{\kappa_\theta(\kappa_x - \kappa_\theta)^2} \Bigg) \\
F_2(t,x,y) = {} & \frac{2\kappa_{x\theta}(e^{-\kappa_x t} - e^{-\kappa_\theta t})}{(\kappa_x - \kappa_\theta)^2} \left((e^{-\kappa_\theta t}\kappa_\theta - e^{-\kappa_x t}\kappa_x)y + \kappa_x\kappa_\theta(e^{-\kappa_x t} - e^{-\kappa_\theta t})x \right) \\
F_3(t,x,y) = {} & \frac{1}{(\kappa_x - \kappa_\theta)^2} \left(\kappa_x\kappa_\theta(e^{-\kappa_\theta t} - e^{-\kappa_x t})x + (e^{-\kappa_\theta t}\kappa_\theta - e^{-\kappa_x t}\kappa_x)y \right)^2 \\
& + \frac{2e^{-(\kappa_x+\kappa_\theta)t}\kappa_x\kappa_\theta\kappa_{x\theta}\sigma_\theta(2\kappa_x\kappa_\theta\rho + (\kappa_x + \kappa_\theta)\rho\sigma_x + \kappa_{x\theta}\sigma_\theta)}{(\kappa_x - \kappa_\theta)^2(\kappa_x + \kappa_\theta)} \\
& - \frac{e^{-2\kappa_\theta t}\kappa_{x\theta}\kappa_\theta\sigma_\theta(2\kappa_x\rho(\kappa_\theta + \sigma_x) + \kappa_{x\theta}\sigma_\theta)}{2(\kappa_x - \kappa_\theta)^2} + \frac{2\kappa_x\kappa_{x\theta}\kappa_\theta\rho\sigma_\theta + \kappa_{x\theta}^2\sigma_\theta^2}{2(\kappa_x + \kappa_\theta)} \\
& + \frac{\kappa_x\sigma_x^2}{2} - \frac{e^{-2\kappa_x t}\kappa_x((\kappa_x - \kappa_\theta)^2\sigma_x^2 + 2\kappa_{x\theta}\kappa_\theta\rho(\kappa_x + \sigma_x)\sigma_\theta + \kappa_{x\theta}^2\sigma_\theta^2)}{2(\kappa_x - \kappa_\theta)^2}.
\end{aligned}
$$

Proof. Denote $\mathcal{E}_* = \mathbb{E}(*)$, where $*$ stands for X, Y, Θ or X^2, Y^2, etc. Let's start with calculating $\mathcal{E}_Y, \mathcal{E}_X$ and $\mathcal{E}_\Theta$ for all $t \in [0, T]$. Since Θ_t is an OU process, we have

$$
\mathcal{E}_\Theta(t) = e^{-\kappa_\theta t}\Theta_0 + \frac{\mu}{\kappa_\theta}(1 - e^{-\kappa_\theta t}), \tag{3.4.7}
$$

and

$$
\mathcal{E}_X(t) = X_0 + \int_0^t (\kappa_{x\theta}\mathcal{E}_\Theta(s) - \kappa_x\mathcal{E}_X(s))\,\mathrm{d}s \tag{3.4.8}
$$

by integrating the SDE for X. Then

$$
\mathcal{E}_X(t) = e^{-\kappa_x t}X_0 + \left(\frac{\kappa_{x\theta}\Theta_0}{\kappa_x - \kappa_\theta} + \frac{\mu\kappa_{x\theta}}{\kappa_\theta(\kappa_x - \kappa_\theta)} \right) \left(e^{-\kappa_\theta t} - e^{-\kappa_x t} \right) + \frac{\mu\kappa_{x\theta}}{\kappa_\theta\kappa_x} \left(1 - e^{-\kappa_x t} \right), \tag{3.4.9}
$$

where $\kappa_\theta = \kappa_x$, we need to take the limit by L'hopital's Rule. Therefore,

$$\begin{aligned}
\mathcal{E}_Y(t) =&\kappa_{x\theta}\mathcal{E}_\Theta(t) - \kappa_x\mathcal{E}_X(t) \\
=&\kappa_{x\theta}e^{-\kappa_\theta t}\Theta_0 + \frac{\mu\kappa_{x\theta}}{\kappa_\theta}(1-e^{-\kappa_\theta t}) - \kappa_x e^{-\kappa_x t}X_0 \\
&-\left(\frac{\kappa_{x\theta}\kappa_x\Theta_0}{\kappa_x-\kappa_\theta} + \frac{\mu\kappa_{x\theta}\kappa_x}{\kappa_\theta(\kappa_x-\kappa_\theta)}\right)\left(e^{-\kappa_\theta t} - e^{-\kappa_x t}\right) - \frac{\mu\kappa_{x\theta}}{\kappa_\theta}\left(1-e^{-\kappa_x t}\right).
\end{aligned} \tag{3.4.10}$$

Applying Itô formula, we have the following SDEs

$$\begin{aligned}
\mathrm{d}Y_t^2 =&2Y_t\mathrm{d}Y_t + \mathrm{d}[Y]_t \\
=&\left(2\kappa_{x\theta}\mu Y_t - 2(\kappa_x+\kappa_\theta)Y_t^2 - 2\kappa_x\kappa_\theta X_tY_t\right)\mathrm{d}t + 2\kappa_{x\theta}\sigma_\theta Y_t dZ_t - 2\kappa_x\sigma_x Y_t\mathrm{d}W_t \\
&+\left(\kappa_{x\theta}^2\sigma_\theta^2 + 2\kappa_{x\theta}\sigma_\theta\kappa_x\sigma_x\rho + \kappa_x^2\sigma_x^2\right)\mathrm{d}t, \\
\mathrm{d}X_tY_t =&X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t + \mathrm{d}[X,Y]_t \\
=&(\kappa_{x\theta}\mu - (\kappa_x+\kappa_\theta)Y_t - \kappa_x\kappa_\theta X_t)X_t dt + \kappa_{x\theta}\sigma_\theta X_t dZ_t - \kappa_x\sigma_x X_t dW_t \\
&+ Y_t^2 dt + \sigma_x Y_t\mathrm{d}W_t + (\rho\kappa_{x\theta}\sigma_x\sigma_\theta - \kappa_x\sigma_x^2)\mathrm{d}t, \\
\mathrm{d}X_t^2 =&2X_t dX_t + \mathrm{d}[X]_t = 2X_tY_t dt + \sigma_x X_t\mathrm{d}W_t + \sigma_x^2\mathrm{d}t.
\end{aligned} \tag{3.4.11}$$

Then $\mathcal{E}_{X^2}(t), \mathcal{E}_{XY}(t), \mathcal{E}_{Y^2}(t)$ would satisfy an ODEs system as below:

$$\begin{aligned}
\mathcal{E}'_{X^2}(t) =&2\mathcal{E}_{XY}(t) + \sigma_x^2, \\
\mathcal{E}'_{XY}(t) =&\mu\kappa_{x\theta}\mathcal{E}_X(t) - (\kappa_x+\kappa_\theta)\mathcal{E}_{XY}(t) - \kappa_x\kappa_\theta\mathcal{E}_{X^2}(t) + \mathcal{E}_{Y^2}(t) \\
&+ \rho\kappa_{x\theta}\sigma_x\sigma_\theta - \kappa_x\sigma_x^2, \\
\mathcal{E}'_{Y^2}(t) =&2\mu\kappa_{x\theta}\mathcal{E}_Y(t) - 2(\kappa_x+\kappa_\theta)\mathcal{E}_{Y^2}(t) - 2\kappa_x\kappa_\theta\mathcal{E}_{XY}(t) \\
&+ \kappa_{x\theta}^2\sigma_\theta^2 + 2\kappa_{x\theta}\sigma_\theta\kappa_x\sigma_x\rho + \kappa_x^2\sigma_x^2.
\end{aligned} \tag{3.4.12}$$

Solve the ODEs (3.4.12), we can verify Eq(3.4.6). □

Remark 3.4.1. 1. The optimal holding position for investor who has access to complete information only depends on the spot processes of Θ_t and X_t, while for investor who only has partial information should depend on the whole price process, i.e. on $X_{0\le s\le t}$.

2. Taking the integration of Eq(3.4.6), we can derive again that the value function satisfying Eq(3.4.4) should be a quadratic polynomial w.r.t the initial value X_0 and Y_0, as it's the case for $\mathbb{E}(Y_t^2)$.

Next, let's consider a simplified model where Θ is simply a Brownian motion, i.e. the parameters $\mu = \kappa_\theta = 0$ and $\sigma_\theta = 1$, which is equivalent the price dynamics with mean reverting drift as discussed in Lakner (1995), etc.

3.4.2 A Special Case: $\Theta_t = Z_t$

Let's consider the case where investor has access to complete market information with exponential and power utility preference. In this case, the dynamic system is reduced to be

$$\begin{aligned} \mathrm{d}X_t &= Y_t \mathrm{d}t + \sigma_x \mathrm{d}W_t, \\ \mathrm{d}Y_t &= -\kappa_x Y_t \mathrm{d}t + \kappa_{x\theta} \mathrm{d}Z_t - \kappa_x \sigma_x \mathrm{d}W_t. \end{aligned} \tag{3.4.13}$$

Then the infinitesimal generator for the Markov system (X, Y, V) is

$$\begin{aligned}\mathcal{L}^{full,H} =& \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} - \kappa_x y\frac{\partial}{\partial y} + yH\frac{\partial}{\partial v} \\ &+ \frac{\sigma_x^2}{2}\frac{\partial^2}{\partial x^2} + (\rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2)\frac{\partial^2}{\partial xy} + \frac{\kappa_{x\theta}^2 - 2\rho\kappa_{x\theta}\kappa_x\sigma_x + \kappa_x^2\sigma_x^2}{2}\frac{\partial^2}{\partial y^2} \\ &+ \frac{H^2\sigma_x^2}{2}\frac{\partial^2}{\partial v^2} + \sigma_x^2 H\frac{\partial^2}{\partial vx} + (\rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2)H\frac{\partial^2}{\partial vy}\end{aligned} \tag{3.4.14}$$

The corresponding HJB equation is

$$\sup \mathcal{L}^{full,H} G = 0, \quad G(T, x, y, v) = U(v). \tag{3.4.15}$$

Then a candidate optimal control would satisfy:

$$H_t^{full,*} = \frac{yG_v + \sigma_x^2 G_{vx} + (\rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2)G_{vy}}{-\sigma_x^2 G_{vv}}. \tag{3.4.16}$$

Substituting back in to the HJB Eq (3.4.15), we have

$$\begin{aligned}&G_t + yG_x - \kappa_x yG_y + \frac{\sigma_x^2}{2}G_{xx} + (\rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2)G_{xy} \\ &\quad + \frac{\kappa_{x\theta}^2 - 2\rho\kappa_{x\theta}\kappa_x\sigma_x + \kappa_x^2\sigma_x^2}{2}G_{yy} - \frac{(yG_v + \sigma_x^2 G_{vx} + (\rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2)G_{vy})^2}{2\sigma_x^2 G_{vv}} = 0.\end{aligned} \tag{3.4.17}$$

One observation is that the coefficients of the above Eq(3.4.17) does not depend on x, let's assume the value function G does not depend on x first. Then Eq(3.4.17) can be simplified as

$$G_t - \kappa_x y G_y + \frac{\kappa_{x\theta}^2 - 2\rho\kappa_{x\theta}\kappa_x\sigma_x + \kappa_x^2\sigma_x^2}{2} G_{yy} - \frac{(yG_v + (\rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2)G_{vy})^2}{2\sigma_x^2 G_{vv}} = 0. \tag{3.4.18}$$

Denote $\tilde{\lambda}^2 = \kappa_{x\theta}^2 - 2\rho\kappa_{x\theta}\kappa_x\sigma_x + \kappa_x^2\sigma_x^2$, and $\bar{\lambda} = \rho\kappa_{x\theta}\sigma_x - \kappa_x\sigma_x^2$, we can rewrite Eq(3.4.18) as

$$G_t - \kappa_x y G_y + \frac{\tilde{\lambda}^2}{2} G_{yy} - \frac{(yG_v + \bar{\lambda}G_{vy})^2}{2\sigma_x^2 G_{vv}} = 0. \tag{3.4.19}$$

With ansatz $G(t, y, v) = e^{f(t,y)}U(v)$, where $f(t, y) = \alpha(t)y^2 + \beta(t)y + D(t), U = U_{\text{exp}}$ or U_{pow}, then Eq(3.4.19) is equivalent to solving

$$\begin{aligned} &(\alpha'(t)y^2 + \beta'(t)y + D'(t)) - \kappa_x y(2\alpha(t)y + \beta(t)) \\ &+\frac{\tilde{\lambda}^2}{2}(2\alpha(t) + (2\alpha(t)y + \beta(t))^2) - \frac{(y + \bar{\lambda}(2\alpha(t)y + \beta(t)))^2}{2\sigma_x^2}\frac{(U')^2}{UU''} = 0, \end{aligned} \tag{3.4.20}$$

where $\frac{(U')^2}{UU''} = 1$, when $U = U_{\text{exp}}$, $\frac{(U')^2}{UU''} = \frac{q-1}{q}$, when $U = U_{\text{pow}}$.

Then $\alpha(t)$ and $\beta(t)$ can be explicitly solved by collecting polynomial coefficients in the exponential and power utility cases separately. In this way, we can figure out that $\beta(t) = 0$ in both cases. Then it suffices to solve the ODE for α.

$$\alpha(t) = \begin{cases} \dfrac{1}{2\kappa_{x\theta}\sigma_x \left(\coth\left(\frac{(t-T)\kappa_{x\theta}}{\sigma_x}\right) - \rho\right)}, & U = U_{\text{exp}}, \\ \dfrac{1-q}{2\sigma_x \left(-qw\coth\left(\frac{(t-T)w}{\sigma_x}\right) + (q-1)\kappa_{x\theta}\rho + \kappa_x\sigma_x\right)}, & U = U_{\text{pow}}, \end{cases} \tag{3.4.21}$$

where $\omega = \sqrt{\frac{(q-1)\kappa_{x\theta}^2 + \kappa_x^2\sigma_x^2}{q}}$.

Therefore, the optimal strategy in both cases are given below:

$$H_t^{full,*} = \frac{Y_t(1+2\bar{\lambda}\alpha(t))}{-\sigma_x^2}\frac{U'}{U''} = \begin{cases} \dfrac{1+2\bar{\lambda}\alpha(t)}{p\sigma_x^2}Y_t, & U = U_{\text{exp}}, \\ \dfrac{1+2\bar{\lambda}\alpha(t)}{q\sigma_x^2}Y_tV_t, & U = U_{\text{pow}}. \end{cases} \tag{3.4.22}$$

From Eq (3.4.22), we conclude in this special case where Θ_t is a standard Brownian motion, the optimal position in both $U = U_{\text{exp}}$, and $U = U_{\text{pow}}$ cases, is myopic, i.e. only depends on the spot drift(and current wealth, respectively). Though the coefficient of $H^{full,*}$ here, unlike in the log case is time dependent, we can find it positive as well.

Proposition 3.4.2. The coefficient function for $H_t^{full,*}$, i.e.

$$\begin{cases} \dfrac{1+2\bar{\lambda}\alpha(t)}{p\sigma_x^2}, & U = U_{\text{exp}}, \\ \dfrac{1+2\bar{\lambda}\alpha(t)}{q\sigma_x^2}, & U = U_{\text{pow}}, \end{cases} \tag{3.4.23}$$

is positive.

Proof. Since $\coth(x)$ is monotonic and $\lim_{x\to 0}\coth(x) = \infty$, it suffices to show that $\frac{1+2\bar{\lambda}\alpha(0)}{p\sigma_x^2} > 0$ in exponential case, and $\frac{1+2\bar{\lambda}\alpha(0)}{q\sigma_x^2} > 0$ in power case. Then it's equivalent to show the numerator $1 + 2\bar{\lambda}\alpha(0) > 0$ in both cases, as the denominator

$p\sigma_x^2$ and $q\sigma_x^2$ are positive. In exponential case,

$$\begin{aligned} 1+2\bar{\lambda}\alpha(0) &= 1+\frac{\kappa_{x\theta}\rho-\kappa_x\sigma_x}{\kappa_{x\theta}\left(\coth\left(-\frac{\kappa_{x\theta}}{\sigma_x}T\right)-\rho\right)} \\ &= \frac{\kappa_{x\theta}\coth\left(\frac{\kappa_{x\theta}}{\sigma_x}T\right)+\kappa_x\sigma_x}{\kappa_{x\theta}\left(\coth\left(\frac{\kappa_{x\theta}}{\sigma_x}T\right)+\rho\right)} > 0, \end{aligned} \tag{3.4.24}$$

since $\coth\left(\frac{\kappa_{x\theta}}{\sigma_x}T\right) > 1$.

In power case,

$$\begin{aligned} 1+2\bar{\lambda}\alpha(0) &= 1+\frac{(1-q)(\kappa_{x\theta}\rho-\kappa_x\sigma_x)}{qw\coth\left(\frac{w}{\sigma_x}T\right)+(q-1)\kappa_{x\theta}\rho+\kappa_x\sigma_x} \\ &= \frac{q\kappa_x\sigma_x+qw\coth\left(\frac{w}{\sigma_x}T\right)}{qw\coth\left(\frac{w}{\sigma_x}T\right)+(q-1)\kappa_{x\theta}\rho+\kappa_x\sigma_x} > 0, \end{aligned} \tag{3.4.25}$$

since $qw\coth\left(\frac{w}{\sigma_x}T\right)+(q-1)\kappa_{x\theta}\rho \geq qw+(q-1)\kappa_{x\theta}\rho \geq (q-1)\kappa_{x\theta}(1+\rho) \geq 0$. □

For cases where investors are only exposed to partial information. In this simplified case, plug $\mu=\kappa_\theta=0, \sigma_\theta=1$ into (3.2.15, 3.2.31), we have the optimal strategy in exponential and power cases as below:

$$H_t^* = \frac{Y_t(1+2\sigma_x^2\bar{\sigma}(t)\alpha(t))}{-\sigma_x^2}\frac{U'}{U''} = \begin{cases} \dfrac{1+2\sigma_x^2\bar{\sigma}(t)\alpha(t)}{p\sigma_x^2}Y_t, & U=U_{\text{exp}}, \\ \dfrac{1+2\sigma_x^2\bar{\sigma}(t)\alpha(t)}{q\sigma_x^2}Y_tV_t, & U=U_{\text{pow}}, \end{cases} \tag{3.4.26}$$

Then similar discussion as Proposition 3.4.2 can be carried out for the partial information case.

