

THE UNIVERSITY OF CHICAGO

ENDOGENOUSLY DECLINING LIQUIDITY AND ROLLOVER-RISK SPILLOVERS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE SOCIAL SCIENCES
DEPARTMENT OF ECONOMICS
AND
THE FACULTY OF THE UNIVERSITY OF CHICAGO
BOOTH SCHOOL OF BUSINESS
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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CHICAGO, ILLINOIS

JUNE 2017

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ACKNOWLEDGMENTS

I would like to express my sincere appreciation to my dissertation committee co-chairs, Professor Fernando Alvarez and Professor Zhiguo He, for their endless support and encouragement of my Ph.D. studies. Their comments and suggestions on my work helped me improve the quality of my dissertation substantially.

My deep gratitude also goes to my other committee members, Professor Lars Peter Hansen and Professor Stavros Panageas, to whom I am indebted for their continuous guidance and support throughout my graduate years. Writing this dissertation would not be possible without their help.

Lastly, I would like to thank my parents and brother for their patience and support throughout my life.

ABSTRACT

This paper studies a short-term debt market in which rollover risks spread across heterogeneous banks because a liquidation price of assets declines over time endogenously. To this aim, the paper uses a general equilibrium approach to analyze interactions between the primary and secondary debt markets. Specifically, in this economy, rollover decisions of creditors and takeover-timing decisions of potential buyers of assets determine the supply and demand for failed assets, respectively. A liquidation price clears the market in equilibrium. In fact, the paper assumes potential buyers have different asset-management abilities and highly skilled buyers are scarce. Thus, a less and less skilled buyer becomes a marginal buyer as time goes by, pushing down the liquidation price. This liquidation-driven pecuniary externality is what propagates rollover risks across different banks. The model further implies injecting emergency funds to only a tiny number of banks may trigger earlier runs on those banks. Extending debt maturities causes a similar consequence. The paper develops a new method to characterize transition dynamics of equilibrium analytically.

CHAPTER 1

INTRODUCTION

A broad consensus is deterioration in the US mortgage market initially provoked the 2007-2009 financial crisis. But, many economists argue what caused the subsequent collapse of the entire US short-term debt market was the spread of disruptions in the housing market to other financial assets. Specifically, Gorton and Metrick (2012) document the repurchase agreement (repo) market experienced systemic runs (or withdrawals) during the crisis, although most of securitized bonds used as repo collaterals had no direct connections to the US housing market. Covitz, Liang, and Suarez (2013) also find the concerns about the burst in the mortgage market shrank the total volume of asset-backed commercial papers (ABCP), backed by either mortgage or non-mortgage types of assets, by \$350 billion in the second half of 2007.

This empirical evidence emphasizes an importance of understanding (i) why rollover risks of financial firms (or banks) issuing collateralized short-term debts are propagated to other banks, (ii) how we can mitigate this rollover-risk spillover effect to prevent the meltdown of the banking system, and (iii) if there are some commonly advocated policies that unintentionally exacerbate financial contagion. To answer these questions, this paper analyzes interactions between creditors in the primary debt market and potential buyers of failed assets (or collaterals) in the secondary market. Moreover, we study these interactions especially in a dynamic setup to examine intertemporal effects of financial contagion.

Specifically, the paper uses a general equilibrium approach to develop a debt-run model with heterogeneous banks in which rollover risks spread across banks because a liquidation price of assets declines over time endogenously. On one hand, rollover decisions of creditors in the primary market determine the quantity of supply of failed assets. On the other hand, asset-takeover decisions of potential buyers in the secondary market determine the quantity of demand for failed assets. As usual, those creditors and potential buyers take the liquidation price of assets as given. The liquidation price is then pinned down so as to clear

the demand and supply of failed assets.

Importantly, as emphasized by Shleifer and Vishny (1992), we assume (i) potential buyers have different asset-management abilities and (ii) each buyer has a limited capacity in purchasing liquidated assets because of some financial constraints. In this circumstance, each potential buyer chooses an optimal time to buy failed assets. As expected, highly skilled buyers choose to purchase assets earlier, while unskilled buyers wait until the liquidation price drops to an attractive level. The demand for failed assets at each point of time is then determined by this takeover-timing decision of potential buyers.

Regarding the supply side, we adopt the model of He and Xiong (2012, hereafter, HX) to develop a dynamic coordination game between creditors within each individual bank. Specifically, each bank invests in a certain asset by issuing short-term debts, collateralized by the asset itself, to many small creditors. Each bank's asset is exposed to only idiosyncratic shocks and creditors are not overlapped across banks. Each creditor has an option to withdraw or roll over her funding whenever her debt claim matures. The maturities of different debts are staggered over time, causing coordination failures among creditors. If any bank fails to repay its debt outstanding, it is forced to liquidate its asset. The supply of failed assets at each point of time is then determined by the aggressiveness of rollover strategies of creditors.