CHAPTER 4

OPTIMAL DYNAMIC FUTURES PORTFOLIOS WITH CONSTRAINTS

In this chapter[1], we study the problem of dynamically trading multiple futures contracts subject to portfolio constraints under a stochastic basis model. We present a utility maximization approach to generate optimal trading strategies for futures portfolio under a stochastic basis model, taking into consideration portfolio constraints. The stochastic basis is modeled by a multidimensional Brownian bridge that converges to zero at maturity, while the underlying asset's spot price is assumed to be a multidimensional geometric Brownian motion. Then, we determine the optimal futures trading strategy by solving a power utility maximization problem with constraints, such as market neutral and dollar neutral. By analyzing and solving the associated Hamilton-Jacobi-Bellman (HJB) equations, we derive the investor's value function and optimal trading strategies. We also provide verification theorem that indicates the solution to value function is indeed the solution to the associated HJB equation. In addition, we also define the investor's certainty equivalent to quantify the value of the futures trading opportunity to the investor. Numerical examples are provide to illustrate how certainty equivalent depend on number of traded futures and different portfolio constraints.

In the literature, early studies of optimal futures trading that incorporates the dynamics of basis include Brennan and Schwartz (1988) and Brennan and Schwartz

1. This chapter contains joint work with Tim Leung and Yang Zhou at Department of Applied Mathematics, University of Washington.

(1990). They assumed that the basis of an index futures follows a scaled Brownian bridge and calculated the value of the embedded timing options to trade the basis. They then used the option prices to devise open-hold-close strategies involving the index futures and the underlying index. Also under a Brownian bridge model, Dai et al. (2011) provided an alternative trading strategy and specification of transaction costs. Another related work by Liu and Longstaff (2004) assumed that the basis follows a scaled Brownian bridge and the investor is subject to a collateral constraint. They derived the closed-form strategy that maximizes the expected logarithmic utility of terminal wealth. For another thing, the stochastic control approach has been applied to stock portfolio optimization dating back to Merton (1971), but much less has been done for dynamic futures portfolio in continuous time. In a number of companion papers (Tourin and Yan, 2013; Leung and Yan, 2018, 2019; Leung and Zhou, 2019; Angoshtari and Leung, 2019, 2020), the utility maximization approach is used to derive dynamic futures trading strategies under various stochastic models without portfolio constraints. Market neutral constraint is taken into consideration to determine optimal pair trading in Angoshtari (2016) and Liu and Timmermann (2013), while Zhao and Palomar (2018) propose a mean-reverting portfolio design with an investment budget constraint. Besides, Li and Papanicolaou (2019) analyze the optimal portfolio for multiple co-integrated assets with a general linear constraint. Compared to these studies, the current paper provides analytic analysis for the futures trading under a multidimensional stochastic basis framework, with and without portfolio constraints. For the constrained case, we can deal with multiple constraints at the same time. Our analysis show how optimal strategies and value function depend on the stochastic basis. Moreover, we decompose the optimal strategies based on different portfolio constraints, and find the optimal "leverage" for value function semi-analytically.

4.1 Model Formulation

We fix a physical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the physical probability measure. The market consists of M risky underlying assets $S_{t,i}$ for $i \in \{1, \dots, M\}$, along with a positive constant rate $r \geq 0$. The asset's spot prices $S_{t,i}$ evolve according to a multidimensional geometric Brownian motion:

$$\mathrm{d}S_{t,i} = S_{t,i} \left(\mu_{i,S} \mathrm{d}t + \sum_{k=1}^{i} \sigma_{i,k,S} \mathrm{d}W_{t,k} \right), \quad i \in \{1, \dots, M\}, \tag{4.1.1}$$

where $\mu_{i,S}$ is the constant drift, $\sigma_{i,k,S}$, for $1 \leq k \leq i$, are constant volatility parameters, and $(W_{t,1}, \dots, W_{t,M})^\top$ is a standard M-dimensional Brownian motion under the measure $\mathbb{P}$.

For each underlying asset $S_{t,i}$, there are N_i futures contracts $F_{t,i,j}$ written on this asset with expiration dates $T_{i,j}$, for $j \in \{1, \dots, N_i\}$. For counting and indexing, we define the order numbers

$$P_{i,j} = \sum_{k=1}^{i-1} N_k + j, \quad i \in \{1, \dots, M\}, \quad j \in \{1, \dots, N_i\},$$

and total number

$$N = \sum_{k=1}^{M} N_k = P_{M,N_M}.$$

Then, we can line up all N futures one by one, where the futures $F_{t,i,j}$ is the $P_{i,j}$-th contract.

Next, we derive the futures price dynamics via the random basis process. To that end, we define the log-value of the random basis for the futures contract $F_{t,i,j}$ by

$$Z_{t,i,j} := \log\left(\frac{F_{t,i,j}}{S_{t,i}}\right) - r(T_{i,j}-t); \quad 0 \le t \le T_{i,j}, \quad i \in \{1,\dots,M\}, \quad j \in \{1,\dots,N_i\}. \tag{4.1.2}$$

Then, we assume the log-basis $Z_{t,i,j}$ evolve according to multidimensional Brownian bridge:

$$\mathrm{d}Z_{t,i,j} = \left(m_{i,j} - \frac{\kappa_{i,j}Z_{t,i,j}}{T_{i,j}-t}\right)\mathrm{d}t + \sum_{k=1}^{P_{i,j}+M} \sigma_{P_{i,j},k,Z}\mathrm{d}W_{t,k}, i \in \{1,\dots,M\}, j \in \{1,\dots,N_i\}, \tag{4.1.3}$$

where drift $m_{i,j}$, coefficient $\kappa_{i,j}$ and volatility parameter $\sigma_{P_{i,j},k,Z}$ are constants for $1 \le k \le P_{i,j}+M$, and $(W_{t,1},\dots,W_{t,N+M})^\top$ is a standard $N+M$ dimensional Brownian motion under the measure $\mathbb{P}$. We define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ being the augmented σ-algebra generated by $\{(W_{u,1},\dots,W_{u,N+M}); 0 \le u \le t\}$ and satisfies the usual conditions. By construction, each log-basis $Z_{t,i,j}$ converges to 0 at the corresponding futures maturity $T_{i,j}$.

From the basis process, we derive the futures price dynamics using Ito's lemma. Precisely, each futures price satisfies the stochastic differential equation (SDE)

$$\mathrm{d}F_{t,i,j} = F_{t,i,j}\left[\left(\theta_{i,j} - \frac{\kappa_{i,j}Z_{t,i,j}}{T_{i,j}-t}\right)\mathrm{d}t + \sum_{k=1}^{P_{i,j}+M} \sigma_{P_{i,j},k,F}\mathrm{d}W_{t,k}\right], \tag{4.1.4}$$

where the drifts $\theta_{i,j}$ are given by

$$\theta_{i,j} = -r + m_{i,j} + \mu_{i,S} + \frac{1}{2}\left(2\sum_{k=1}^{i}\sigma_{i,k,S}\sigma_{P_{i,j},k,Z} + \sum_{k=1}^{P_{i,j}+M}\sigma^2_{P_{i,j},k,Z}\right), \tag{4.1.5}$$

and volatility parameters $\sigma_{P_{i,j},k,F}$ satisfy

$$\sigma_{P_{i,j},k,F} = \begin{cases} \sigma_{i,k,S} + \sigma_{P_{i,j},k,Z}, & 1 \le k \le i, \\ \sigma_{P_{i,j},k,Z}, & i < k \le P_{i,j} + M, \end{cases} \tag{4.1.6}$$

for $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N_i\}$.

In order to rewrite SDEs (4.1.1), (4.1.3), and (4.1.4) in matrix form, we denote the vectors of assets, log-bases, and futures, respectively, as

$$\mathbf{S}_t := (S_{t,1}, \ldots, S_{t,M})^\top, \tag{4.1.7}$$

$$\mathbf{Z}_t := (Z_{t,1,1}, \ldots, Z_{t,1,N_1}, Z_{t,2,1}, \ldots, Z_{t,M,N_M})^\top, \tag{4.1.8}$$

$$\mathbf{F}_t := (F_{t,1,1}, \ldots, F_{t,1,N_1}, F_{t,2,1}, \ldots, F_{t,M,N_M})^\top. \tag{4.1.9}$$

Also, we define the coefficients vectors by

$$\boldsymbol{\mu} := (\mu_{1,S}, \ldots, \mu_{M,S})^\top \in \mathbb{R}^M,$$

$$\boldsymbol{\theta} := (\theta_{1,1}, \ldots, \theta_{1,N_1}, \theta_{2,1}, \ldots, \theta_{M,N_M})^\top \in \mathbb{R}^N,$$

$$\mathbf{m} := (m_{t,1,1}, \ldots, m_{t,1,N_1}, m_{t,2,1}, \ldots, m_{t,M,N_M})^\top \in \mathbb{R}^N,$$

$$\boldsymbol{K}(t) := \operatorname{diag}\left(\frac{\kappa_{1,1}}{T_{1,1} - t}, \ldots, \frac{\kappa_{1,N_1}}{T_{1,N_1} - t}, \frac{\kappa_{2,1}}{T_{2,1} - t}, \ldots, \frac{\kappa_{M,N_M}}{T_{M,N_M} - t}\right) \in \mathbb{R}^{N\times N},$$

and the vectors of independent standard Brownian motions by

$$\mathbf{W}_{t,1} := (W_{t,1}, \ldots, W_{t,M})^\top \in \mathbb{R}^M,$$
$$\mathbf{W}_{t,2} := (W_{t,M+1}, \ldots, W_{t,N+M})^\top \in \mathbb{R}^N.$$

Next, we define the volatility parameter matrix $\widetilde{\mathbf{\Sigma}}_{\mathbf{S}} \in \mathbb{R}^{M\times M}$ for the M underlying assets by

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{S}} = \begin{bmatrix} \sigma_{1,1,S} & 0 & \dots & 0 \\ \sigma_{2,1,S} & \sigma_{2,2,S} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{M,1,S} & \sigma_{M,2,S} & \dots & \sigma_{M,M,S} \end{bmatrix}.$$

The volatility parameter matrices $\widetilde{\mathbf{\Sigma}}_{\mathbf{ZS}} \in \mathbb{R}^{N\times M}$, $\widetilde{\mathbf{\Sigma}}_{\mathbf{Z}} \in \mathbb{R}^{N\times N}$ for N log-bases are defined by

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{ZS}} = \begin{bmatrix} \sigma_{1,1,Z} & \dots & \sigma_{1,M,Z} \\ \sigma_{2,1,Z} & \dots & \sigma_{2,M,Z} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1,Z} & \dots & \sigma_{N,M,Z} \end{bmatrix}, \quad \widetilde{\mathbf{\Sigma}}_{\mathbf{Z}} = \begin{bmatrix} \sigma_{1,M+1,Z} & 0 & \dots & 0 \\ \sigma_{2,M+1,Z} & \sigma_{2,M+2,Z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N,M+1,Z} & \sigma_{N,M+2,Z} & \dots & \sigma_{N,M+N,Z} \end{bmatrix}.$$

The volatility parameter matrices $\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}} \in \mathbb{R}^{N\times M}$, $\widetilde{\mathbf{\Sigma}}_{\mathbf{F}} \in \mathbb{R}^{N\times N}$ for N futures are defined by

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}} = \begin{bmatrix} \sigma_{1,1,F} & \cdots & \sigma_{1,M,F} \\ \sigma_{2,1,F} & \cdots & \sigma_{2,M,F} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1,F} & \cdots & \sigma_{N,M,F} \end{bmatrix}, \quad \widetilde{\mathbf{\Sigma}}_{\mathbf{F}} = \begin{bmatrix} \sigma_{1,M+1,F} & 0 & \cdots & 0 \\ \sigma_{2,M+1,F} & \sigma_{2,M+2,F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N,M+1,F} & \sigma_{N,M+2,F} & \cdots & \sigma_{N,M+N,F} \end{bmatrix}.$$

With these notations, the SDEs (4.1.1),(4.1.3) and (4.1.4) can be written in matrix form:

$$\begin{aligned} \mathrm{d}\mathbf{S}_t &= \mathrm{diag}(\mathbf{S}_t)\left[\boldsymbol{\mu}\mathrm{d}t + \widetilde{\mathbf{\Sigma}}_{\mathbf{S}}\mathrm{d}\boldsymbol{W}_{t,1}\right], \\ \mathrm{d}\mathbf{Z}_t &= (\mathbf{m} - \boldsymbol{K}(t)\mathbf{Z}_t)\mathrm{d}t + \widetilde{\mathbf{\Sigma}}_{\mathbf{ZS}}\mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\mathbf{\Sigma}}_{\mathbf{Z}}\mathrm{d}\boldsymbol{W}_{t,2}, \\ \mathrm{d}\mathbf{F}_t &= \mathrm{diag}(\mathbf{F}_t)\left[(\boldsymbol{\theta} - \boldsymbol{K}(t)\mathbf{Z}_t)\mathrm{d}t + \widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\mathbf{\Sigma}}_{\mathbf{F}}\mathrm{d}\boldsymbol{W}_{t,2}\right]. \end{aligned}$$

4.2 Futures Portfolio Optimization

We now consider the futures trading problem under the stochastic basis model. The portfolio consists of futures with different underlying assets and different maturities. The underlying assets are not traded. As such, we are in an incomplete market setting where not all risks can be traded away.

Let $\pi_{t,i,j}$ be the trader's position, measured in cash amount, on the futures $F_{i,j}$

at time t. The trader's portfolio value is denoted by $X_t^{\boldsymbol{\pi}}$ and evolves according to

$$\frac{\mathrm{d}X_t^{\boldsymbol{\pi}}}{X_t^{\boldsymbol{\pi}}} = rdt + \sum_{i=1}^{M}\sum_{j=1}^{N_i} \pi_{t,i,j}\frac{\mathrm{d}F_{t,i,j}}{F_{t,i,j}} \tag{4.2.1}$$

$$= rdt + \boldsymbol{\pi}_t^{\top}\left[(\boldsymbol{\theta} - \boldsymbol{K}(t)\mathbf{Z}_t)\mathrm{d}t + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}\mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}}\mathrm{d}\boldsymbol{W}_{t,2}\right], \tag{4.2.2}$$

where we have defined the strategy vector

$$\boldsymbol{\pi}_t := (\pi_{t,1,1}, \dots, \pi_{t,1,N_1}, \pi_{t,2,1}, \dots, \pi_{t,M,N_M})^{\top}.$$

The trader's risk preferences are encapsulated by the power utility function

$$U(x) = \frac{x^p}{p}, \tag{4.2.3}$$

with a constant parameter $p < 0$. The relative risk aversion parameter γ is given by

$$\gamma = -\frac{xU''(x)}{U'(x)} = 1 - p.$$

We only consider the case that $p < 0$, which means the power utilities are less risk seeking than logarithmic utility. In particular, when $0 < p < 1$, the analysis is more intricate, and involves identifying the so-called *nirvana solutions* where the expected utility becomes infinite. For details on these solutions under the stochastic basis framework, we refer the reader to Angoshtari and Leung (2019) and Angoshtari and Leung (2020).

We consider futures portfolios with and without constraints. The portfolio optimization problem leads to the study of the associated Hamilton-Jacobi-Bellman equation (HJB), which is reduced to a system of ODEs. In addition, we provide the verification theorem for our utility maximization problem.

Let's begin with general strategies without constraints, which we use superscript 'no' to denote 'no constraints'.

4.2.1 The Portfolio Optimization Problem without Constraints

Without portfolio constraints, the trader seeks an admissible strategy $\boldsymbol{\pi} \in \mathcal{A}^{no}$, that maximizes the expected utility of wealth at T, where $0 < T < T_{i,j}$ for all $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N_i\}$. It means trading stops strictly before the expiry of the futures contracts. Then, the convergence between asset's price $S_{t,i}$ and futures price $F_{t,i,j}$ is not realized in the market. This non-convergence has practical relevance since speculative futures trades are always closed out before the delivery date.

Before defining the set of admissible trading strategies, we construct an auxiliary process for a given strategy $\boldsymbol{\pi}$ by

$$\begin{aligned}\boldsymbol{Y}_t^{\boldsymbol{\pi}} = \int_0^t & \left(-\boldsymbol{Z}_s^\top \boldsymbol{H}^{no}(t) + \boldsymbol{g}^{no}(t)^\top\right) \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}} \mathrm{d}\boldsymbol{W}_{s,1} + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{Z}} \mathrm{d}\boldsymbol{W}_{s,2}\right) \\ & + p\boldsymbol{\pi}_s^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}} \mathrm{d}\boldsymbol{W}_{s,1} + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}} \mathrm{d}\boldsymbol{W}_{s,2}\right),\end{aligned}$$

where $\boldsymbol{H}^{no}(t), \boldsymbol{g}^{no}(t)$ are deterministic functions that only depends on the model parameters, which will appear later in Theorem 4.2.1 by solving the corresponding ODEs. Then, the admissible strategy set is given as:

Definition 4.2.1 (Admissibility)**.** We denote $\mathcal{A}^{no}$ the set of all $\mathbb{F}$-adapted processes $\{\boldsymbol{\pi}_t\}_{0\le t\le T}$, such that

(i) $\mathbb{E}\left(\int_0^T |\boldsymbol{\pi}_t^\top \boldsymbol{Z}_t| + \|\boldsymbol{\pi}_t\|^2 \mathrm{d}t\right) < \infty$;

(ii) $X_t^{\boldsymbol{\pi}} \in \mathcal{D}$, $\mathbb{P}$-a.s., for all $t \in [0,T]$, where $\mathcal{D} = \mathbb{R}^+$ and $(X_t^{\boldsymbol{\pi}})_{0\le t\le T}$ is given by (4.2.1);

(iii) $\mathbb{E}\left(\exp\left(\boldsymbol{Y}_T^{\boldsymbol{\pi}} - \frac{1}{2}\langle \boldsymbol{Y}^{\boldsymbol{\pi}}\rangle_T\right)\right) = 1.$

Condition *(i)* is a general integrability condition to ensure the existence of the wealth process. Condition *(ii)* is to assure that the wealth should be positive almost surely, and Condition *(iii)* can be found in many places, e.g. Kuroda and Nagai (2002), which is equivalent to say the stochastic exponential of $\boldsymbol{Y}_t^{\boldsymbol{\pi}}$ is a martingale.

Then the value function is defined as

$$V^{no}(t, \boldsymbol{z}, x) = \sup_{\boldsymbol{\pi}\in\mathcal{A}^{no}} \mathbb{E}[U(X_T^{\boldsymbol{\pi}})|\boldsymbol{Z}_t = \boldsymbol{z}, X_t^{\boldsymbol{\pi}} = x]. \tag{4.2.4}$$

To solve the portfolio optimization problem, we define the volatility matrices $\boldsymbol{\Sigma}_{\mathbf{Z}} \in \mathbb{R}^{N\times N}$, $\boldsymbol{\Sigma}_{\mathbf{FZ}} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{\Sigma}_{\mathbf{F}} \in \mathbb{R}^{N\times N}$ matrix, as

$$\boldsymbol{\Sigma}_{\mathbf{Z}} = \widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}}^\top + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{Z}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{Z}}^\top, \tag{4.2.5}$$

$$\boldsymbol{\Sigma}_{\mathbf{FZ}} = \widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}}^\top + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{Z}}^\top, \tag{4.2.6}$$

$$\boldsymbol{\Sigma}_{\mathbf{F}} = \widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}^\top + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}}^\top. \tag{4.2.7}$$

Then, we define the linear operator

$$\mathcal{L}\cdot = rx\partial_x \cdot + (\mathbf{m} - \boldsymbol{K}(t)\boldsymbol{z})^\top \nabla_{\boldsymbol{z}} \cdot + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\Sigma}_{\mathbf{Z}}\nabla^2_{\boldsymbol{z}}\cdot), \tag{4.2.8}$$

where $\nabla_{\boldsymbol{z}}\cdot = (\partial_{z_{1,1}}\cdot, \ldots, \partial_{z_{1,N_1}}\cdot, \partial_{z_{2,1}}\cdot, \ldots, \partial_{z_{M,N_M}}\cdot)^\top$ is the nabla operator and Hessian operator $\nabla^2_{\boldsymbol{z}}\cdot$ satisfies

$$\nabla^2_{\boldsymbol{z}}\cdot = \begin{bmatrix} \partial^2_{z_{1,1}}\cdot & \partial_{z_{1,1}}\partial_{z_{1,2}}\cdot & \ldots & \partial_{z_{1,1}}\partial_{z_{N,N_M}}\cdot \\ \partial_{z_{1,2}}\partial_{z_{1,1}}\cdot & \partial^2_{z_{1,2}}\cdot & \ldots & \partial_{z_{1,2}}\partial_{z_{N,N_M}}\cdot \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{z_{N,N_M}}\partial_{z_{1,1}}\cdot & \partial_{z_{N,N_M}}\partial_{z_{1,2}}\cdot & \ldots & \partial^2_{z_{N,N_M}}\cdot \end{bmatrix},$$

and column-valued-function $\boldsymbol{a}^{no}(t, \boldsymbol{z}, x)$

$$\boldsymbol{a}^{no}(t, \boldsymbol{z}, x) = (\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z})x\partial_x u^{no} + \boldsymbol{\Sigma}_{\mathbf{FZ}}x\nabla_{\boldsymbol{z}}\partial_x u^{no}. \tag{4.2.9}$$

Then, we obtain the HJB equation for $u^{no}(t, \boldsymbol{z}, x)$,

$$\partial_t u^{no} + \mathcal{L}u^{no} + \max_{\boldsymbol{\pi}\in\mathcal{A}^{no}}\left\{\boldsymbol{\pi}^\top \boldsymbol{a}^{no}(t, \boldsymbol{z}, x) + \frac{x^2\partial_{xx}u^{no}}{2}\boldsymbol{\pi}^\top\boldsymbol{\Sigma}_{\mathbf{F}}\boldsymbol{\pi}\right\} = 0, \tag{4.2.10}$$

for $(t, \boldsymbol{z}, x) \in [0, T) \times \mathbb{R}^N \times \mathcal{D}$, where operator $\mathcal{L}$ is defined in (4.2.8). The terminal condition is

$$u^{no}(T, \boldsymbol{z}, x) = \frac{x^p}{p},$$

for $(\boldsymbol{z}, x) \in \mathbb{R}^N \times \mathcal{D}$.

To solve the HJB equation (4.2.10), we have:

Theorem 4.2.1. Define $\boldsymbol{\Sigma}^{no} \in \mathbb{R}^{N\times N}$ by

$$\boldsymbol{\Sigma}^{no} = \boldsymbol{\Sigma}_{\mathbf{Z}} + \frac{p}{(1-p)}\boldsymbol{\Sigma}_{\mathbf{FZ}}^{\top}\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}. \tag{4.2.11}$$

Then, the following statements hold.

1. The matrix Riccati differential equation below has a unique solution that is positive semi-definite for all $t \in [0, T]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}^{no}(t) &= \left(\boldsymbol{K}(t) + \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)\boldsymbol{H}^{no}(t)\\ &\quad + \boldsymbol{H}^{no}(t)\left(\boldsymbol{K}(t) + \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)^{\top}\\ &\quad + \boldsymbol{H}^{no}(t)\boldsymbol{\Sigma}^{no}\boldsymbol{H}^{no}(t) + \frac{p}{1-p}\boldsymbol{K}(t)^{\top}\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{K}(t),\\ \boldsymbol{H}^{no}(T) &= \mathbf{0}_{N\times N},\end{aligned} \tag{4.2.12}$$

   where $\mathbf{0}_{N\times N}$ denotes the zero matrix of dimension $N \times N$.

2. The solution of the HJB equation(4.2.10) is given by

$$u^{no}(t, \boldsymbol{z}, x) = U(x)\exp\left(-\frac{1}{2}\boldsymbol{z}^{\top}\boldsymbol{H}^{no}(t)\boldsymbol{z} + \boldsymbol{z}^{\top}\boldsymbol{g}^{no}(t) + f^{no}(t)\right),$$

   for $(t, \boldsymbol{z}, x) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^+$, where $\boldsymbol{H}^{no}(t) \in \mathbb{R}^{N\times N}$ satisfies the matrix Riccati differential equation (4.2.24), $\boldsymbol{g}^{no}(t) \times \mathbb{R}^N$ satisfies the following ODE system,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{g}^{no}(t) &= \boldsymbol{K}(t)\boldsymbol{g}^{no}(t) + \boldsymbol{H}^{no}(t)\boldsymbol{m} + \left(\boldsymbol{H}^{no}(t)\boldsymbol{\Sigma} + \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)\boldsymbol{g}^{no}(t) \\
&\quad + \frac{p}{1-p}\left(\boldsymbol{K}(t) + \boldsymbol{H}^{no}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^{\top}\right)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\theta}, \qquad (4.2.13) \\
\boldsymbol{g}^{no}(T) &= \mathbf{0}_{N\times 1},
\end{aligned}
$$

and $f^{no}(t) \in \mathbb{R}$ follows the ODE,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}f^{no}(t) &= -\boldsymbol{m}^{\top}\boldsymbol{g}^{no}(t) + \frac{1}{2}tr\left(\boldsymbol{\Sigma}_{\mathbf{Z}}\boldsymbol{H}^{no}(t)\right) - \frac{1}{2}\boldsymbol{g}^{no}(t)^{\top}\boldsymbol{\Sigma}_{\mathbf{Z}}\boldsymbol{g}^{no}(t) - rp \\
&\quad - \frac{p}{2(1-p)}\left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{g}^{no}(t)\right)^{\top}\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{g}^{no}(t)\right), \qquad (4.2.14) \\
f^{no}(T) &= 0.
\end{aligned}
$$

3. The optimal strategy is given by

$$
\boldsymbol{\pi}^{*}(t, \boldsymbol{z}) = \frac{1}{1-p}\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\left(\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\big(\boldsymbol{g}^{no}(t) - \boldsymbol{H}^{no}(t)\boldsymbol{z}\big)\right). \qquad (4.2.15)
$$

Proof. To show the existence and uniqueness of positive semi-definite solution to Riccati equation (4.2.12), it suffices to show $\boldsymbol{\Sigma}^{no}$ is positive semi-definite, which stands out as a lemma below.

Lemma 4.2.1. $\boldsymbol{\Sigma}^{no}$ is positive semi-definite.

By (4.2.11), we can decompose $\mathbf{\Sigma}^{no}$ into two components, i.e.

$$\mathbf{\Sigma}^{no} = \frac{1}{(1-p)}\mathbf{\Sigma_Z} - \frac{p}{(1-p)}(\mathbf{\Sigma_Z} - \mathbf{\Sigma}_{\mathbf{FZ}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma_{FZ}}).$$

Since $p < 0$, the first term is obviously positive semi-definite. It suffices to show that $\mathbf{\Sigma_Z} - \mathbf{\Sigma}_{\mathbf{FZ}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma_{FZ}}$ is positive semi-definite. Define $\boldsymbol{A_S} = \widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top} - \widetilde{\mathbf{\Sigma}}_{\mathbf{ZS}}^{\top}$. Then by (4.2.6), (4.2.5) and (4.2.7), we have

$$\mathbf{\Sigma}_{\mathbf{FZ}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma_{FZ}} = (\mathbf{\Sigma_F} - \widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\boldsymbol{A_S})^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}(\mathbf{\Sigma_F} - \widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\boldsymbol{A_S}).$$

Therefore, we obtain

$$\mathbf{\Sigma_Z} - \mathbf{\Sigma}_{\mathbf{FZ}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma_{FZ}} = \boldsymbol{A}_{\boldsymbol{S}}^{\top}(\boldsymbol{I} - \widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}})\boldsymbol{A_S}.$$

Then, we only need to show $\boldsymbol{I} - \widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}$ is the non-negative matrix. Suppose $\boldsymbol{v}$ is an eigenvector for $\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}$ such that

$$\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\boldsymbol{v} = \lambda\boldsymbol{v}.$$

Then, we have

$$\begin{aligned}\lambda\boldsymbol{v}^{\top}\boldsymbol{v} = \boldsymbol{v}^{\top}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\boldsymbol{v} &= \boldsymbol{v}^{\top}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma_F}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\boldsymbol{v} \\ &\geq \boldsymbol{v}^{\top}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}^{\top}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\mathbf{\Sigma}}_{\mathbf{FS}}\boldsymbol{v} = \lambda^2\boldsymbol{v}^{\top}\boldsymbol{v},\end{aligned}$$

where the inequality is from the (4.2.7), the definition of $\mathbf{\Sigma_F}$. We obtain $\lambda \in [0, 1]$, i.e. the eigenvalues for $\boldsymbol{I} - \widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}^{\top}\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}$ are also in $[0,1]$, which therefore, should be positive semi-definite.

Now, we proceed with the proof of Theorem 4.2.1. According to Appendix A in Angoshtari and Leung (2020), the Riccati Equation (4.2.12) has a unique symmetric non-negative definite solution, since $\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}$ and $\boldsymbol{\Sigma}^{no}$ are positive semi-definite.

Next, performing optimization in (4.2.10) with the first-order condition for the optimal strategy $\boldsymbol{\pi}_t^*$,

$$\boldsymbol{a}^{no}(t,\boldsymbol{z},x) + (x^2\partial_{xx}u)\boldsymbol{\Sigma}_{\mathbf{F}}\boldsymbol{\pi}_t^* = 0.$$

Then, we obtain

$$\boldsymbol{\pi}_t^* = -\frac{\boldsymbol{\Sigma}_{\boldsymbol{F}}^{-1}\boldsymbol{a}^{no}(t,\boldsymbol{z},x)}{x^2\partial_{xx}u}. \tag{4.2.16}$$

Plugging (4.2.16) back to the HJB (4.2.10), we have

$$\partial_t u^{no} + \mathcal{L}u^{no} - \frac{1}{2x^2\partial_{xx}u^{no}}\boldsymbol{a}^{no}(t,\boldsymbol{z},x)^{\top}\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{a}^{no}(t,\boldsymbol{z},x) = 0.$$

With the ansatz $u^{no}(t,\boldsymbol{z},x) = U(x)\exp\left(-\frac{1}{2}\boldsymbol{z}^{\top}\boldsymbol{H}^{no}(t)\boldsymbol{z} + \boldsymbol{z}^{\top}\boldsymbol{g}^{no}(t) + f^{no}(t)\right)$, where $\boldsymbol{H}^{no}(t) \in \mathbb{R}^{N\times N}$ is a symmetric matrix and $\boldsymbol{g}^{no}(t) \in \mathbb{R}^N$, we obtain the matrix Riccati equation (4.2.12) for $\boldsymbol{H}^{no}(t)$, ODE system (4.2.13) for $\boldsymbol{g}^{no}(t)$ and ODE (4.2.14) for $f^{no}(t)$. In addition, $\partial_{xx}u = p(p-1)u \leq 0$. □

It's common practice to seek dynamic futures trading strategies with portfolio constraints, among which dollar neutrality or market neutrality are most popular. We are going to give a rigorous definition of portfolio constraints, and introduce a neutrality concept that includes both dollar and market neutrality.

4.2.2 Constrained Futures Portfolio

We now incorporate portfolio constraints of the form

$$\boldsymbol{\Gamma}^\top \boldsymbol{\pi} = \boldsymbol{c}, \tag{4.2.17}$$

for $\boldsymbol{\Gamma} \in \mathbb{R}^{N\times d}$ and $\boldsymbol{c} \in \mathbb{R}^d$. Furthermore, we assume these d constraints are linearly independent, i.e. $\text{rank}(\boldsymbol{\Gamma}) = d$. Otherwise, some constraints are either redundant or infeasible. The admissible set $\mathcal{A}$ for constrained case are almost the same as given in Definition 4.2.1, except that $\boldsymbol{H}^{no}, \boldsymbol{g}^{no}$ in the auxiliary process $\boldsymbol{Y}^{\boldsymbol{\pi}}$ should be replaced by $\boldsymbol{H}, \boldsymbol{g}$, given below, respectively. The trader seeks a constrained admissible strategy $\boldsymbol{\pi} \in \{\boldsymbol{\pi} \in \mathcal{A} | \boldsymbol{\Gamma}^\top \boldsymbol{\pi} = \boldsymbol{c}\}$ over the trading horizon $[0, T]$, that maximizes the expected utility of wealth at T.

Then, the value function is defined as

$$V(t, \boldsymbol{z}, x) = \sup_{\boldsymbol{\pi} \in \mathcal{A}, \boldsymbol{\Gamma}^\top \boldsymbol{\pi} = \boldsymbol{c}} \mathbb{E}[U(X_T^{\boldsymbol{\pi}}) | \boldsymbol{Z}_t = \boldsymbol{z}, X_t^{\boldsymbol{\pi}} = x], \tag{4.2.18}$$

and column-vector-valued function

$$\boldsymbol{a}(t, \boldsymbol{z}, x) = (x\partial_x u)(\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z}) + x\boldsymbol{\Sigma}_{\mathbf{FZ}}(\partial_x \nabla_{\boldsymbol{z}} u). \tag{4.2.19}$$

Then, the corresponding Hamilton-Jacobi-Bellman equation (HJB) follows

$$\partial_t u + \mathcal{L}u + \max_{\boldsymbol{\pi} \in \mathcal{A}, \boldsymbol{\Gamma}^\top \boldsymbol{\pi} = \boldsymbol{c}} \left\{ \boldsymbol{\pi}^\top \boldsymbol{a}(t, \boldsymbol{z}, x) + \frac{x^2 \partial_{xx} u}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma}_{\mathbf{F}} \boldsymbol{\pi} \right\} = 0, \tag{4.2.20}$$

for $(t, \boldsymbol{z}, x) \in [0, T) \times \mathbb{R}^N \times \mathcal{D}$, where the set $\mathcal{D}$ contains all possible values for the wealth x. The terminal condition is

$$u(T, \boldsymbol{z}, x) = U(x),$$

for $(\boldsymbol{z}, x) \in \mathbb{R}^N \times \mathcal{D}$.

The following result provides the solution to the HJB equation (4.2.20).

Theorem 4.2.2. Assume the trader's utility is given by (4.2.3). Define $\boldsymbol{D}_{\boldsymbol{\Gamma}} \in \mathbb{R}^{d \times d}$, $\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}} \in \mathbb{R}^{N \times N}$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$ as

$$\boldsymbol{D}_{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}^\top \boldsymbol{\Sigma}_{\mathbf{F}}^{-1} \boldsymbol{\Gamma}, \tag{4.2.21}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}} = \boldsymbol{\Sigma}_{\mathbf{F}}^{-1} \boldsymbol{\Gamma} \boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1} \boldsymbol{\Gamma}^\top \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}, \tag{4.2.22}$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\mathbf{Z}} + \frac{p}{(1-p)} \boldsymbol{\Sigma}_{\mathbf{FZ}}^\top (\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}) \boldsymbol{\Sigma}_{\mathbf{FZ}}. \tag{4.2.23}$$

We note that $\boldsymbol{D}_{\boldsymbol{\Gamma}}$ is invertible due to the assumption that $\text{rank}(\boldsymbol{\Gamma}) = d$. Then, the following statements hold.