In equilibrium, the liquidation price declines endogenously over time.¹ Specifically, recall highly skilled buyers optimally choose to purchase assets earlier than unskilled buyers and then exit the market; as a result, a less and less skilled buyer becomes a marginal buyer as time goes by, pushing down the liquidation price. This price impact on the liquidation value causes a feedback effect to the creditors in the primary market and spreads rollover risks of distressed banks to other healthy banks. That is, if some banks are hit by a negative shock at time 0 and so have troubles refinancing their debts, the creditors of other banks also choose

1. More precisely, the aggregate component of the liquidation price declines over time (to be discussed later).

to run more aggressively by anticipating the future drop in the liquidation price. Those creditors actually respond to the local negative shock immediately even before the distressed banks go bankrupt because, as mentioned, the creditors foresee the liquidation price will drop. These earlier withdrawals then depress the liquidation price further and reinforce the feedback effect. Bernardo and Welch (2004) also study a similar type of a feedback effect in a two-period setup; but they consider a stock market, not a short-term debt market.

This paper develops a new method to characterize transition dynamics of equilibrium analytically. The main difficulty we encounter is our economy may exhibit strategic substitutability as well as strategic complementarity because a liquidation price declines over time. We again emphasize strategic substitutability arises in our model because of the dynamic feature of the liquidation price. Indeed, suppose some creditors of any single bank, say, bank i , choose to withdraw their funding more aggressively. This change in their rollover strategies reduces the expected lifespan of bank i , without affecting the future liquidation price because any single bank is infinitesimally small. Interestingly, the reduced lifespan of the bank can either harm or benefit its creditors. On one hand, the bank's creditors may not receive the promised payments fully because the bank is expected to default earlier. Thus, more aggressive withdrawals of some creditors can increase the withdrawal incentives of other creditors in bank i .

On the other hand, if bank i fails earlier, its creditors can sell off their collateral before the (aggregate) liquidation price drops in the future. As a result, earlier withdrawals of some creditors can actually benefit other creditors and so reduce their incentives to run. If this benefit dominates the above-mentioned loss, the economy exhibits (local) strategic substitutability. The presence of these two opposite economic forces generally makes the equilibrium characterization hard; see Goldstein and Pauzner (2005).² Moreover, we need to pin down an entire transition path of equilibrium because a time-varying liquidation price

2. In Goldstein and Pauzner (2005), strategic substitutability arises because if so many depositors run, further withdrawals dilute the payments per each withdrawing creditor, which is different from the economic force driving strategic substitutability in our model.

makes our economy non-stationary. Nevertheless, the present paper shows our economy has at least one equilibrium by using an iterative elimination of dominated strategies from only one side. Unfortunately, whether this economy obtains a unique equilibrium is unknown at the moment.³

The paper contributes to the bank-run literature as follows. Most of papers in this literature focus on coordination failure problems within a single bank and do not discuss systemic runs in depth. For instance, see Diamond and Dybvig (1983), Rochet and Vives (2004), Goldstein and Pauzner (2005), Cheng and Milbradt (2012), He and Xiong (2012), Schroth, Suarez, and Taylor (2014), He and Manela (2016). Of course, there are some papers that study financial contagion in a banking sector. But, those papers analyze the interactions between banks through their direct capital linkages (Allen and Gale (2000), Lagunoff and Schreft (2001), Dasgupta (2004), Liu (2016)), overlapped risk-averse creditors (Goldstein and Pauzner (2004)), and fears about common information (Chen (1999), Oh (2013)). None of these papers study the effect of asset liquidation in the secondary market, even though Altman, et al. (2005) and Benmelech and Bergman (2011) provide empirical evidence for the contagion effect of collateral liquidation. In fact, Choi (2014), Ahnert (2016) and Eisenbach (2016) study some interactions between the primary and secondary markets. But those papers use a reduced-form demand function for failed assets for simplicity and focus on other important economic issues. The present paper endogenizes both primary and secondary markets by using a general equilibrium approach. Uhlig (2010) studies systemic runs by assuming outside buyers are either ambiguity-averse or less informed. But, his model assumes each bank has a representative creditor to abstract from a within-bank coordination problem.

Another body of the literature studying the macroeconomic effects of fire sales and collateral constraints includes, for instance, Shleifer and Vishny (1992), Kiyotaki and Moore

3. In general, an economy exhibiting a strong feedback effect tends to have multiple equilibria; see Angeletos and Werning (2006) and Liu (2016). But, unfortunately, specifying precise conditions under which multiple equilibria indeed arise seems hard for our model.

(1997), Gromb and Vayanos (2002), Brunnermeier and Pedersen (2009), and Caballero and Simsek (2010). But, the main interests of those papers are capital structure, investment, asset price dynamics, or portfolio choice, and so on. Instead, our model applies the basic concept of inefficient fire sales in this literature to the bank-run framework.

This paper is also related to Carlin, Lobo, and Viswanathan (2007), He and Milbradt (2014), and Oehmke (2014) in the sense all those papers study how trading frictions in the secondary market affect banks or collateral sellers. Lastly, this paper makes a methodological contribution to the global-game literature by characterizing transition dynamics of equilibrium of the economy which may not exhibit (global) strategic complementarity; see Carlsson and van Damme (1993), Morris and Shin (1998), Frankel and Pauzner (2000), and Angeletos, Hellwig, and Pavan (2007), who pioneered this literature.

The paper is organized as follows. Chapter 2 develops the model. Chapter 3 characterizes an equilibrium. Chapter 3.6 discusses model implications. Chapter 4 concludes. All technical proofs are included in the Appendix.

CHAPTER 2

THE MODEL

There is a continuum of banks of measure 1, indexed by $i \in [0, 1]$, in the primary debt market. Each bank has an investment opportunity in one unit of a long-term asset. No single investor with a deep pocket exists in this economy; thus, the owner of each bank finances the bank's asset by (i) issuing short-term debts to a measure 1 of its own creditors and (ii) injecting her own money to some extent as well. In addition, the secondary market consists of potential buyers of assets with different asset-management skill levels.

All agents are risk neutral and have a discount rate of ρ . All information is publicly observable. There is only one type of consumption goods, simply called cash. Time is continuous.

2.1 Primary Debt Market

2.1.1 Assets

All the assets are ex-ante identical. Each asset i (that is, bank i 's asset) produces constant cash flows (or dividends) of cdt at each moment of time. Each asset i matures at a random time that arrives according to a Poisson process with intensity ϕ , independently across any other assets. If asset i matures, say, at time $\tau(i, \phi)$, it produces a lump-sum final payoff $\exp(u_{\tau(i, \phi)}^i)$, where u_t^i indicates the time- t value of the final payoff. We hereafter call u_t^i the fundamental of bank i at time t . Moreover, u_t^i evolves according to

$$du_t^i = \nu dt + \sigma dZ_t,$$

where ν measures the drift term, $\sigma > 0$ measures the volatility, and Z_t^i is a standard Brownian motion, independent across the assets. Each bank exits the market once its asset matures.

In this setup, the unlevered (or first-best) value of asset i is given by

$$\begin{aligned} F(u_t^i) &= E_t \left[\int_t^\infty \phi e^{-\phi(s-t)} \left(\int_t^s e^{-\rho(r-t)} c dr + e^{-\rho(s-t)} e^{u_s^i} \right) ds \right] \\ &= \frac{c}{\rho + \phi} + \frac{\phi e^{u_t^i}}{\rho + \phi - \nu - \frac{\sigma^2}{2}}. \end{aligned}$$

The first (second) term in the right-hand side indicates the present value of the interim cash flows (final output).

We use $m(t, u)$ to denote a cross-sectional distribution of the banks at time t . That is,

$$m(t, u) = \#\{i : u_t^i \in [u, u + du)\}.$$

An initial distribution of the banks is arbitrarily given as $m_0(u)$. We discuss later how $m(t, u)$ evolves over time.

2.1.2 Debt Contracts

Each bank issues short-term debts to its creditors of measure 1 with equal seniority. Different banks do not share their creditors, although, technically, this assumption is not necessary because creditors are assumed to be risk neutral. Each debt of any individual bank matures at a random time with Poisson intensity λ , independently of any other debts within and across banks. Accordingly, a fraction λdt of the debts outstanding of each bank mature at each moment of time. Each debt pays a coupon payment $c dt$ at each moment of time and a lump-sum face value P at the debt-maturity date. Note that each bank transfers all of its cash dividends to its creditors as the coupon payments. That is, each bank does not hold any cash reserves during the interim periods. We make this assumption to abstract from the bank's cash management problem.

Each creditor of bank i can choose whether to withdraw or roll over her funding at her debt-maturity date. If she withdraws (or runs), she receives the face value P unless bank

i defaults. If she rolls over, however, she receives nothing today, but her debt contract is renewed under the same contract terms; thus she can keep receiving the coupons until the next maturity date (unless the bank defaults prematurely).

Lastly, if bank i 's asset itself matures at time $\tau(i, \phi)$, the bank pays the final payoff up to the face value, $\min\{\kappa^i \exp(u_{\tau(i, \phi)}^i), \kappa^i P\}$, to its creditors equally and then leaves the market.

2.1.3 Debt Repayment and Default

Recall a fraction of λdt of the creditors of each bank i reach their maturity dates at each moment of time. Suppose a fraction ξ_t^i of those maturing creditors choose to run at each date t , where $\xi_t^i \in [0, 1]$. (In fact, we later assume all creditors in the economy use a symmetric rollover strategy, so ξ_t^i will be either 0 or 1.) But, because bank i does not hold any cash reserves, the bank relies on its parent company to repay $\xi_t^i \lambda P dt$ dollars. (We may regard the parent company as a credit-line provider, government, or insurer as well.) That is, the parent company commits to provide emergency funds $\xi_t^i \lambda P dt$ to the bank at time t , whenever possible. But, the parent company does not have an infinite amount of cash, either. To model this scarcity of emergency funds in a simplest possible way, we assume each bank i fails to find emergency cash with an instantaneous probability $\theta \xi_t^i \lambda P dt$ at time t . Formally speaking, bank i defaults with a non-stationary Poisson intensity $\theta \xi_t^i \lambda P$, which arrives independently of other banks. Note that the default probability is proportional to the amount of money to be paid.

If any bank defaults on its debt repayments, it is forced to liquidate its entire asset immediately in the secondary market. The bank then distributes the liquidation proceeds (up to the face value) to all of its creditors equally and exits the market. We discuss later how the liquidation price of failed assets is determined. Meanwhile, if the bank successfully defends any withdrawing creditor, it immediately sells the same debt contract to a new creditor at the fair debt price. This assumption releases us from tracking the number of the remaining creditors of the bank. We discuss the fair debt price later.