1. The matrix Riccati differential equation below has a unique solution that is

positive definite for all $t \in [0, T]$,

$$\begin{aligned}
\boldsymbol{H}'(t) &= \left(\boldsymbol{K}(t) + \frac{p}{1-p}\boldsymbol{K}(t)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)\boldsymbol{H}(t) \\
&\quad + \boldsymbol{H}(t)\left(\boldsymbol{K}(t) + \frac{p}{1-p}\boldsymbol{K}(t)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)^{\top} \qquad (4.2.24)\\
&\quad + \boldsymbol{H}(t)\boldsymbol{\Sigma}\boldsymbol{H}(t) + \frac{p}{1-p}\boldsymbol{K}(t)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{K}(t), \\
\boldsymbol{H}(T) &= \mathbf{0}_{N\times N},
\end{aligned}$$

where $\mathbf{0}_{N\times N}$ denotes the zero matrix of dimension $N \times N$.

2. The solution of the HJB equation(4.2.20) is given by

$$u(t, \boldsymbol{z}, x) = U(x)\exp\left(-\frac{1}{2}\boldsymbol{z}^{\top}\boldsymbol{H}(t)\boldsymbol{z} + \boldsymbol{z}^{\top}\boldsymbol{g}(t) + f(t)\right),$$

for $(t, \boldsymbol{z}, x) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^+$, where $\boldsymbol{H}(t) \in \mathbb{R}^{N\times N}$ satisfies the matrix Riccati differential equation (4.2.24), $\boldsymbol{g}(t) \times \mathbb{R}^N$ satisfies the following ODE system

$$\begin{aligned}
\boldsymbol{g}'(t) &= \left(\boldsymbol{K}(t) + \boldsymbol{H}(t)\boldsymbol{\Sigma} + \frac{p}{1-p}\boldsymbol{K}(t)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)\boldsymbol{g}(t) + \boldsymbol{H}(t)\boldsymbol{m} \\
&\quad + p\left(\boldsymbol{K}(t) + \boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^{\top}\right)\left(\frac{(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\theta}}{1-p} + \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c}\right), \qquad (4.2.25)\\
\boldsymbol{g}(T) &= \mathbf{0}_{N\times 1},
\end{aligned}$$

and $f(t) \in \mathbb{R}$ follows the ODE,

$$
\begin{aligned}
f'(t) = & -\boldsymbol{m}^\top \boldsymbol{g}(t) + \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}_{\mathbf{Z}}\boldsymbol{H}(t)\right) - \frac{1}{2}\boldsymbol{g}(t)^\top \boldsymbol{\Sigma}_{\mathbf{Z}}\boldsymbol{g}(t) - rp \\
& - \frac{p}{2(1-p)}\left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{g}(t)\right)^\top \left(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}\right)\left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{g}(t)\right) \qquad (4.2.26) \\
& - p\left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{g}(t)\right)^\top \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c} + \frac{p(1-p)}{2}\boldsymbol{c}^\top \boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c}, \\
f(T) = & \; 0.
\end{aligned}
$$

3. The optimal strategy is given by

$$
\boldsymbol{\pi}^*(t,\boldsymbol{z}) = \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c} + \frac{1}{1-p}\left(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}\right)\Big(\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\big(\boldsymbol{g}(t) - \boldsymbol{H}(t)\boldsymbol{z}\big)\Big). \qquad (4.2.27)
$$

Again, we begin with two lemmas.

Lemma 4.2.2. $\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}$ is positive semi-definite.

Proof. By (4.2.22), we have

$$
\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}} = \boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}.
$$

Since $\boldsymbol{\Sigma}_{\mathbf{F}}$ is symmetric positive definite matrix, there exists a invertible matrix $\boldsymbol{B}$ such that $\boldsymbol{B}\boldsymbol{B}^\top = \boldsymbol{\Sigma}_{\mathbf{F}}$. Then, we define an auxiliary matrix $\boldsymbol{A} = \boldsymbol{B}^{-1}\boldsymbol{\Gamma} \in \mathbb{R}^{N\times d}$. Thus, we obtain $\boldsymbol{D}_{\boldsymbol{\Gamma}} = \boldsymbol{A}^\top\boldsymbol{A}$. Then, it suffices to show

$$
\boldsymbol{B}^\top(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1})\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top,
$$

is non-negative definite. Since, $\boldsymbol{A}\boldsymbol{v}$ is the eigenvector of $\boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top$, for any $\boldsymbol{v} \in \mathbb{R}^d$ and rank$(\boldsymbol{A}) = d$, the eigenvalues for $\boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top$ are 0 or 1. Therefore, $\boldsymbol{I} - \boldsymbol{A}(\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top$ is positive semi-definite. □

Lemma 4.2.3. $\boldsymbol{\Sigma}$ is positive semi-definite.

Proof. By (4.2.23), we can decompose $\boldsymbol{\Sigma}$ into two components, i.e.

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{no} - \frac{p}{(1-p)}\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}\boldsymbol{\Sigma}_{\mathbf{FZ}}.$$

According to Lemma 4.2.1, $\boldsymbol{\Sigma}^{no} \geq 0$. Besides, since $p < 0$, the second term is also positive semi-definite. Therefore, $\boldsymbol{\Sigma}$ is positive semi-definite. □

Now, we present the proof of Theorem 4.2.2. To show that the Riccati Equation (4.2.24) has a unique symmetric positive semi-definite solution, it suffices to demonstrate that $\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}$ and $\boldsymbol{\Sigma}$ being the coefficients of quadratic and constant terms are positive semi-definite, which are proved in Lemma 4.2.2 and Lemma 4.2.3.

Next, we turn to solve the HJB equation (4.2.20) by the method of Lagrange multiplier, similarly seen in Li and Papanicolaou (2019). Let $\boldsymbol{\lambda}(t) = (\lambda_1(t), \ldots, \lambda_d(t))^\top$ be the (vector) Lagrange multiplier. We define the Lagrangian function corresponding to the constrained case by

$$L(t, \boldsymbol{\pi}, \boldsymbol{\lambda}) = \boldsymbol{\pi}^\top\boldsymbol{a}(t, \boldsymbol{z}, x) + \frac{x^2\partial_{xx}u}{2}\boldsymbol{\pi}^\top\boldsymbol{\Sigma}_{\mathbf{F}}\boldsymbol{\pi} - \boldsymbol{\lambda}(t)^\top(\boldsymbol{\Gamma}^\top\boldsymbol{\pi} - \boldsymbol{c}).$$

Then, it suffices to solve the system of equations:

$$\begin{cases} \nabla_{\boldsymbol{\pi}} L = \boldsymbol{a}(t,\boldsymbol{z},x) + (x^2\partial_{xx}u)\boldsymbol{\Sigma}_{\mathbf{F}}\boldsymbol{\pi} - \boldsymbol{\Gamma}\boldsymbol{\lambda}(t) = 0, \\ \nabla_{\boldsymbol{\lambda}} L = \boldsymbol{\Gamma}^\top\boldsymbol{\pi} - \boldsymbol{c} = 0. \end{cases}$$

Therefore, we obtain

$$\boldsymbol{\pi}_t = -\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\frac{\boldsymbol{a}(t,\boldsymbol{z},x) - \boldsymbol{\Gamma}\boldsymbol{\lambda}(t)}{x^2\partial_{xx}u}, \tag{4.2.28}$$

and

$$\boldsymbol{\lambda}(t) = (\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{a}(t,\boldsymbol{z},x) + (x^2\partial_{xx}u)\boldsymbol{c}). \tag{4.2.29}$$

Inserting $\boldsymbol{\lambda}$ back to formula (4.2.28), we have

$$\boldsymbol{\pi}_t = \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c} - \frac{(\boldsymbol{\Sigma}_{\boldsymbol{F}}^{-1} - \boldsymbol{\Sigma}_{\Gamma})\boldsymbol{a}(t,\boldsymbol{z},x)}{x^2\partial_{xx}u}, \tag{4.2.30}$$

where $\boldsymbol{D}_{\boldsymbol{\Gamma}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}$ are given by (4.2.21) and (4.2.22), respectively. We can verify that the optimal strategies satisfy the constraints,

$$\boldsymbol{\Gamma}^\top\boldsymbol{\pi}_t = \boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c} - \frac{\boldsymbol{\Gamma}^\top(\boldsymbol{\Sigma}_{\boldsymbol{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{a}(t,\boldsymbol{z},x)}{x^2\partial_{xx}u} = \boldsymbol{c}.$$

Plugging the candidate $\boldsymbol{\pi}_t$ back to the HJB(4.2.20), we obtain

$$\begin{aligned} \partial_t u + \mathcal{L}u - \frac{\boldsymbol{a}(t,\boldsymbol{z},x)^\top(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{a}(t,\boldsymbol{z},x)}{2x^2\partial_{xx}u} &+ \boldsymbol{c}^\top\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{a}(t,\boldsymbol{z},x) \\ &+ \frac{x^2\partial_{xx}u}{2}\boldsymbol{c}^\top\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c} = 0. \end{aligned} \tag{4.2.31}$$

With the ansatz $u(t,\boldsymbol{z},x) = U(x)\exp\left(-\frac{1}{2}\boldsymbol{z}^\top\boldsymbol{H}(t)\boldsymbol{z} + \boldsymbol{z}^\top\boldsymbol{g}(t) + f(t)\right)$, where $\boldsymbol{H}(t) \in$

$\mathbb{R}^{N\times N}$ is a symmetric matrix and $\boldsymbol{g}(t) \in \mathbb{R}^N$, we obtain the matrix Riccati equation (4.2.24) for $\boldsymbol{H}(t)$, ODE system (4.2.25) for $\boldsymbol{g}(t)$ and ODE (4.2.26) for $f(t)$.

Plugging the solved equation $u(t, \boldsymbol{z}, x)$ into (4.2.30) gives us the formula of optimal strategy. The verification will be provided as Theorem 4.2.3.

We now examine the optimal strategies more closely under the two common constraints.

Example 4.2.1 (Dollar Neutral)**.** A trading portfolio is said to be dollar neutral if

$$\mathbf{1}_{N\times 1}^\top \boldsymbol{\pi} = 0,$$

where $\mathbf{1}_{N\times 1}$ is an all-ones vector of dimension N. This amounts to set $\boldsymbol{\Gamma} = \mathbf{1}_{N\times 1}$ and $\boldsymbol{c} = 0$ in (4.2.17). Then, according to Theorem 4.2.2, we obtain the optimal strategy for dollar neutral portfolio,

$$\boldsymbol{\pi}_t^* = \frac{1}{1-p}(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\Big(\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\big(\boldsymbol{g}(t) - \boldsymbol{H}(t)\boldsymbol{z}\big)\Big).$$

When $N = 1$, meaning we trade only one futures, the strategy $\boldsymbol{\pi}$ must be zero for dollar neutral constraint.

Example 4.2.2 (Market Neutral)**.** A trading portfolio is said to be market neutral if

$$dX_t^{\boldsymbol{\pi}}\, dS_{i,t} = 0,$$

for $i \in \{1, \ldots, M\}$, which reduces to the market neutral constraints

$$(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}} \widetilde{\boldsymbol{\Sigma}}_{\mathbf{S}}^{\top})^{\top} \boldsymbol{\pi} = \mathbf{0}_{M \times 1}.$$

This amounts to set $\boldsymbol{\Gamma} = \widetilde{\boldsymbol{\Sigma}}_{\mathbf{SF}} \widetilde{\boldsymbol{\Sigma}}_{\mathbf{S}}^{\top}$ and $\boldsymbol{c} = \mathbf{0}$ in (4.2.17). Then according to Theorem 4.2.2, we obtain the optimal strategy for market neutral portfolio,

$$\boldsymbol{\pi}_t^* = \frac{1}{1-p} (\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}) \Big(\boldsymbol{\theta} - \boldsymbol{K}(t) \boldsymbol{z} + \boldsymbol{\Sigma}_{\mathbf{FZ}} \big(\boldsymbol{g}(t) - \boldsymbol{H}(t) \boldsymbol{z} \big) \Big).$$

Since $\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}$, $\boldsymbol{H}(t)$ and $\boldsymbol{g}(t)$ depend on the choice of portfolio constraints, $\boldsymbol{\Gamma}$ and $\boldsymbol{c}$, the optimal strategy presented in two examples are different even though they have the same formula. It also implies the admissible strategy set $\mathcal{A}$ depends on the portfolio constraints.

Next, we introduce a more general class of constraints.

Definition 4.2.2 ($\boldsymbol{\Gamma}$-Neutral)**.** We say a strategy $\boldsymbol{\pi}$ is $\boldsymbol{\Gamma}$-Neutral if it satisfies the following equality:

$$\boldsymbol{\Gamma}^{\top} \boldsymbol{\pi} = \mathbf{0}.$$

Remark 4.2.1. The definition of $\boldsymbol{\Gamma}$-Neutral is universal whenever the constraints are imposed on strategies that are describing either invest fractions or dollar amounts.

With the definition of $\boldsymbol{\Gamma}$-Neutral strategy, we can integrate the dollar neutral and market neutral conditions to be $\boldsymbol{\Gamma}$-Neutral that $\boldsymbol{\Gamma} = \mathbf{1}$ or $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{SF}} \widetilde{\boldsymbol{\Sigma}}_{\mathbf{S}}^{\top}$. Moreover, we can decompose the optimal strategies for general constraints $\boldsymbol{\Gamma}^{\top} \boldsymbol{\pi} = \boldsymbol{c}$ into two components, one of which is dominated by the $\boldsymbol{\Gamma}$-Neutral case, and the remaining component reveals the hedging demand for $\boldsymbol{c} \neq 0$. With this idea, we decompose the coefficient functions $\boldsymbol{g}(t)$ and $f(t)$ firstly as following:

Lemma 4.2.4. There exists $\boldsymbol{\Psi}(t)$, $\boldsymbol{\beta}(t)$, $\boldsymbol{\Lambda}(t)$, which are $N \times d, 1 \times d$ and $d \times d$ matrices, respectively, such that

$$\boldsymbol{g}(t) = \boldsymbol{g}_0(t) + \boldsymbol{\Psi}(t)\boldsymbol{c}, \tag{4.2.32}$$

$$f(t) = f_0(t) + \boldsymbol{\beta}(t)\boldsymbol{c} + \boldsymbol{c}^\top \boldsymbol{\Lambda}(t)\boldsymbol{c}, \tag{4.2.33}$$

where $\boldsymbol{g}_0(t), f_0(t)$ are the unique solutions to ODEs (4.2.25), (4.2.26) when $\boldsymbol{c} = \mathbf{0}$. Moreover, $\boldsymbol{\Lambda}(t)$ is positive semi-definite.

Proof. The existence of $\boldsymbol{\Psi}(t)$, $\boldsymbol{\beta}(t)$, $\boldsymbol{\Lambda}(t)$, is guaranteed by solving an ODEs system below. Firstly, we observe from the matrix Riccati equation (4.2.24) that $\boldsymbol{H}(t)$ doesn't depend on $\boldsymbol{c}$. Then, plugging the decomposition (4.2.32) and (4.2.33) into ODE system (4.2.25) and ODE (4.2.26), and collecting the coefficients of $\boldsymbol{c}$, we obtain the ODE system for $\boldsymbol{g}_{\mathbf{0}}(t) \in \mathbb{R}^N$,

$$\begin{aligned}
\boldsymbol{g}_0'(t) = \boldsymbol{K}(t)\boldsymbol{g}_0(t) + \boldsymbol{H}(t)\boldsymbol{m} + \left(\boldsymbol{H}(t)\boldsymbol{\Sigma} - \frac{p}{1-p}\boldsymbol{K}(t)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)\boldsymbol{g}_0(t) \\
- \frac{p}{1-p}\left(\boldsymbol{K}(t) - \boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top\right)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\mu}_{\boldsymbol{F}}, \\
\boldsymbol{g}_0(T) = \mathbf{0}_{N\times 1},
\end{aligned}$$

the ODE for $f_0(t) \in \mathbb{R}$,

$$
\begin{aligned}
f_0'(t) &= -\boldsymbol{m}^\top \boldsymbol{g}_0(t) + \frac{1}{2} tr\left(\boldsymbol{\Sigma}_{\mathbf{Z}} \boldsymbol{H}(t)\right) - \frac{1}{2} \boldsymbol{g}_0(t)^\top \boldsymbol{\Sigma}_{\mathbf{Z}} \boldsymbol{g}_0(t) - rp \\
&\quad - \frac{p}{2(1-p)} \left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}} \boldsymbol{g}_0(t)\right)^\top \left(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}\right) \left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}} \boldsymbol{g}_0(t)\right), \\
f_0(T) &= 0,
\end{aligned}
$$

and ODEs for $\boldsymbol{\Psi}(t), \boldsymbol{\beta}(t), \boldsymbol{\Lambda}(t)$,

$$
\begin{aligned}
\boldsymbol{\Psi}'(t) &= \boldsymbol{K}(t)\boldsymbol{\Psi}(t) + \left(\boldsymbol{H}(t)\boldsymbol{\Sigma} - \frac{p}{1-p}\boldsymbol{K}(t)(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}}\right)\boldsymbol{\Psi}(t), \\
&\quad - p\left(\boldsymbol{K}(t) - \boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top\right)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}, \\
\boldsymbol{\beta}'(t) &= -\boldsymbol{m}^\top\boldsymbol{\Psi}(t) - \boldsymbol{g}_0(t)\boldsymbol{\Sigma}\boldsymbol{\Psi}(t) - \frac{p}{1-p}\boldsymbol{\mu}_{\boldsymbol{F}}^\top(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}} \\
&\quad - p(\boldsymbol{\mu}_{\boldsymbol{F}} + \boldsymbol{g}_0(t)\boldsymbol{\Sigma}_{\mathbf{FZ}})^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^\top, \\
\boldsymbol{\Lambda}'(t) &= -\frac{1}{2}\boldsymbol{\Psi}(t)^\top\boldsymbol{\Sigma}\boldsymbol{\Psi}(t) - \frac{p}{2}\boldsymbol{\Psi}(t)^\top\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1} - \frac{p}{2}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{\Gamma}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{\Psi}(t) \\
&\quad + \frac{p(1-p)}{2}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}.
\end{aligned}
$$

Besides, we can check that $\boldsymbol{g}_0, f_0$ solve the corresponding ODEs for $\boldsymbol{\Gamma}$-Neutral constraint.