2.2 Secondary Debt Market

Recall any defaulted bank has to sell off its asset immediately to potential buyers in the secondary market. To produce an inelastic demand for assets in a tractable way, we assume (i) potential buyers have different asset-management skill levels and (ii) each potential buyer can purchase an asset of only one bank because of some financial or non-financial constraints. Also, we assume each potential buyer finances a failed asset by issuing equity only.

Specifically, there is a continuum of potential buyers, indexed by $j \in [0, \infty)$, whose skill levels range from 0 to $\bar{\gamma}$. We use $q(\gamma)$ to denote the measure potential buyers whose skill levels are larger than γ as shown in Figure 2.1. For simplicity, we assume $q(\gamma)$ is continuous and strictly decreasing (we can relax these conditions without any difficulty). We then use $\gamma(q)$ to indicate the inverse function of $q(\gamma)$. In particular, we will use $\gamma(q) = \bar{\gamma}(1 + q)^{-\eta}$ for some $0 < \eta < 1$ when calibrating the model.

Importantly, whenever each potential buyer j with skill level $\gamma^j \in [0, \bar{\gamma}]$ purchases asset i at time t , she needs to pay

$$(1 - \alpha)F(u_t^i) + \bar{\gamma} - \gamma^j \tag{2.1}$$

as liquidation costs, where $\alpha \in (0, 1)$ is an exogenous constant. The first term measures a proportional liquidation cost and the second term, $\bar{\gamma} - \gamma^j$, measures a buyer-specific fixed cost. We can interpret these costs as a loss of customers, restructuring charges, legal fees, agency costs, and so on. Thus, by purchasing asset i at time t at some price, buyer j is expected to earn $\alpha F(u_t^i) - \bar{\gamma} + \gamma^j$ during her future lifetime. Each buyer is in fact allowed to resell her asset after purchasing it, but we show shortly doing so is not optimal.

Also, note that the first term in (2.1) is proportional to the unlevered value of asset i , $F(u_t^i)$, because each buyer finances an asset by issuing equity only. In fact, the parameter α does not play any significant role in generating contagion; but we introduce this parameter to make our model consistent with the benchmark model of HX (2012). That is, if $\gamma^j = \bar{\gamma}$

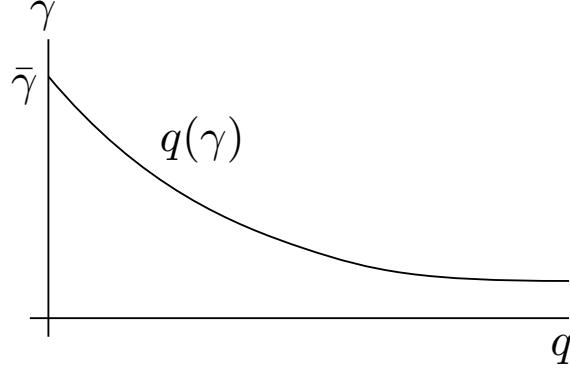


Figure 2.1: A skill-level distribution function $q(\gamma)$.

for each j , our model reduces to HX (2012).¹

The secondary market behaves competitively; that is, all market participants are price takers. In particular, we postulate each asset i is traded at time t in the secondary market at the price of

$$\alpha F(u_t^i) - \bar{\gamma} + p_t, \quad (2.2)$$

where p_t indicates an aggregate component of the liquidation price. Postulating this form of the price is reasonable because each potential buyer with skill level γ^j is expected to earn $\alpha F(u_t^i) - \bar{\gamma} + \gamma^j$ by purchasing asset i at time t . For simplicity, we do not consider other forms of a liquidation price. Also, without loss of generality, we can assume p_t is deterministic, differentiable, and less than $\bar{\gamma}$ because this property will hold in equilibrium. Lastly, we will call p_t the liquidation price rather than the aggregate component of the liquidation price, if there is no serious confusion.

The expected net profit accrued to potential buyer j is then given by

$$\alpha F(u_t^i) - \bar{\gamma} + \gamma^j - \left(\alpha F(u_t^i) - \bar{\gamma} + p_t \right) = \gamma^j - p_t.$$

Here, we have used the fact each buyer has no incentive to resell her asset after purchasing it,

1. The reason we introduce a buyer-specific liquidation cost, $\bar{\gamma} + \gamma^j$, in an additive form rather than in a multiplicative form is to disentangle the buyer-specific liquidation cost from the idiosyncratic value of each asset, $F(u_t^i)$. Doing so simplifies our analysis substantially without losing any related economic insights.

because for each time $s \geq t$, the expected discounted value of holding the asset forever from time s , $F(u_s^i)$, is larger than the selling price of the asset, $\alpha F(u_s^i) - \bar{\gamma} + p_s$. Importantly, note that the above net profit does not depend on the idiosyncratic component of the liquidation price, $\alpha F(u_t^i)$. This fact means each potential buyer only needs to make a decision about *when* to buy an asset, not about *which* asset to buy. This result is important because numerous banks of different fundamentals will be liquidated at each point of time in our economy (to be discussed). In this regard, by saying the secondary market clears at each time, we mean the number of failed assets coincides with the number of potential buyers willing to buy some assets at each time. Lastly, recall each bank sells its asset only when it fails to repay debts outstanding.