As for matrix $\mathbf{\Lambda}(t)$, we have

$$\begin{aligned}\mathbf{\Lambda}'(t) &= -\frac{1}{2}\mathbf{\Psi}(t)^\top \mathbf{\Sigma}\mathbf{\Psi}(t) - \frac{p}{2}\mathbf{\Psi}(t)^\top \mathbf{\Sigma}_{\mathbf{FZ}}\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Gamma}\boldsymbol{D}_{\mathbf{\Gamma}}^{-1} \\ &\quad - \frac{p}{2}\boldsymbol{D}_{\mathbf{\Gamma}}^{-1}\mathbf{\Gamma}^\top\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma}_{\mathbf{FZ}}^\top\mathbf{\Psi}(t) + \frac{p(1-p)}{2}\boldsymbol{D}_{\mathbf{\Gamma}}^{-1} \\ &\leq \frac{p}{2(1-p)}\mathbf{\Psi}(t)^\top\mathbf{\Sigma}_{\mathbf{FZ}}^\top\mathbf{\Sigma}_{\mathbf{\Gamma}}\mathbf{\Sigma}_{\mathbf{FZ}}\mathbf{\Psi}(t) \\ &\quad - \frac{p}{2}\mathbf{\Psi}(t)^\top\mathbf{\Sigma}_{\mathbf{FZ}}^\top\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Gamma}\boldsymbol{D}_{\mathbf{\Gamma}}^{-1} - \frac{p}{2}\boldsymbol{D}_{\mathbf{\Gamma}}^{-1}\mathbf{\Gamma}^\top\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma}_{\mathbf{FZ}}\mathbf{\Psi}(t) + \frac{p(1-p)}{2}\boldsymbol{D}_{\mathbf{\Gamma}}^{-1} \\ &= \frac{p}{2(1-p)}\left(\mathbf{\Gamma}^\top\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma}_{\mathbf{FZ}}\mathbf{\Psi}(t) - (1-p)\right)^\top \boldsymbol{D}_{\mathbf{\Gamma}}^{-1}\left(\mathbf{\Gamma}^\top\mathbf{\Sigma}_{\mathbf{F}}^{-1}\mathbf{\Sigma}_{\mathbf{FZ}}\mathbf{\Psi}(t) - (1-p)\right),\end{aligned}$$

where the first inequality comes from (4.2.23) that

$$\mathbf{\Sigma} = \mathbf{\Sigma}_{\mathbf{Z}} + \frac{p}{(1-p)}\mathbf{\Sigma}_{\mathbf{FZ}}^\top(\mathbf{\Sigma}_{\mathbf{F}}^{-1} - \mathbf{\Sigma}_{\mathbf{\Gamma}})\mathbf{\Sigma}_{\mathbf{FZ}} \geq -\frac{p}{1-p}\mathbf{\Sigma}_{\mathbf{FZ}}^\top\mathbf{\Sigma}_{\mathbf{\Gamma}}\mathbf{\Sigma}_{\mathbf{FZ}}.$$

Since $\mathbf{\Lambda}(T) = \mathbf{0}$ and we've shown that $\mathbf{\Lambda}'(t)$ is semi-negative definite, we prove $\mathbf{\Lambda}(t)$ is semi-positive definite. □

Though $\boldsymbol{g}(t), f(t)$ are given by solving an ODEs system, which may not be fully explicit, the decomposition in Lemma 4.2.4 reveals some good properties to analyze the influences for different $\boldsymbol{c}$, since those $\mathbf{\Psi}(t)$, $\boldsymbol{\beta}(t)$, $\mathbf{\Lambda}(t)$ functions are independent with $\boldsymbol{c}$. And since the ODEs for $\boldsymbol{H}(t)$ do not depend on $\boldsymbol{c}$, in other words, $\boldsymbol{H}(t)$ is the same as that in $\mathbf{\Gamma}$-Neutral case. Hence, we have the following decomposition for optimal strategy:

Corollary 4.2.1. We can decompose the optimal strategy into two parts,

$$\boldsymbol{\pi}^*(t,\boldsymbol{z}) = \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1}\boldsymbol{c} + \frac{1}{1-p}(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\Big(\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\big(\boldsymbol{g}(t) - \boldsymbol{H}(t)\boldsymbol{z}\big)\Big)$$
$$= \underbrace{\frac{1}{1-p}(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\Big(\boldsymbol{\theta} - \boldsymbol{K}(t)\boldsymbol{z} + \boldsymbol{\Sigma}_{\mathbf{FZ}}\big(\boldsymbol{g}_0(t) - \boldsymbol{H}(t)\boldsymbol{z}\big)\Big)}_{\boldsymbol{\Gamma}\text{-Neutral holding position}}$$
$$+ \underbrace{\Big(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}_{\boldsymbol{\Gamma}}^{-1} + \frac{1}{1-p}(\boldsymbol{\Sigma}_{\mathbf{F}}^{-1} - \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}})\boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{\Phi}(t)\Big)\boldsymbol{c}}_{\text{hedging demand for } \boldsymbol{c}}.$$

The first component is exactly the optimal strategy corresponds to $\boldsymbol{\Gamma}$-Neutral constraint. And the remaining component has linear dependence on $\boldsymbol{c}$, which are called hedging demand as it is extra holding positions required for general linear constraints.

As we can see, the admissible strategies for constrained cases and non-constrained case may vary. But we can observe that the optimal strategy in constrained case is also admissible in unconstrained case, i.e the intuitive comparison relationship between their value functions $V \leq V^{no}$ holds. But in our paper, we provide a direct way to look into the relationship for candidates of value functions between the constrained HJB solution $u(t,\boldsymbol{z},x)$ and the unconstrained HJB solution $u^{no}(t,\boldsymbol{z},x)$:

Proposition 4.2.1. Let's define the auxiliary functions below,

$$\widetilde{\boldsymbol{H}}(t) = \boldsymbol{H}^{no}(t) - \boldsymbol{H}(t), \widetilde{\boldsymbol{g}}(t) = \boldsymbol{g}^{no}(t) - \boldsymbol{g}(t), \widetilde{f}(t) = f^{no}(t) - f(t),$$

then we obtain the equation between $u(t,\boldsymbol{z},x)$ and $u^{no}(t,\boldsymbol{z},x)$:

$$u(t,\boldsymbol{z},x) = u^{no}(t,\boldsymbol{z},x)\exp\left(\frac{1}{2}\left(\boldsymbol{z}^\top, 1\right)\begin{pmatrix}\widetilde{\boldsymbol{H}}(t), & -\widetilde{\boldsymbol{g}}(t)\\ -\widetilde{\boldsymbol{g}}(t)^\top, & -2\widetilde{f}(t)\end{pmatrix}\begin{pmatrix}\boldsymbol{z}\\ 1\end{pmatrix}\right).$$

The matrix $\begin{pmatrix}\widetilde{\boldsymbol{H}}(t), & -\widetilde{\boldsymbol{g}}(t)\\ -\widetilde{\boldsymbol{g}}(t)^\top, & -2\widetilde{f}(t)\end{pmatrix}$ is semi-positive definite. Therefore,

$$u(t,\boldsymbol{z},x) \leq u^{no}(t,\boldsymbol{z},x).$$

Proof. The ODEs for the auxiliary functions $\widetilde{\boldsymbol{H}}(t) = \boldsymbol{H}^{no}(t) - \boldsymbol{H}(t), \widetilde{\boldsymbol{g}}(t) = \boldsymbol{g}^{no}(t) - \boldsymbol{g}(t)$ and $\widetilde{f}(t) = f^{no}(t) - f(t)$ can be described in multiple ways. We choose the one without involving $\boldsymbol{H}^{no}(t), \boldsymbol{g}(t)$ or $f^{no}(t)$. Precisely, we have

$$\begin{aligned}
\widetilde{\boldsymbol{H}}'(t) &= \left(\boldsymbol{K}(t) - \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}} + \boldsymbol{H}(t)\boldsymbol{\Sigma}^{no}\right)\widetilde{\boldsymbol{H}}(t)\\
&\quad + \widetilde{\boldsymbol{H}}(t)\left(\boldsymbol{K}(t) - \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}} + \boldsymbol{H}(t)\boldsymbol{\Sigma}^{no}\right)^\top\\
&\quad + \widetilde{\boldsymbol{H}}(t)\boldsymbol{\Sigma}^{no}\widetilde{\boldsymbol{H}}(t) + \frac{p}{1-p}(\boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top - \boldsymbol{K}(t))\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}(\boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{H}(t) - \boldsymbol{K}(t)),\\
\widetilde{\boldsymbol{g}}'(t) &= \boldsymbol{K}(t)\widetilde{\boldsymbol{g}}(t) + \widetilde{\boldsymbol{H}}(t)\boldsymbol{m} + \widetilde{\boldsymbol{H}}(t)\boldsymbol{\Sigma}^{no}\boldsymbol{g} + \widetilde{\boldsymbol{H}}\boldsymbol{\Sigma}^{no}\widetilde{\boldsymbol{g}} + \boldsymbol{H}\boldsymbol{\Sigma}^{no}\widetilde{\boldsymbol{g}}\\
&\quad - \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}\widetilde{\boldsymbol{g}} + \frac{p}{1-p}\widetilde{\boldsymbol{H}}\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\mu}_{\boldsymbol{F}}\\
&\quad + \frac{p}{1-p}(\boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top - \boldsymbol{K}(t))\boldsymbol{\Sigma}_{\boldsymbol{\Gamma}}(\boldsymbol{\Sigma}_{\mathbf{FZ}}\boldsymbol{g} + \boldsymbol{\mu}_{\boldsymbol{F}} - (1-p)\boldsymbol{\Sigma}_{\mathbf{F}}\boldsymbol{\pi}),
\end{aligned} \tag{4.2.34}$$

$$\begin{aligned}
\widetilde{f}'(t) = & -\boldsymbol{m}^\top \widetilde{\boldsymbol{g}}(t) + \frac{1}{2} tr\left(\boldsymbol{\Sigma}_{\mathbf{Z}} \widetilde{\boldsymbol{H}}(t)\right) \\
& - \frac{1}{2}\widetilde{\boldsymbol{g}}(t)^\top \boldsymbol{\Sigma}^{\boldsymbol{no}} \widetilde{\boldsymbol{g}}(t) - \boldsymbol{g}^\top \boldsymbol{\Sigma}^{\boldsymbol{no}} \widetilde{\boldsymbol{g}} - \frac{p}{1-p}\boldsymbol{\mu}_{\boldsymbol{F}}^\top \boldsymbol{\Sigma}_{\mathbf{F}}^{-1} \boldsymbol{\Sigma}_{\mathbf{FZ}} \widetilde{\boldsymbol{g}} - \frac{p}{2(1-p)} \times \\
& \quad \left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}} \boldsymbol{g}(t) - (1-p)\boldsymbol{\Sigma}_{\mathbf{F}} \boldsymbol{\pi}\right)^\top \boldsymbol{\Sigma}_{\boldsymbol{\Gamma}} \left(\boldsymbol{\theta} + \boldsymbol{\Sigma}_{\mathbf{FZ}} \boldsymbol{g}(t) - (1-p)\boldsymbol{\Sigma}_{\mathbf{F}} \boldsymbol{\pi}\right), \\
\widetilde{\boldsymbol{H}}'(T) = & \ \mathbf{0}_{N\times N}, \\
\widetilde{\boldsymbol{g}}'(T) = & \ \mathbf{0}_{N\times 1}, \\
\widetilde{f}'(T) = & \ 0.
\end{aligned}$$

Now, we denote $\boldsymbol{M}(t) = \begin{pmatrix} \widetilde{\boldsymbol{H}}(t), & -\widetilde{\boldsymbol{g}}(t) \\ -\widetilde{\boldsymbol{g}}(t)^\top, & -2\widetilde{f}(t) \end{pmatrix}$, and show the matrix function satisfying some proper Riccati differential equation as following by direct calculation:

$$\begin{aligned}
\boldsymbol{M}'(t) = & \boldsymbol{M}(t) \begin{pmatrix} \boldsymbol{\Sigma}^{no}, & \mathbf{0} \\ \mathbf{0}, & 0 \end{pmatrix} \boldsymbol{M}(t) \\
& + \begin{pmatrix} \boldsymbol{K}(t) + \boldsymbol{H}\boldsymbol{\Sigma}^{no} - \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}, & \mathbf{0} \\ -\boldsymbol{m}^\top - \boldsymbol{g}^\top \boldsymbol{\Sigma}^{no} - \frac{p}{1-p}\boldsymbol{\mu}_{\boldsymbol{F}}^\top \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}, & 0 \end{pmatrix} \boldsymbol{M}(t) \\
& + \boldsymbol{M}(t) \begin{pmatrix} \boldsymbol{K}(t) + \boldsymbol{H}\boldsymbol{\Sigma}^{no} - \frac{p}{1-p}\boldsymbol{K}(t)\boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}, & \mathbf{0} \\ -\boldsymbol{m}^\top - \boldsymbol{g}^\top \boldsymbol{\Sigma}^{no} - \frac{p}{1-p}\boldsymbol{\mu}_{\boldsymbol{F}}^\top \boldsymbol{\Sigma}_{\mathbf{F}}^{-1}\boldsymbol{\Sigma}_{\mathbf{FZ}}, & 0 \end{pmatrix}^\top + \begin{pmatrix} Q_1, & Q_2 \\ Q_2^\top, & Q_4 \end{pmatrix},
\end{aligned} \tag{4.2.35}$$

where

$$Q_1 = \frac{p}{1-p}(\boldsymbol{\Sigma_{FZ}}\boldsymbol{H}(t) - \boldsymbol{K}(t))^\top \boldsymbol{\Sigma_\Gamma}(\boldsymbol{\Sigma_{FZ}}\boldsymbol{H}(t) - \boldsymbol{K}(t)),$$
$$Q_2 = -\frac{p}{1-p}(\boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top - \boldsymbol{K}(t))\boldsymbol{\Sigma_\Gamma}(\boldsymbol{\Sigma_{FZ}}\boldsymbol{g} + \boldsymbol{\mu_F} - (1-p)\boldsymbol{\Sigma_F}\boldsymbol{\pi}),$$
$$Q_4 = -tr(\boldsymbol{\Sigma_Z}\widetilde{\boldsymbol{H}}) + \frac{p}{(1-p)}\left(\boldsymbol{\theta} + \boldsymbol{\Sigma_{FZ}}\boldsymbol{g}(t) - (1-p)\boldsymbol{\Sigma_F}\boldsymbol{\pi}\right)^\top \boldsymbol{\Sigma_\Gamma}\left(\boldsymbol{\theta} + \boldsymbol{\Sigma_{FZ}}\boldsymbol{g}(t) - (1-p)\boldsymbol{\Sigma_F}\boldsymbol{\pi}\right).$$

Moreover, (4.2.34) implies that $\widetilde{\boldsymbol{H}}$ is positive semi-definite. Therefore, there exists matrix B such that $\widetilde{\boldsymbol{H}} = B^\top B$, then $-tr(\boldsymbol{\Sigma_Z}\widetilde{\boldsymbol{H}}) = -tr(B\boldsymbol{\Sigma_Z}B^\top) \leq 0$. Combining $p < 0$, we know that

$$\begin{pmatrix} Q_1, & Q_2 \\ Q_2^\top, & Q_4 \end{pmatrix} = \begin{pmatrix} \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & -tr(\boldsymbol{\Sigma_Z}\widetilde{\boldsymbol{H}}) \end{pmatrix} + \frac{p}{1-p}\boldsymbol{N}(t)\boldsymbol{\Sigma_\Gamma}\boldsymbol{N}(t)^\top,$$

is negative semi-definite, where $\boldsymbol{N}(t) = \begin{pmatrix} \boldsymbol{K}(t) - \boldsymbol{H}(t)\boldsymbol{\Sigma}_{\mathbf{FZ}}^\top \\ \boldsymbol{\theta}^\top + \boldsymbol{g}(t)^\top \boldsymbol{\Sigma}_{\mathbf{FZ}}^\top - (1-p)\boldsymbol{\pi}^\top\boldsymbol{\Sigma_F} \end{pmatrix}$. In addition, $\boldsymbol{\Sigma}^{no}$ is positive semi-definite. Therefore, $\boldsymbol{M}(t)$ is the corresponding unique positive semi-definite solution for Riccati equation (4.2.35). □

Next, we verify that the value function (4.2.18) coincides with the solution of HJB equation (4.2.20) from Theorem 4.2.2. We also identify the optimal trading strategy.

Theorem 4.2.3 (Verification Theorem)**.** 1. The value function in (4.2.4) is equal to the function u^{no} given in Theorem 4.2.1. Furthermore, the optimal trading strategy is given by (4.2.15).

2. The value function in (4.2.18) is equal to the function u given in Theorem 4.2.2. Furthermore, the optimal trading strategy is given by (4.2.27).