2.3 Individual Creditor's Problem

To describe an individual creditor's problem, we first need to specify admissible rollover strategies of creditors. For tractability, this paper focuses on symmetric time-dependent threshold strategy. That is, we assume there is a time-dependent deterministic function, z_t , such that each creditor of bank i chooses to run at her debt-maturity date τ if and only if u_τ^i lies below z_τ ; see Figure 2.2. Without loss of generality, we assume z_t is continuous in t and has a finite limit as t goes to ∞ , because this property will hold in equilibrium. In addition, we particularly use $z_t \equiv \infty$ to indicate a rollover strategy under which each creditor unconditionally chooses to withdraw her funding at her first maturity date. Similarly, we use $z_t \equiv -\infty$ to indicate a rollover strategy under which each creditor unconditionally chooses to roll over her funding at each of her maturity date.

The definition of this rollover strategy implies a fraction λdt of the creditors of each bank i choose to run at each time t if and only if u_t^i lies below z_t . In other words, we have $\xi_t^i = 1_{z_t < u_t^i}$, implying each bank i defaults with non-stationary Poisson intensity $\theta \lambda P 1_{u_t^i < z_t}$.

The debt value of bank i at time t , denoted by $V(t, u_t^i)$, is defined as the expected discounted value of the future payoffs accrued to each creditor of bank i from time t . Each

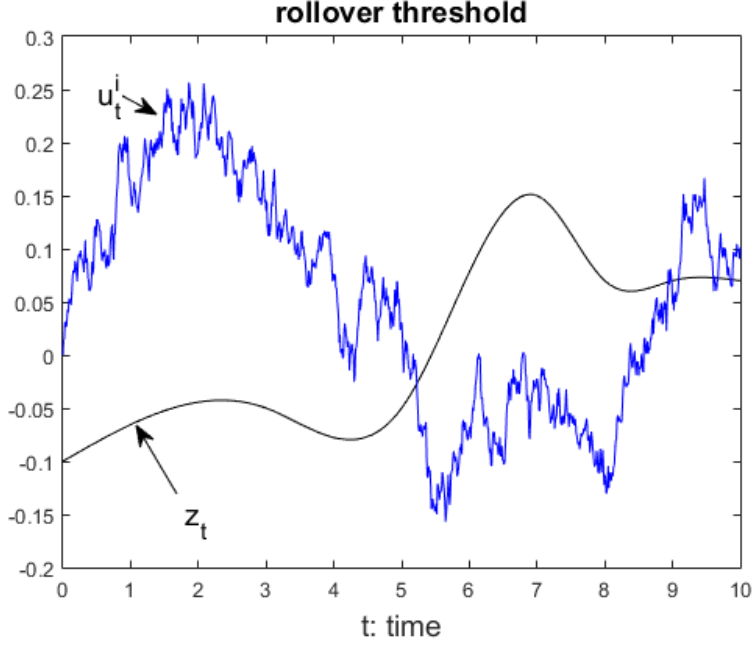


Figure 2.2: A rollover strategy $\{z_t\}$.

creditor takes the liquidation price path $\{p_t\}$ and the rollover strategy of other creditors $\{z_t\}$ as given. In this setting, the debt value satisfies the following recursive equation:

$$V(t, u_t^i) = cdt + \phi dt \min\{e^{u_t^i}, P\} + \theta \lambda P 1_{u_t^i < z_t} dt \min\{\alpha F(u) - \bar{\gamma} + p_t, P\} + \\ \lambda dt \max_{\text{run, roll over}} \{P, V(t, u_t^i)\} + (1 - \phi dt - \theta \lambda P 1_{u_t^i < z_t} dt - \lambda dt) e^{-\rho dt} E_t[V(t + dt, u_{t+dt}^i)].$$

The first term in the right-hand side denotes the coupon payment; the second term denotes the final payoff; the third term denotes the liquidation proceeds; the fourth term denotes the rollover option value; the fifth term denotes the expected discounted value of the future payoffs. Before we proceed, we discuss the fourth term more carefully. If any time- t maturing creditor of bank i chooses to run, her expected payoff is equal to

$$P(1 - \theta \lambda P 1_{u_t^i < z_t} dt) + \min\{\alpha F(u_t^i) - \bar{\gamma} + p_t, P\} \theta \lambda P 1_{u_t^i < z_t} dt,$$

where we have used the fact this individual creditor's rollover decision does not affect the

bank's default probability. Meanwhile, if she decides to roll over her funding, her expected payoff is equal to

$$V(t, u_t^i)(1 - \theta \lambda P 1_{u_t^i < z_t} dt) + \min\{\alpha_t F(t, u_t^i), P\} \theta \lambda P 1_{u_t^i < z_t} dt.$$

Thus, multiplying the above two terms by λdt and eliminating dt^2 -order terms, the rollover option value reduces to $\max\{P, V(t, u_t^i)\}$.

Now, by using Ito's lemma, we show $V(t, u)$ satisfies the following HJB equation:

$$0 = c + \phi(\min\{e^u, P\} - V(t, u)) + \theta \lambda P 1_{u < z_t} (\min\{\alpha F(u) - \bar{\gamma} + p_t, P\} - V(t, u)) + \lambda \max_{\text{run, roll}} \{P - V(t, u), 0\} - \rho V(t, u) + \nu V_u(t, u) + \frac{\sigma^2}{2} V_{uu}(t, u) + V_t(t, u). \quad (2.3)$$

Appendix 5.1 shows the existence and uniqueness of a solution to this equation.

2.4 Individual Potential Buyer's Problem

A potential buyer's problem is relatively simple. Each potential buyer chooses an optimal time τ to buy an asset, given any liquidation price path $\{p_t\}$. That is, suppose some individual buyer of skill level γ (the index j is omitted) has not yet purchased any asset until time t . Then, her profit maximization problem at time t is given by

$$\max_{\tau \geq t} e^{-\rho(\tau-t)} (\gamma - p_\tau).$$

The necessary FOC for this problem is given by²

$$\frac{dp_s}{ds} \Big|_{s=\tau} - \rho(\gamma - p_\tau). \quad (2.4)$$

2. If $\gamma < \min_{t \geq 0} p_t$, the solution is given by $\tau = \infty$. That is, a potential buyer, whose skill level is lower than the minimum possible liquidation price, will never purchase an asset. Also, the solution to this problem is time consistent.

By rewriting this condition as

$$\gamma - p_\tau = \frac{1}{1 + \rho dt}(\gamma - p_{\tau+dt}),$$

we can say this buyer has no incentive to deviate from her purchasing time τ . We show later the optimal purchasing time τ satisfying FOC (2.4) is indeed optimal when $\{p_t\}$ is given by an equilibrium price.

We now show a more skilled buyer chooses to buy an asset earlier than a less skilled buyer, given any liquidation price path. Indeed, suppose τ_1 (resp. τ_2) is an optimal purchasing time for a buyer of skill level γ_1 (resp. γ_2), where $\gamma_1 > \gamma_2$ and $\tau_1 > \tau_2$. This assumption means

$$e^{-\rho\tau_1}(\gamma_1 - p_{\tau_1}) \geq e^{-\rho\tau_2}(\gamma_1 - p_{\tau_2}) \quad \text{and} \quad e^{-\rho\tau_2}(\gamma_2 - p_{\tau_2}) \geq e^{-\rho\tau_1}(\gamma_2 - p_{\tau_1}),$$

which implies $e^{-\rho(\tau_1-\tau_2)}(\gamma_1 - \gamma_2) \geq \gamma_1 - \gamma_2$, a contradiction. We will use this fact when characterizing an equilibrium.

2.5 Aggregate Default Rate

Recall $m(t, u)$ denotes the cross-sectional distribution of banks. Then, given any rollover strategy $\{z_t\}$, the default rate of banks (or the supply flow of failed banks) at each time t is given by

$$\frac{dq_t}{dt} = \theta\lambda P \int_{-\infty}^{z_t} m(t, u) du. \quad (2.5)$$

So, to pin down the default rate, we need to discuss how $m(t, u)$ evolves. Using the fact that (i) a fraction ϕdt of the existing banks exit the market at each time and (ii) each bank i defaults with Poisson intensity $\theta\lambda P 1_{u_t^i < z_t}$, the Kolmogorov forward equation implies $m(t, u)$ satisfies

$$m_t(t, u) = -\phi m(t, u) - \theta\lambda P 1_{u < z_t} m(t, u) - \nu m_u(t, u) + \frac{\sigma^2}{2} m_{uu}(t, u). \quad (2.6)$$

We will sometime use $m(t, u; z)$ to denote the dependence of $m(t, u)$ on the strategy $\{z_t\}$. By the way, $m(t, u)$ vanishes in the long run for each u , meaning that we need to interpret our model as a short-run behavior of a crisis even though we consider an infinite-horizon model. Introducing newly born banks and a temporary price impact would be an interesting future research topic.

2.6 Equilibrium

Definition 2.6.1. *An equilibrium in this economy is defined as a pair of a rollover strategy and a liquidation price, $\{z_t^*, p_t^*\}$, such that (i) the strategy $\{z_t^*\}$ itself is individually optimal for each creditor of any single bank who takes $\{z_t^*, p_t^*\}$ as given and (ii) the supply and demand for failed assets coincide with each other, where each potential buyer takes $\{p_t^*\}$ as given.*

In fact, condition (i) can be written as

$$V(t, u^i; z^*, p^*) \begin{cases} < P, & \text{if } u^i < z_t^* \\ = P, & \text{if } u^i = z_t^* \\ > P, & \text{if } u^i > z_t^*, \end{cases} \quad (2.7)$$

which means for each i , each creditor of bank i , whose debt matures at time t , has an incentive to run if and only if the bank's current fundamental u_t^i is larger than z_t^* itself.

CHAPTER 3

MODEL ANALYSIS

In this section, we solve for an equilibrium of the economy under some parameter restrictions. We begin with characterizing a partial equilibrium in the secondary market because analyzing the secondary market is a bit easier than analyzing the primary market.