Proof. Since the unconstrained scenario is just a special constrained case where $\boldsymbol{\Gamma} = \mathbf{0}$, and $\boldsymbol{c} = \mathbf{0}$, it suffices to prove the first statement. Let u be the solution given in Theorem 4.2.2. We prove the following two assertions:

(a) With any admissible strategy $\boldsymbol{\pi} \in \mathcal{A}$, satisfying linear constraints $\boldsymbol{\pi}^\top \boldsymbol{\Gamma} = \boldsymbol{c}$, we have

$$u(t, \boldsymbol{z}, x) \geq \mathbb{E}_{t,\boldsymbol{z},x}[U\left(X_T^{\boldsymbol{\pi}}\right)],$$

for all $(t, \boldsymbol{z}, x) \in [0, T] \times \mathbb{R}^N \times D$, where $\mathbb{E}_{t,\boldsymbol{z},x}[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot | X_t^{\boldsymbol{\pi}} = x, \boldsymbol{Z}_t = \boldsymbol{z}]$ and $X_T^{\boldsymbol{\pi}}$ is the terminal wealth.

(b) There exists an admissible strategy $\boldsymbol{\pi}^* \in \mathcal{A}$, satisfying $\boldsymbol{\pi}^{*\top} \boldsymbol{\Gamma} = \boldsymbol{c}$ such that

$$u(t, \boldsymbol{z}, x) = \mathbb{E}_{t,\boldsymbol{z},x}[U\left(X_T^{\boldsymbol{\pi}^*}\right)],$$

for $(t, \boldsymbol{z}, x) \in [0, T] \times \mathbb{R}^N \times D$.

Combining the above two statements, (a) implies $u \geq V$, and (b) implies $u \leq V$, therefore, $u = V$ as desired.

(a) Given $\pi \in \mathcal{A}$, satisfying the constraint $\boldsymbol{\Gamma}^\top \pi = \boldsymbol{c}$, we apply Ito's formula to

$u(t, \boldsymbol{z}, x)$, to get

$$\begin{aligned}
\mathrm{d}u(t, \boldsymbol{Z}_t, X_t^{\boldsymbol{\pi}}) = &\Big\{ u_t + (\mathbf{m} - \boldsymbol{K}(t)\boldsymbol{Z}_t)\nabla_{\boldsymbol{z}} u + \frac{1}{2} tr(\boldsymbol{\Sigma}_{\mathbf{Z}} \nabla_{\mathbf{z}}^2 u) + rxu_x \\
&+ \boldsymbol{\pi}^\top (\boldsymbol{\mu}_{\mathbf{F}} + \boldsymbol{K}(t)\boldsymbol{Z}_t) x u_x + \boldsymbol{\pi}^\top \boldsymbol{\Sigma}_{\mathbf{FZ}} \nabla_{\boldsymbol{z}}(u_x) x + \frac{1}{2} \boldsymbol{\pi}^\top \boldsymbol{\Sigma}_{\mathbf{F}} \boldsymbol{\pi} x^2 \partial_{xx} u \Big\} \mathrm{d}t \\
&+ (\nabla_{\boldsymbol{z}} u)^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}} \mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{Z}} \mathrm{d}\boldsymbol{W}_{t,2} \right) + x u_x \boldsymbol{\pi}^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}} \mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}} \mathrm{d}\boldsymbol{W}_{t,2} \right).
\end{aligned}$$

According to the HJB equation (4.2.20) for $u(t, \boldsymbol{z}, x)$, we have

$$\begin{aligned}
\mathrm{d}u(t, \boldsymbol{Z}_t, X_t^{\boldsymbol{\pi}}) &\leq (\nabla_{\boldsymbol{z}} u)^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}} \mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{Z}} \mathrm{d}\boldsymbol{W}_{t,2} \right) \\
&\quad + x u_x \boldsymbol{\pi}^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}} \mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}} \mathrm{d}\boldsymbol{W}_{t,2} \right) \\
&= u(t, \boldsymbol{Z}_t, X_t^{\boldsymbol{\pi}}) \Big((\boldsymbol{g}(t) - \boldsymbol{H}(t)\boldsymbol{Z}_t)^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}} \mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\boldsymbol{Z}} \mathrm{d}\boldsymbol{W}_{t,2} \right) \\
&\quad + p \boldsymbol{\pi}^\top \left(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}} \mathrm{d}\boldsymbol{W}_{t,1} + \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}} \mathrm{d}\boldsymbol{W}_{t,2} \right) \Big) \\
&= u(t, \boldsymbol{Z}_t, X_t^{\boldsymbol{\pi}}) \mathrm{d}\boldsymbol{Y}_t^{\boldsymbol{\pi}}.
\end{aligned}$$

This results in the inequality:

$$U\left(X_T^{\boldsymbol{\pi}}\right) \leq u(t, \boldsymbol{Z}_t, X_t^{\boldsymbol{\pi}}) \mathcal{E}\left(\boldsymbol{Y}_\cdot^{\boldsymbol{\pi}} - \boldsymbol{Y}_t^{\boldsymbol{\pi}}\right)_T.$$

Taking the conditional expectation for both sides completes the proof of (a). Moreover, the equality holds when $\boldsymbol{\pi} = \boldsymbol{\pi}^*$.

(b) It suffices to show that $\boldsymbol{\pi}^*$ is admissible. We combine the integral form of $\boldsymbol{Z}_t$ according to Remark 3.5 in Angoshtari and Leung (2020) and integration by part technique to check that $\boldsymbol{Z}_t$ satisfies Benes condition below, thus, the corresponding integrand of $\boldsymbol{Y}^{\boldsymbol{\pi}^*}$ also satisfies Benes condition. That is, there exists some constant K such that

$$\begin{aligned}
\|(\boldsymbol{Z}_t^\top \boldsymbol{H}(t) + \boldsymbol{g}(t)^\top)\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{ZS}}\|_{L^1} &\leq K(1 + \max_{0\leq s\leq t} \|(\boldsymbol{W}_{s,1}, \boldsymbol{W}_{s,2})\|_{L^1}),\\
\|(\boldsymbol{Z}_t^\top \boldsymbol{H}(t) + \boldsymbol{g}(t)^\top)\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{Z}}\|_{L^1} &\leq K(1 + \max_{0\leq s\leq t} \|(\boldsymbol{W}_{s,1}, \boldsymbol{W}_{s,2})\|_{L^1}),\\
\|\boldsymbol{\pi}_t^{*\top}\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{FS}}\|_{L^1} &\leq K(1 + \max_{0\leq s\leq t} \|(\boldsymbol{W}_{s,1}, \boldsymbol{W}_{s,2})\|_{L^1}),\\
\|\boldsymbol{\pi}_t^{*\top}\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{F}}\|_{L^1} &\leq K(1 + \max_{0\leq s\leq t} \|(\boldsymbol{W}_{s,1}, \boldsymbol{W}_{s,2})\|_{L^1}),
\end{aligned}$$

where $\boldsymbol{\pi}^*$ is given by (4.2.27). See Beneš (1971) or p.200 in Karatzas and Shreve (1991), which verifies the admissibility conditions.

□

4.3 Certainty Equivalent

We can interpret value functions from another perspective by its corresponding certainty equivalent (CE). We denote by $CE(t, \boldsymbol{z}, x)$ ($CE^{no}(t, \boldsymbol{z}, x)$ for unconstrained case, respectively) the certainty equivalent value of the trader at the state $(t, \boldsymbol{z}, x)$, which is defined as following:

Definition 4.3.1 (Certainty Equivalent(CE))**.** Certainty equivalent is the guaranteed cash amount that would yield the same utility as that from dynamically trading futures according to (4.2.18). This amounts to applying the inverse of the utility function to the value function. Precisely, we have

$$CE(t,\boldsymbol{z},x) = x\exp\left(-\frac{1}{2p}\boldsymbol{z}^\top \boldsymbol{H}(t)\boldsymbol{z} + \frac{1}{p}\boldsymbol{z}^\top \boldsymbol{g}(t) + \frac{1}{p}f(t)\right),$$

specifically, for $\boldsymbol{\Gamma}-$Neutral case, we denote

$$CE_0(t,\boldsymbol{z},x) = x\exp\left(-\frac{1}{2p}\boldsymbol{z}^\top \boldsymbol{H}_0(t)\boldsymbol{z} + \frac{1}{p}\boldsymbol{z}^\top \boldsymbol{g}_0(t) + \frac{1}{p}f_0(t)\right),$$

and for unconstrained case,

$$CE^{no}(t,\boldsymbol{z},x) = x\exp\left(-\frac{1}{2p}\boldsymbol{z}^\top \boldsymbol{H}^{no}(t)\boldsymbol{z} + \frac{1}{p}\boldsymbol{z}^\top \boldsymbol{g}^{no}(t) + \frac{1}{p}f^{no}(t)\right).$$

where $\boldsymbol{g}_0, f_0$ are defined in Lemma 4.2.4 ,and $\boldsymbol{H}_0(t) = \boldsymbol{H}(t)$ as explained.

After introducing certainty equivalent to measure the dynamically trading value for both cases, we can rewrite the relation given in Theorem 4.2.1 to be with certainty equivalent.

Corollary 4.3.1. We have the following equality between constrained and unconstrained case:

$$CE(t,\boldsymbol{z},x) = CE^{no}(t,\boldsymbol{z},x)\exp\left(\frac{1}{2p}\left(\boldsymbol{z}^\top, 1\right)\begin{pmatrix} \widetilde{\boldsymbol{H}}(t), & -\widetilde{\boldsymbol{g}}(t) \\ -\widetilde{\boldsymbol{g}}(t)^\top, & -2\widetilde{f}(t)\end{pmatrix}\begin{pmatrix}\boldsymbol{z} \\ 1\end{pmatrix}\right).$$

Obviously, $CE(t,\boldsymbol{z},x) \leq CE^{no}(t,\boldsymbol{z},x)$.

Theorem 4.3.1. We can decompose the certainty equivalent into two parts,

$$CE(t, \boldsymbol{z}, x) = \underbrace{CE_0(t, \boldsymbol{z}, x)}_{\boldsymbol{\Gamma}\text{-Neutral CE}} \underbrace{\exp\left(\frac{1}{p}\boldsymbol{z}^\top \boldsymbol{\Psi}(t)\boldsymbol{c} + \frac{1}{p}\boldsymbol{\beta}(t)\boldsymbol{c} + \frac{1}{p}\boldsymbol{c}^\top \boldsymbol{\Lambda}(t)\boldsymbol{c}\right)}_{\text{opportunity multiplier by } \boldsymbol{c}}.$$

The first part in the certainty equivalent corresponds to $\boldsymbol{\Gamma}$-Neutral case, and the second part is a multiplier in quadratic form w.r.t $\boldsymbol{c}$.

Proof. Plugging in the decomposition given in Lemma 4.2.4 for $\boldsymbol{g}$ and $\boldsymbol{f}$, we can directly get the decomposition for certainty equivalents as above. □

Now we are capable of seeking the best investment proportion at time t, with investment horizon T. By Theorem 4.3.1, the second part is exponential of a quadratic form, whose convexity is guaranteed by semi-positiveness of $\boldsymbol{\Lambda}(t)$ and $p < 0$. So the corresponding cash equivalent $CE(t, \boldsymbol{z}, x)$ has a global maximum.

Definition 4.3.2. The $\boldsymbol{c}$ for maximizing the certainty equivalent in power utility is denoted by

$$\boldsymbol{c}^*(t, \boldsymbol{z}, x) = \arg\max_{\boldsymbol{c} \in \mathbb{R}^d} \; CE(t, \boldsymbol{z}, x).$$

If $\boldsymbol{\Lambda}(t)$ is strictly positive, then $\boldsymbol{c}^*(t)$ is unique.

With Theorem 4.3.1, seeking the optimal $\boldsymbol{c}$ is equivalent to maximize the 'opportunity multiplier generated by $\boldsymbol{c}$' part. And if $\boldsymbol{\Lambda}(t)$ is strictly positive, then the formula for the unique $\boldsymbol{c}^*$ is given by

$$\boldsymbol{c}^*(t, \boldsymbol{z}, x) = -\boldsymbol{\Lambda}(t)^{-1}\left(\Psi(t)^\top \boldsymbol{z} + \boldsymbol{\beta}(t)^\top\right).$$

Remark 4.3.1. 1. $\boldsymbol{c}^*$ does not depend on the initial wealth x. Moreover, it has linear dependence on the basis $\boldsymbol{z}$.

2. When $\boldsymbol{\Gamma} = \mathbf{1}$, then $\boldsymbol{c}$ means the constraint on investment fraction of the wealth. In this case, $\boldsymbol{c}^*$ is the optimal proportion of investment set up at the beginning to maximize the corresponding certainty equivalent, which is the counterpart of Kelly Criterion in power utility optimization.

We provide some numerical examples for certainty equivalents in Section 4.4.

4.4 Numerical Illustration

In this section, we simulate the asset's spot prices, futures prices and optimal strategies for our basis model. We also generate empirical wealth distribution at terminal time and certainty equivalents for different constraints and parameters. Primarily, we consider a market with two different assets S_1 and S_2 and three futures that $F_{1,1}$ is written on S_1, while $F_{2,1}$ and $F_{2,2}$ are written on S_2. Their maturities are $T_{1,1} = T_{2,1} = 2/12$ year and $T_{2,2} = 3/12$ year, respectively. Then, our trading horizon is set to be $T = 1/12$ year, strictly less than the futures maturities. We use 'months' or 'trading day' as the x axis in figures. For clarification, we assume there be 252 trading days in a year and 21 trading days for a month. Therefore, our trading horizon is 21 trading days in total. For other model parameters, we list them as follows.

$$r = 0.01, \quad \boldsymbol{\mu} = (0.1, 0.2)^\top, \quad \boldsymbol{m} = (0,0,0)^\top, \quad (\kappa_{1,1}, \kappa_{2,1}, \kappa_{2,2}) = (0.5, 0.5, 0.5),$$

$$\widetilde{\boldsymbol{\Sigma}}_{\mathbf{S}} = \begin{pmatrix} 0.5 & 0 \\ 0.3 & 0.4 \end{pmatrix}, \quad \widetilde{\boldsymbol{\Sigma}}_{\mathbf{ZS}} = \begin{pmatrix} -0.25 & 0 \\ -0.15 & -0.2 \\ -0.15 & -0.2 \end{pmatrix}, \quad \widetilde{\boldsymbol{\Sigma}}_{\mathbf{Z}} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}.$$

We obtain the values for $\boldsymbol{\theta}$, $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}$ and $\widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}}$ by equations (4.1.2), (4.1.5) and (4.1.6):

$$\boldsymbol{\theta} = \begin{pmatrix} 0.17625 \\ 0.26625 \\ 0.22625 \end{pmatrix}, \quad \widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}} = \begin{pmatrix} 0.25 & 0 \\ 0.15 & 0.2 \\ 0.15 & 0.2 \end{pmatrix}, \quad \widetilde{\boldsymbol{\Sigma}}_{\mathbf{F}} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}.$$

In Figure 4.1, we show a set of simulated paths for asset price $\boldsymbol{S}_t$, futures price $\boldsymbol{F}_t$ and log-bases $\boldsymbol{Z}_t$. In the top figure, we plot price paths for asset S_1 and its 2-month futures $F_{1,1}$. The price paths for asset S_2 and its two futures ($F_{2,1}$ and $F_{2,2}$) are presented in the middle figures. The bottom figure shows the simulated paths for log-bases Z_t. The initial prices for both assets are \$10. The log-bases $\boldsymbol{Z}_t$ is a multidimensional Brownian bridge that starts from $\boldsymbol{Z}_0 = (0.02, 0.02, 0.02)^\top$ and converges to zero, which guarantees that each futures price is equal to corresponding asset's price at its maturity, which are $T_{1,1} = T_{2,1} = 2$ months and $T_{2,2} = 3$ months, respectively. We show the optimal strategies for multiple portfolios in Figure 4.2. In the left panels, we illustrate optimal unconstrained strategy and optimal dollar neutral strategy for three-futures portfolio. For the dollar neutral strategy, the sum

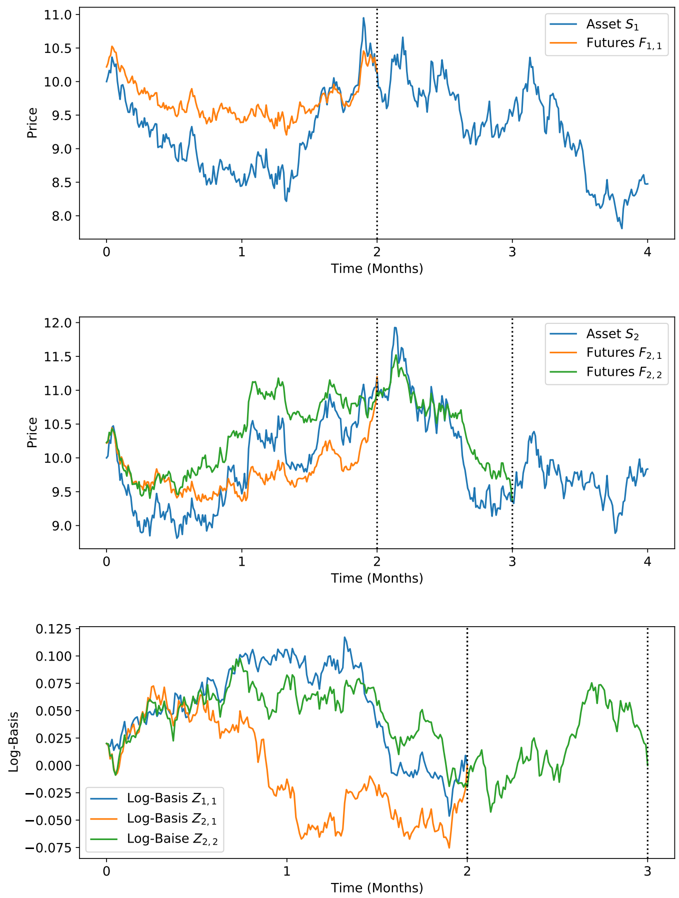

Figure 4.1: Simulated path for assets prices $\boldsymbol{S}_t$, futures prices $\boldsymbol{F}_t$ and log-bases $\boldsymbol{Z}_t$. Top: asset S_1 with its 2-month futures $F_{1,1}$. Middle: asset S_2 with its 2-month futures $F_{2,1}$ and 3-month futures $F_{2,2}$. Bottom: log-basis $\boldsymbol{Z}_t$. Initial value: $\boldsymbol{S}_0 = (10, 10)^\top$ and $\boldsymbol{Z}_0 = (0.02, 0.02, 0.02)^\top$.

of positions for all futures has to be zero. The two sets of strategy look alike, which reasons the comprehensive use of dollar neutral strategy in the industry. Also, we observe that it always put opposite positions between two sets of futures, i.e. $F_{2,1}$ vs $F_{1,1}$ and $F_{2,2}$, which shows that the long-short strategy is optimal. Same phenomenon can be observed from the right panels as well, which presents the optimal strategies for two-futures portfolios, consisted of $F_{1,1}$ and $F_{2,1}$. In the bottom left figure, due to the dollar neutral constraint, the investor is forced to invest equal fraction of wealth in opposite positions on two futures. Besides, in our model, the optimal strategies depend on the log-bases $\boldsymbol{Z}_t$, unlike the optimal strategies obtained by Leung and Yan (2018, 2019), which are time-deterministic.