3.1 Parameter Restrictions

We impose the following restrictions on the model parameters:

$$\nu + \sigma^2/2 < \rho + \phi, \tag{3.1}$$

$$\rho < c/P < \rho + \phi, \tag{3.2}$$

$$\alpha < \bar{\alpha} \text{ for some } \bar{\alpha}, \tag{3.3}$$

$$\bar{\gamma} < \frac{c + \lambda P}{\rho + \phi + \lambda}. \tag{3.4}$$

Condition (3.1) rules out the explosion of the first-best value of an asset. The first inequality in condition (3.2) assumes the coupon rate c/P is larger than the risk-free rate ρ to assure the debt value is larger than the face value P when the current fundamental is sufficiently large, regardless of the strategies of other creditors and the liquidation price. The second inequality in condition (3.2) assumes the coupon rate c/P is smaller than $\rho + \phi$ to ensure the debt value is lower than the face value P when the current fundamental is sufficiently small, regardless of the strategies of other creditors and the liquidation price. Lemma 3.3.1 verifies these two assertions. Condition (3.3) says the parameter α must be small enough so that asset liquidation becomes costly. We impose this condition to assure the existence of a lower dominance region; see Lemma 3.3.4 for details. Lastly, condition (3.4) is needed for the liquidation price $\alpha F(u) - \bar{\gamma} + p_t$ to remain positive for any (t, u) .

3.2 Partial Equilibrium in the Secondary Market

We have already seen given any rollover strategy $\{z_t\}$, the cumulative supply of failed banks is given by

$$q_t = \theta \lambda P \int_0^t \int_{-\infty}^{z_s} m(s, u) ds.$$

We also recall more skilled buyers choose to buy assets earlier than less skilled buyers for any given liquidation price path $\{p_t\}$. Then, because one buyer is allowed to buy only one bank's asset, for the demand and supply of failed assets to equal each other, the skill level of a marginal buyer at time t , denoted by γ_t , must be given by

$$\gamma_t = \gamma(q_t), \quad \forall t.$$

From FOC (2.4), the equilibrium price p_t must then satisfy

$$\frac{dp_t}{dt} = -\rho(\gamma_t - p_t), \quad \forall t. \quad (3.5)$$

We can actually rewrite this equation as follows:

$$\underbrace{\gamma_t - p_t}_{\text{premium for today's buyer}} = \underbrace{\gamma_t - \gamma_{t+dt}}_{\text{skill-level difference}} + (1 - \rho dt) \underbrace{\gamma_{t+dt} - p_{t+dt}}_{\text{premium for tomorrow's buyer}}.$$

That is, the premium for the waiting option of today's buyer, $\gamma_t - p_t$, equals the skill-level difference between today's and tomorrow's buyers plus the discounted premium for the waiting option of tomorrow's buyer. The solution to equation (3.5) is given by

$$p_t = \gamma_t + \int_t^{\infty} e^{-\rho(s-t)} \frac{d\gamma_s}{ds} ds. \quad (3.6)$$

Here, we have used the fact $p_t \leq \gamma_t$ for each t , which implies $e^{-\rho t} p_t = 0$, to pin down the initial price p_0 . If this fact does not hold at some time t , the demand cannot meet the

supply at that time. Also, formula (3.6) implies p_t converges to γ_t as t goes ∞ , meaning the premium for the

Lastly, we show a purchasing time t is indeed optimal for the potential buyer of skill level $\gamma = \gamma_t$. Recall this buyer solves the following problem at date 0:

$$\max_s e^{-\rho s}(\gamma_t - p_s).$$

The first-order derivative of this net-profit function satisfies

$$e^{-\rho s}(-\rho(\gamma_t - p_s) - \frac{dp_s}{ds}) = \rho e^{-\rho s}(\gamma_s - \gamma_t) \begin{cases} \geq 0, & \forall s \leq t \\ \leq 0, & \forall s \geq t, \end{cases}$$

because γ_s is decreasing in s , which verifies the desired optimality. We omit to check the time consistency because checking it is easy.

3.3 Properties of the Debt-Value Function

Before we show the existence of equilibrium, we discuss some important properties of the debt-value function. The main difficulty in characterizing an equilibrium is our economy may exhibit (local) strategic substitutability as well as strategic complementarity within individual banks. To see why, given any declining liquidation price p_t , let us analyze the interactions between creditors within some bank i . Specifically, imagine all creditors of bank i , except one particular creditor (say, creditor j) choose to run more aggressively and thus trigger earlier default of bank i . Interestingly, a shortened lifespan of the bank can either harm or benefit creditor j .

On the one hand, creditor j will receive the coupon payments for a shorter period because the bank is expected to fail earlier. On the other hand, creditor j can liquidate the bank's collateral earlier before the liquidation price p_t drops further in the future. Put differently, more aggressive withdrawals of other creditors make creditor j exit from her locked-in debt

contract before the bank enters into a distressed aggregate economic condition. In this regard, our economy exhibits strategic substitutability if this benefit of earlier redemptions of other creditors indeed dominates the above-mentioned loss. The following lemmas help us characterize an equilibrium in the presence of these two opposite economic forces.