In Figure 4.3, we present empirical distributions of terminal wealth for different strategy constraints, different number of futures and different risk parameter p. Like Figure 4.2, we use the combination of $F_{1,1}$ and $F_{2,1}$ as the two-futures portfolio. Recall that the market neutral strategy for the two-futures portfolio is zero, we do not present it in second and fourth figures. Among three strategies, the market neutral strategy is the most centralized, while no constraint strategy is the most diversified with the heaviest tail. It essentially reflects that no constraint strategy has highest degree of freedom and the investor is more risk when no constraint is posted. For another thing, the terminal wealth distribution for more risk averse investor $(p = -1)$, is more centralized than those for less risk averse investor $(p = -0.5)$, see top 2 figures vs lower 2 figures. It is because the optimal strategy π^* is inversely proportional to $1-p$ according to Theorem 4.2.2. Therefore, less risk averse investor is likely to bet more on futures, which leads to more diversification for the associated

Figure 4.2: Optimal strategies.

Top left: unconstrained three-futures portfolio. Top right: unconstrained two-futures portfolio. Bottom left: dollar neutral three-futures portfolio. Bottom right: dollar neutral two futures portfolio.

wealth distribution.

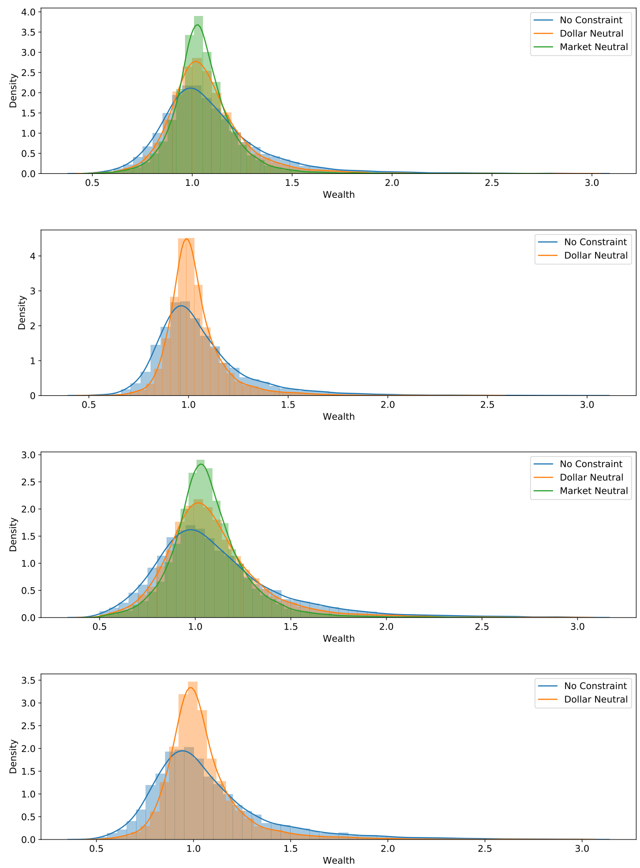

Figure 4.3: The distribution of terminal wealth.
From top to bottom: (i) three-futures portfolio with $p = -1$; (ii) two-futures portfolio with $p = -1$; (iii) three-futures portfolio with $p = -0.5$; (iv) two-futures portfolio with $p = -0.5$.

We also provide the averages, standard deviations and quartiles for distributions in Figure 4.3 in the Table 4.1. We present them as the annualized average log-return, annualized standard deviation and three quartiles. We also present the Sharpe ratio for each portfolio, which is ratio between annualized average log-return and annualized standard deviation. The statistics for more risk averse investor ($p = -1$) is shown in the upper table, while the other one ($p = -0.5$) is shown in lower table. As discuss in the Figure 4.3, the no constraint strategy is most profitable and also most risky, while the market neutral strategy is the most conservative and stable. Take the three-futures portfolio with $p = -1$ as an example. The no constraint strategy achieves 82.14% average log-return with 79.66% standard deviation. On the contrary, the market neutral strategy brings 36.11% average log-return and bears 42.52% standard deviation. The average log-return and standard deviation for dollar neutral strategy lie in the middle, which are 66.58% and 61.36%, respectively. However, in terms of the Sharpe ratio, the dollar neutral strategy performs best in this portfolio. It achieves 1.07, which is at least 0.05 higher than other two strategies. When $p = -0.5$, the dollar neutral strategy also performs the best for three-futures portfolio. It achieves 0.97, while the other two are 0.89 and 0.74 respectively. Besides, compared to the upper table ($p = -1$), the less risk averse investor ($p = -0.5$) has higher average return, bears higher volatility but achieves lower Sharpe ratio. Take two-futures portfolios as an example. The less risk averse investor could achieve 43.75% for average log-return and 63.26% for standard deviation under dollar neutral constraint, while the more risk averse investor only gets 35.93% and 47.40% for average log-return and standard deviation respectively. However, the more risk averse investor obtains 0.74 for Sharpe ratio in the dollar neutral strategy, which is 0.06 higher than the Sharpe for less risk averse investor with dollar neutral constraint.

Next, we show some examples for certainty equivalents. In Figure 4.4, we plot the certainty equivalents for dollar constraint portfolios, where the strategy sum $\mathbf{1}_{N\times 1}^{\top}\boldsymbol{\pi}$ is set to be a fixed parameter c and N is the number of futures in the

$p=-1$	3 Futures			2 Futures	
	NC	DN	MN	NC	DN
Annualized Average Log-Return	82.14%	66.58%	36.11%	56.42%	35.93%
Annualized Standard Deviation	79.66%	61.36%	42.52%	71.54%	47.40%
Lower Quartile	-93.53%	-57.04%	-34.76%	-103.42%	-56.49%
Median	53.84%	55.04%	37.36%	17.28%	11.33%
Upper Quartile	231.52%	178.04%	113.81%	181.31%	96.84%
Sharpe Ratio	1.02	1.07	0.83	0.77	0.74

$p=-0.5$	3 Futures			2 Futures	
	NC	DN	MN	NC	DN
Annualized Average Log-Return	96.33%	80.01%	42.59%	65.68%	43.75%
Annualized Standard Deviation	106.61%	81.45%	55.87%	96.03%	63.26%
Lower Quartile	-137.66%	-81.51%	-48.50%	-148.12%	-79.03%
Median	61.69%	66.27%	46.98%	17.18%	13.37%
Upper Quartile	297.97%	228.26 %	144.95%	234.45%	126.98%
Sharpe Ratio	0.89	0.97	0.74	0.67	0.68

Table 4.1: Annualized average log-return, annualized standard deviation, Sharpe ratio and quartiles for wealth distributions in Figure 4.3.
'NC', 'DN' and 'MN' stand for 'no constraint', 'dollar neutral' and 'market neutral', respectively. Upper: more risk averse investor ($p=-1$). Lower: less risk averse investor ($p=-0.5$).

portfolio. Specifically, when $c=0$, it is dollar neutral strategy. We show the certainty equivalents for more risk averse investor ($p=-1$) on the left and the certainty equivalents for less risk averse parameter ($p=-0.5$) on the right. The green and red curves represent the certainty equivalents for three-futures portfolio and two-futures portfolio, respectively. Since, with respect to dollar constraint, the admissible strategy for two-futures portfolio is always the admissible strategy for three-futures portfolio, the three-futures portfolio's certainty equivalent will be always larger than two-futures portfolio's certainty equivalent as shown in the figures. For comparison, we use the blue dashed lines marking the certainty equivalent for no constraint three-futures portfolio and naturally it is larger than the certainty equivalent for any other portfolios. We also provide the dark dashed lines as the initial wealth for each portfolio. If portfolio's certainty equivalent is lower than its initial wealth, then it is not worthy to trade. Most importantly, we mark down the optimal parameter c^* for each portfolio by crosses. For more risk averse investor, the best parameters c^* for three-futures and two-futures portfolios are $c_1^* = 0.763$ and $c_2^* = 0.712$, which bring best certainty equivalents $CE_1^* = 1.063$ and $CE_2^* = 1.037$, respectively. On the contrary, for the less risk averse investor, they could achieve best certainty equivalents $CE_3^* = 1.084$ and $CE_4^* = 1.052$ for two portfolios by letting constraint parameter be $c_3^* = 1.103$ and $c_4^* = 1.018$, respectively. Naturally, the less risk averse investor could achieve higher certainty equivalent than more risk averse investor for each portfolio.

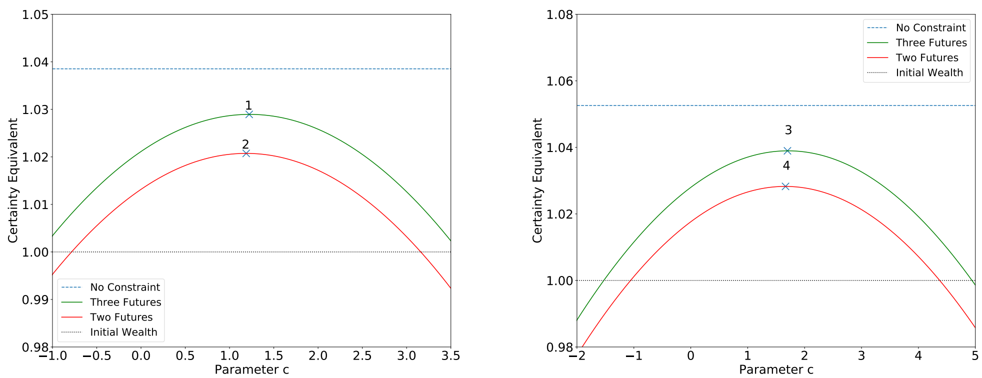

Figure 4.4: Certainty equivalent (CE) as the function of constraint parameter c. Left: the risk parameter $p = -1$. Right: the risk parameter $p = -0.5$. The blue dashed lines show the certainty equivalents for three-futures portfolio with no constraints. The green curves show the CE for three-futures portfolio with budget constraint. The red curves show the CE for two-futures portfolio with budge constraint. The black dashed line represents the initial wealth. The crosses mark the optimal parameter c^* and maximum certainty equivalent $CE*$. Optimal parameters: $c_1^* = 0.763$, $c_2^* = 0.712$, $c_3^* = 1.103$ and $c_4^* = 1.018$. Best certainty equivalent: $CE_1^* = 1.063$, $CE_2^* = 1.037$, $CE_3^* = 1.084$ and $CE_4^* = 1.052$.

In Figure 4.5, we present the certainty equivalents as functions of parameter $\boldsymbol{c}$ for market constraint three-futures portfolios with different parameter p. For the market constraint, the strategy satisfies $(\widetilde{\boldsymbol{\Sigma}}_{\mathbf{FS}}\widetilde{\boldsymbol{\Sigma}}_{\mathbf{S}}^{\top})^{\top}\boldsymbol{\pi} = \boldsymbol{c}$ for some fixed $\boldsymbol{c} \in \mathbb{R}^{M\times 1}$, where $M = 2$ is the number of assets in our example. On the left, we show the certainty equivalent for more risk averse investor ($p = -1$), while on the right, we show the certainty equivalent for less risk averse investor ($p = -0.5$). The dashed contours denote the certainty equivalent equal to the initial wealth $X_0 = 1$. For portfolio with constraints lies in the contour, it is worth to trade, since its certainty equivalent is higher than the initial wealth. Moreover, the contour region for less risk averse investor is larger than the contour region for more risk averse investor, which also reflects that the less risk averse investor could achieve higher certainty equivalent with same portfolio. As for the optimal parameters $\boldsymbol{c}^*$ and best certainty equivalent CE^*, we mark them by crosses that investors could achieve $CE_1^* = 1.020$ and $CE_2^* = 1.026$ by letting the constraint parameter be $\boldsymbol{c}_1^* = (0.091, 0.152)^\top$ and $\boldsymbol{c}_2^* = (0.131, 0.212)^\top$, respectively.

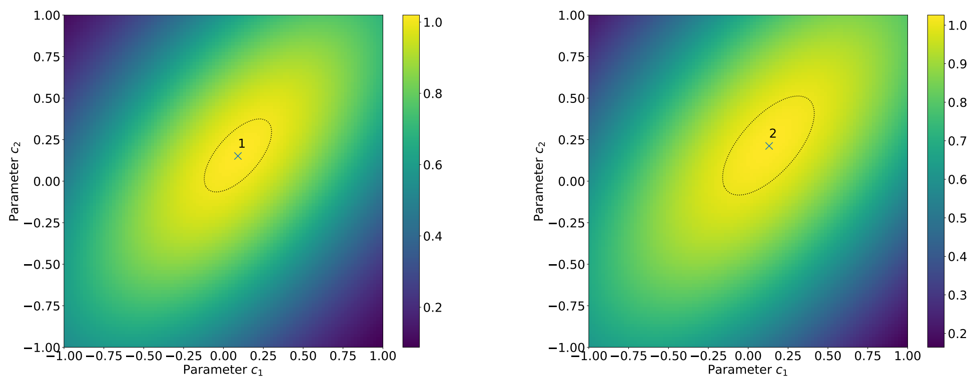

Figure 4.5: Certainty equivalents (CE) for market-constraint three-futures portfolios with different risk parameter p.

Left: the risk parameter $p = -1$. Right: the risk parameter $p = -0.5$. The dashed contours denote the certainty equivalent equal to the initial wealth $X_0 = 1$ and the optimal parameters $\boldsymbol{c}^*$ are marked down by crosses. Optimal parameter: $\boldsymbol{c}_1^* = (0.091, 0.152)^\top$ and $\boldsymbol{c}_2^* = (0.131, 0.212)^\top$. Best certainty equivalent: $CE_1^* = 1.020$ and $CE_2^* = 1.026$.

Appendices

APPENDIX A

ADMISSIBLE CONTROLS AND CONTROLLED STATE PROCESSES

A.1 Admissible Controls

The *control set* A is a Borel subset of $\mathbb{R}^d$, and $\mathcal{U}$ is the set of all progressively measurable processes. The *admissible controls* $\mathcal{A} \subset \mathcal{U}$ is composed with L^2-integrable controls $\alpha : [0, T] \times \Omega \to A$, satisfying the following two conditions.

Given α_1 and α_2 in $\mathcal{A}$ and a stopping time $\tau \in \mathcal{T}$, we define the τ-concatenation of α_1 and α_2 by:

$$\alpha_1 \overset{\tau}{\oplus} \alpha_2 := \alpha_1 \mathbf{1}_{[0,\tau)} + \alpha_2 \mathbf{1}_{[\tau,T]}.$$

A. *Stability under concatenation:*

$$\alpha_1 \overset{\tau}{\oplus} \alpha_2 \in \mathcal{A}, \text{ for all } \alpha_1, \alpha_2 \in \mathcal{A}, \tau \in \mathcal{T}.$$

B. *Stability under measurable selection:* For any $\tau \in \mathcal{T}$, and any measurable map

$$\phi : (\Omega, \mathcal{F}_\tau) \to (\mathcal{A}, \mathcal{B}_{\mathcal{A}}),$$

there exists $\alpha \in \mathcal{A}$ such that

$$\phi = \alpha \quad \text{on } [\tau, T] \times \Omega, \;\; Leb \times \mathbb{P} \; a.s.$$

Lemma A.1.1. Suppose that $\mathcal{A}$ is a separable metric space. Then condition **B** holds.

Proof. The key point is to show that for any ϕ given above, define

$$\alpha(t,\omega) := \phi(\omega)(t,\omega)\mathbf{1}_{t\geq\tau}(\omega) + \tilde{\alpha}_t(\omega)\mathbf{1}_{t<\tau}(\omega), \tag{A.1.1}$$

for some $\tilde{\alpha} \in \mathcal{A}$ is progressively measurable.

We can divide the proof into two steps:

(a) For simple function $\phi = \sum_{k=1}^{\infty} \alpha_k \mathbf{1}_{B_k}$ for some $\alpha_k \in \mathcal{A}$ and disjoint subsets $B_k \in \mathcal{F}_\tau$. To show that the concatenated α_k given by (A.1.1) is progressively measurable, it suffices to show that for any $t \in [0,T]$, and Borel subset $B \subset A$, we have

$$U := \{(s,\omega) \in [0,t] \times \Omega |\ \alpha(s,\omega) \in B\} \tag{A.1.2}$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable. Let

$$U_k := \{(s,\omega) \in [0,t] \times B_k |\ \tau \leq s \text{ and } \alpha_k(s,\omega) \in A\},\ \ k \geq 1,$$

and

$$U_0 := \{(s,\omega) \in [0,t] \times \Omega |\ s < \tau \text{ and } \tilde{\alpha}_k(s,\omega) \in A\},$$

then U_k is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable, for all $k \geq 0$.

Then $U = \cup_{k\geq 0} U_k$ is also $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

(b) Since $\mathcal{A}$ is separable metric space, for any given map ϕ, we can approximate it by simple functions ϕ_k, then the corresponding concatenated α_k given by (A.1.1), is progressively measurable by (a), also goes to the one for ϕ, should also be progressively measurable.

The separability discussion on $\mathcal{A}$ can be found p.8 in Soner and Touzi (2003). The following two lemmas are crucial in proving the existence of ε-optimal control in Theorem 2.1.1. The proofs can be found in Bertsekas and Shreve (1978). □

Lemma A.1.2 (**Von Neumann** Measurable Selection Theorem)**.** Let X and Y be Borel sets and A an analytic subset of $X \times Y$. And the projection from A to X defined as

$$\pi_X(A) := \{x \in X | \; \exists y \in Y \text{ such that } (x, y) \in A\}. \tag{A.1.3}$$

Then there exists a analytically measurable function

$$\phi : \pi_X(A) \to Y \tag{A.1.4}$$

such that

$$\mathrm{Gr}(\phi) := \{(x, \phi(x)) | x \in \pi_X(A))\} \subset A. \tag{A.1.5}$$

Lemma A.1.3. Let X be a Polish space, then every Borel subset is analytic, and every analytic subset is universally measurable, i.e. $\mathcal{B}(X) \subset \mathcal{A}(X) \subset \mathcal{BU}(X)$.

Lemma A.1.4 (ε-optimal control)**.** For any product measure μ on $[0, T] \times \mathbb{R}^m$, induced by a Lebesgue measure on $[0, T]$ and probability measure on $\mathbb{R}^m$, there exists a Borel measurable function

$$\phi_\mu : ([0,T] \times \mathbb{R}^m, \mathcal{B}([0,T] \times \mathbb{R}^m)) \to (\mathcal{A}, \mathcal{B}(\mathcal{A})), \tag{A.1.6}$$

such that $\phi_\mu(t,x)$ is an ε-optimal control for starting point $X(t) = x$, for each $(t,x) \in [0,T] \times \mathbb{R}^m$, μ-a.s.

Proof. The proof is divided into three steps:

(a) For any $\varepsilon > 0$, define set

$$G_\varepsilon := \{(t,x,\alpha) \in [0,T] \times \mathbb{R}^m \times \mathcal{A} |\ v(t,x) - J(t,x,\alpha) < \epsilon\}. \tag{A.1.7}$$

We want to prove G_ε is Borel measurable, hence analytic by Lemma A.1.3. It suffices to show that $J(t,x,\alpha)$ is Borel measurable, and since $v(t,x)$ is the supremum within a separable metric space $\mathcal{A}$ of $J(t,x,\alpha)$, it's also Borel measurable. And the measurability of $J(t,x,\alpha)$ is guaranteed by the measurability of α and X.

(b) Since G_ε is Borel measurable, applying Lemma A.1.2, there exists an analytically measurable function

$$\phi_\varepsilon : [0,T] \times \mathbb{R}^m \to \mathcal{A}, \tag{A.1.8}$$

such that $\mathrm{Gr}(\phi_\varepsilon) \subset G_\varepsilon$, i.e. $\phi_\varepsilon(t,x)$ is an ε-optimal control, for all $(t,x) \in [0,T] \times \mathbb{R}^m$.

(c) Let $\mathcal{P}$ be the set of all probability measures on $[0,T] \times \mathbb{R}^m$. For any $\tilde{\mu} \in \mathcal{P}$, denote $\mathcal{B}_{\tilde{\mu}}([0,T] \times \mathbb{R}^m)$ to be the $\tilde{\mu}$-completion σ-algebra of $\mathcal{B}([0,T] \times \mathbb{R}^m)$. Then the universal σ-algebra $\mathcal{U} = \cap_{\tilde{\mu} \in \mathcal{P}} \mathcal{B}_{\tilde{\mu}}([0,T] \times \mathbb{R}^m) \subset \mathcal{B}_{\mu}([0,T] \times \mathbb{R}^m)$. By Lemma A.1.3, any analytic function is also universal measurable, thus there exists a Borel measurable map $\phi_\mu = \phi_\varepsilon$, μ-a.s.

For more details, see Cerqueti (2009). □

A.2 Controlled State Processes

For $\tau \in \mathcal{T}, L^p_m(\theta)$ is the set of all p integrable, $\mathbb{R}^m$-valued random variables which are measurable with respect to $\mathcal{F}(\theta)$. We also introduce the set $\mathcal{S}$ of all pairs $(\tau, \xi) \in \mathcal{T} \times L^2_m(\theta)$. The *controlled state process* $X^{t,x,\alpha}$ is assumed to satisfy the following conditions:

E1. *Initial data:* $X^{\theta,\xi,\alpha} = 0$ on $[0,\theta)$ and $X^{\theta,\xi,\alpha}(\theta) = \xi$.

E2. *Consistency with deterministic initial data:* for all $(t,x) \in \mathcal{S}$,

$$\mathbb{E}\left(f(X_s^{\theta,\xi,\alpha})|(\theta,\xi) = (t,x)\right) = \mathbb{E}\left(X_s^{t,x,\alpha}\right)$$

for any bounded Borel function f and $s \geq t$.

E3. *Pathwise uniqueness:* for all $\tau \in \mathcal{T}$ with $\theta \leq \tau$ a.s., we have

$$X^{\theta,\xi,\alpha} = X^{\tau,\zeta,\alpha} \text{ on } [\tau, T] \text{ where } \zeta := X^{\theta,\xi,\alpha}(\tau).$$

E4. *Causality* if two admissible controls α_1, α_2 are equal between two stopping times $\theta \leq \tau$ in $\mathcal{T}$, then,

$$X^{\theta,\xi,\alpha_1} = X^{\theta,\xi,\alpha_2} \text{ on } [\theta, \tau].$$

E5. *Measurability:* the map

$$(t, x, \alpha) \in [0, T] \times \mathbb{R}^m \times \mathcal{A} \mapsto X^{t,x,\alpha}$$

is Borel measurable.

A.3 The Dynamic Programming Equation

Let $\mathcal{S}_m$ be the set of all $m \times m$ symmetric matrices, and define map $H : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{S}_m$ by

$$H(t, x, p, \gamma) := \sup_{a \in A} \left\{ b(t, x, a) \cdot p + \frac{1}{2} Tr \left(\sigma\sigma^\top(t, x, a)\gamma \right) \right\} \tag{A.3.1}$$

Proposition A.3.1. Assume the value function v in (1.2.2) is $C^{1,2}([0, T) \times \mathbb{R}^m, \mathbb{R})$, and $H(t, x, \nabla_x v, \nabla_x^2 v) < \infty$. Assume further that H is upper semicontinuous, then for $(t, x) \in [0, T) \times \mathbb{R}^m$

$$v_t + H(t, x, \nabla_x v, \nabla_x^2 v) \geq 0. \tag{A.3.2}$$

Proof. Let $(t_0, x_0) \in [0, T) \times \mathbb{R}^m$ be fixed, assume to the contrary that

$$v_t + H(t, x, \nabla_x v, \nabla_x^2 v) < 0,$$

then

1. Given $\epsilon > 0$, define a smooth function $\phi \geq v$ by

$$\phi(t, x) = v(t, x) + \varepsilon \left(|t - t_0|^2 + \|x - x_0\|^4 \right).$$

   Then $(v - \phi)(t_0, x_0) = 0, (\nabla_x v - \nabla_x \phi)(t_0, x_0) = 0, (v_t - \phi_t)(t_0, x_0) = 0$ and $(\nabla_x^2 v - \nabla_x^2 \phi)(t_0, x_0) = 0$, and it follows the continuity of H, we have

$$h(t, x) := \phi_t + H(t, x, \nabla_x \phi, \nabla_x^2 \phi) < 0, \tag{A.3.3}$$

   for small nbhd $\mathcal{N}_r$ around (t_0, x_0) in $[0, T) \times \mathbb{R}^m$.

2. Let

$$-\eta := \max_{\partial \mathcal{N}_r} (v - \phi) < 0.$$

   and for an arbitrary control $\alpha \in \mathcal{A}(t_0, x_0)$, we define the stopping time

$$\tau := \inf \left\{ t > t_0 : (t, X_t^{(t_0, x_0, \alpha)}) \notin \mathcal{N}_r \right\},$$

   then $(\tau, X_\tau^{(t_0, x_0, \alpha)}) \in \partial \mathcal{N}_r$, it follows that

$$\phi(\tau, X_\tau^{(t_0, x_0, \alpha)}) \geq \eta + v(\tau, X_\tau^{(t_0, x_0, \alpha)}). \tag{A.3.4}$$

3. Apply Itô's formula to ϕ, we have

$$
\begin{aligned}
v(t_0, x_0) &= \phi(t_0, x_0) \\
&= \mathbb{E}\left(\phi(\tau, X_\tau^{(t_0,x_0,\alpha)}) - \int_0^\tau \left(\partial_t + \mathcal{L}^{\alpha_s}\right)\phi(s, X_s^{(t_0,x_0,\alpha)})\mathrm{d}s\right) \\
&\geq \mathbb{E}\left(\phi(\tau, X_\tau^{(t_0,x_0,\alpha)})\right) \geq \eta + \mathbb{E}\left(v(\tau, X_\tau^{(t_0,x_0,\alpha)})\right)
\end{aligned}
$$

by (A.3.3) and (A.3.4). Contradicted with Corollary 2.1.1 by the arbitrariness of control α.

$\square$

APPENDIX B

KALMAN-BUCY FILTER

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given some observation process Y_t, and the filtration $(\mathcal{F}_t^Y)_{t\geq 0}$ generated by Y_t, we want to filter out the signal process X_t, with respect to $\mathcal{F}_t$, i.e. find $\hat{X}_t = \mathbb{E}\left(X_t|\mathcal{F}_t^Y\right)$. Kalman-Bucy filter is a linear filtering model dealing with Gaussian signal and linear observation function. To find more introduction on filtering theory, please see Xiong (2008).

We consider the signal-observation given by the following system

$$\begin{cases} \mathrm{d}X_t = (\tilde{b}_t + b_t X_t)\mathrm{d}t + c_t \mathrm{d}W_t + \sigma_t \mathrm{d}B_t, \\ \mathrm{d}Y_t = (\tilde{h}_t + h_t X_t)\mathrm{d}t + \mathrm{d}W_t, Y_0 = 0, \end{cases} \tag{B.0.1}$$

where X_0 is a normal random vector with mean $\hat{X}_0$ and covariance matrix $\gamma_0 \in \mathbb{R}^d \times \mathbb{R}^d$, (W_t, B_t) is an $m{+}d$-dimensional Brownian motion, the coefficients $\tilde{b}_t, b_t, \tilde{c}_t, \sigma_t, \tilde{h}_t, h_t$ are deterministic matrices (or vectors) of dimensions $d \times 1, d \times d, d \times m, d \times d, m \times 1, m \times d$, respectively.

Theorem B.0.1 (Kalman-Bucy filtering)**.** The filtered process $\hat{X}_t = \mathbb{E}\left(X_t|\mathcal{F}_t^Y\right)$ can be expressed as below:

$$\hat{X}_t = \hat{X}_0 + \int_0^t (\tilde{b}_s + b_s \hat{X}_s)\mathrm{d}s + \int_0^t (c_s + \gamma_s h_s^\top)\mathrm{d}\nu_s, \tag{B.0.2}$$

where $\nu_t = Y_t - \int_0^t (\hat{h}_s + h_s \hat{X}_s)\mathrm{d}s$, is a d-dimensional Brownian motion adapted to

$(\mathcal{F}_t^Y)_{t\geq 0}$, and γ is the covariance matrix for X_t and $\hat{X}_t$, i.e.

$$\gamma_t^{ij} = \mathbb{E}(X_t^i X_t^j) - \mathbb{E}(\hat{X}_t^i \hat{X}_t^j),$$

satisfying the following matrix Riccati Equation:

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = \gamma_t b_t^\top + b_t \gamma_t + c_t c_t^\top + \sigma_t \sigma_t^\top - (c_t + \gamma_t h_t^\top)(c_t + \gamma_t h_t^\top)^\top. \tag{B.0.3}$$

This is a direction application of Kushner-FKK equation to Gaussian linear system. All of the proofs can be found in chapter 9 of Xiong (2008).

REFERENCES

B. Angoshtari. On the market-neutrality of optimal pairs-trading strategies. *ArXiv e-prints*, 2016.

B. Angoshtari and T. Leung. Optimal dynamic basis trading. *Annals of Finance*, 15(3):307–335, 2019.

B. Angoshtari and T. Leung. Optimal trading of a basket of futures contracts. *Annals of Finance*, 16(2):253–280, 2020.

V. E. Beneš. Existence of optimal stochastic control laws. *SIAM Journal on Control*, 9(3):446–472, 1971.

Michael Boguslavsky and Elena Boguslavskaya. Arbitrage under power. *RISK magazine*, pp.6973, 2004.

Fred Espen Benth and Kenneth Hvistendahl Karlsen. A Note on Merton's Portfolio Selection Problem for the Schwartz Mean-Reversion Model. *Stochastic Analysis and Applications*, 23(4):687–704, 2007.

Simon Brendle. Portfolio selection under incomplete information. *Stochastic Processes and their Applications*, 116(5):701–723, 2006.

Simon Brendle. Portfolio selection under incomplete information. *Applied Mathematics and Optimization*, 58:257-274, 2008.

D.P. Bertsekas and S.E. Shreve. Stochastic Optimal Control : The Discrete Time Case. *Mathematics in Science and Engineering* 139, Academic Press, 1978.

Michael J Brennan and Eduardo S Schwartz. Optimal arbitrage strategies under basis variability. In M. Sarnat, editor, *Essays in Financial Economics*. North Holland, 1988.

Michael J. Brennan and Eduardo S. Schwartz. Arbitrage in stock index futures. *Journal of Business*, 63(1):S7–S31, 1990.

Roy Cerqueti. Dynamic Programming via Measurable Selection. *Pacific Journal of Optimization*, 2009.

Min Dai, Yifei Zhong, and Yue Kuen Kwok. Optimal arbitrage strategies on stock index futures under position limits. *Journal of Futures Markets*, 31(4):394–406, 2011.

I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*. Springer-Verlag, 1991.

F. Klebaner and R. Liptser. *When a Stochastic Exponential is a True Martingale. Extension of a Method of Beneš*. Theory of Probability and Its Applications 58(1), 2011.

Kazutaka Kuroda and Hideo Nagai. Risk-sensitive portfolio optimization on infinite time horizon. *Stochastics and Stochastics Reports*, 73:309–331, 2002.

Peter Lakner. Utility maximization with partial information. *Stochastic Processes and their Applications*, 56(2):247–273, 1995.

Peter Lakner. Optimal trading strategy for an investor: the case of partial information. *Stochastic Processes and their Applications*, 76(1):77–97, 1998.

Sangmin Lee and Andrew Papanicolaou. Pairs Trading of Two Assets with Uncertainty in Co-integrations Level of Mean Reversion. *International Journal of Theoretical and Applied Finance*, 19(8), 2016.

Tim Leung and Brian Ward. Dynamic index tracking and risk exposure control using derivatives. *Applied Mathematical Finance*, 25(2):180–212, 2018.

Tim Leung and Raphael Yan. Optimal dynamic pairs trading of futures under a two-factor mean-reverting model. *International Journal of Financial Engineering*, 5(3):1850027, 2018.

Tim Leung and Raphael Yan. A stochastic control approach to managed futures portfolios. *International Journal of Financial Engineering*, 6(1):1950005, 2019.

Tim Leung and Yang Zhou. Dynamic optimal futures portfolio in a regime-switching market framework. *Internation Journal of Financial Engineering*, 6(4):1950034, 2019.

T. N. Li and Andrew Papanicolaou. Dynamic optimal portfolios for multiple co-integrated assets. *Working paper*, 08 2019.

J. Liu and F. A. Longstaff. Losing money on arbitrage: Optimal dynamic portfolio choice in markets with arbitrage opportunities. *The Review of Financial Studies*, 17(3):611–641, 2004.

Liu, J. and Timmermann, A. (2013). Optimal convergence trade strategies. *Review of Financial Studies*, 26(4):1048–1086.

Matt Lorig, Zhou Zhou and Bin Zou. A Mathematical Analysis of Technical Analysis. *Applied Mathematical Finance*,26(1):38-69, 2019.

Supakorn Mudchanatongsuk, James A. Primbs and Wilfred Wong. Optimal Pairs Trading: A Stochastic Control Approach. *2008 American Control Conference*.

Merton, R. Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, 03:373–413, 1971.

Huyên Pham. Continuous-time Stochastic Control and Optimization with Financial Applications. *Springer*, 2009.

W.T.Reid. Riccati Differential Equations. *Academic Press*, 1972.

Ralph S.J.Koijen, Juan Carlos Rodriguez and Alessandro Sbuelz. Momentum and Mean-Reversion in Strategic Asset Allocation. *EFA 2006 Zurich Meetings*, 2009.

H. Mete Soner and Nizar Touzi. Dynamic Programming for Stochastic Target Problems and Geometric Flows. *Journal of the European Mathematical Society*, 2003.

Agnes Tourin and Raphael Yan. Dynamic pairs trading using the stochastic control approach. *Journal of Economic Dynamics and Control*, 37:1947–2156, 2013.

Nizar Touzi. Option Stochastic Control, Stochastic Target Problems, and Backward SDE. *Springer*, 2013.

J. Xiong. An Introduction to Stochastic Filtering Theory. *Oxford Graduate Texts in Mathematics, 18. Oxford University Press*, 2008.

Z. Zhao and D. P. Palomar. Mean-reverting portfolio with budget constraint. *IEEE Transactions on Signal Processing*, 66(9):2342–2357, 2018.