Lemma 3.3.1. *For any pair of a rollover strategy and liquidation price, $\{z_t, p_t\}$, we have*

$$\lim_{u \rightarrow -\infty} V(t, u; z, p) < P < \lim_{u \rightarrow \infty} V(t, u; z, p), \quad \forall t. \quad (3.7)$$

Proof. See Appendix 5.2. □

This lemma says if the parameter restrictions imposed in Section 3.1 hold, the debt value crosses the face value P at some interior point $u \in \mathbb{R}$ so that we can rule out uninteresting cases in which either $z_t \equiv -\infty$ or $z_t \equiv \infty$ becomes an equilibrium strategy. The next lemma discusses when (local) strategic complementarity arises.

Lemma 3.3.2. *For any two pairs of a rollover strategy and liquidation-price path, $\{z_t^a, p_t^a\}$ and $\{z_t^b, p_t^b\}$, such that $z_t^a \leq z_t^b$ and $p_t^a \geq p_t^b$ for each t , suppose one of the following conditions holds: either*

$$V(t, u; z^b, p^b) \geq \min\{\alpha F(u) - \bar{\gamma} + p_t^b, P\}, \quad \forall t, u \quad \text{such that} \quad z_t^a \leq u \leq z_t^b \quad (3.8)$$

or

$$V(t, u; z^a, p^a) \geq \min\{\alpha F(u) - \bar{\gamma} + p_t^a, P\}, \quad \forall t, u \quad \text{such that} \quad z_t^a \leq u \leq z_t^b. \quad (3.9)$$

Then, we have

$$V(t, u; z^a, p^a) \geq V(t, u; z^b, p^b), \quad \forall t, u. \quad (3.10)$$

To clarify, either $z_t^a \equiv -\infty$ or $z_t^b \equiv \infty$ is allowed above.

Proof. See Appendix 5.3. □

We can understand this lemma as follows. First, without loss of generality, we can assume $p_t^a = p_t^b$ for each t because the depressed liquidation price from $\{p_t^a\}$ to $\{p_t^b\}$ merely lowers the debt value. We now imagine all creditors of some bank i increase their rollover strategy from $\{z_t^a\}$ to $\{z_t^b\}$, given the liquidation-price path $\{p_t^a\}$. Then, the instantaneous default probability of bank i increases from 0 to $\theta\lambda P dt$ at time t , whenever the bank's fundamental u_t^i lies between z_t^a and z_t^b . A direct negative effect of this increased chance of default is given by a foregone continuation value of debt, $V(t, u; z^b, p^b)$. Meanwhile, a direct positive effect of the heightened default chance is given by the immediate liquidation value of asset i , $\min\{\alpha F(u) - \bar{\gamma} + p_t^b, P\}$. Thus, if the negative effect dominates the positive effect as in (3.8), the debt value is lowered as in (3.10), meaning (local) strategic complementarity obtains.¹ We can similarly understand condition (3.9) by considering another case in which all creditors decrease their strategy from $\{z_t^b\}$ to $\{z_t^a\}$.

The next lemma describes when the debt value strictly increases in the bank's fundamental.

Lemma 3.3.3. *For any $\{z_t, p_t\}$ and constant $\epsilon > 0$, suppose*

$$V(t, u; z, p) \geq \min\{\alpha F(u) - \bar{\gamma} + p_t, P\}, \quad \forall t, u \text{ such that } z_t \leq u \leq z_t + \epsilon. \quad (3.11)$$

Then, $V(t, u; z, p)$ is strictly increasing in u . To clarify, if $z_t \equiv -\infty$ or $z_t \equiv \infty$, assumption (3.11) is vacuous, meaning both $V(t, u; z \equiv -\infty, p)$ and $V(t, u; z \equiv \infty, p)$ are strictly increasing in u .

Proof. See Appendix 5.4. □

We explain this lemma as follows. Without loss of generality, suppose the initial fundamental of some bank i increases from u_0^i to $u_0^i + \delta$, where $0 < \delta < \epsilon$, given the liquidation-

1. Formally, strategic complementarity is defined in terms of utility differential. In our model, the utility differential is given by $V(t, u; z, p) - P$, where the first term corresponds to the continuation value accrued to a rolling-over creditor and the second term corresponds to the face value paid to a withdrawing creditor. But, because the second term is a constant, to see if strategic complementarity holds, we only need to check if $V(t, u; z^a, p) \geq V(t, u; z^b, p)$ when $z_t^a \leq z_t^b$ for each t .

price path $\{p_t\}$. Then, any realized path of the fundamental, u_t^i , also increases to $u_t^i + \delta$. Thus, the instantaneous default probability decreases from $\theta\lambda P dt$ to 0 at time t , whenever $z_t \leq u_t^i + \delta \leq z_t + \delta$. A direct positive effect of this decreased default is given by the continuation value of debt, $V(t, u_t^i + \delta; z, p)$. Meanwhile, a direct negative effect of the reduced default chance is given by the immediate liquidation value of asset i , $\min\{\alpha F(u_t^i + \delta) - \bar{\gamma} + p_t, P\}$. As a result, if the positive effect dominates the negative effect as in (3.11), the debt value $V(t, u; z, p)$ increases in u . In fact, the increased fundamental not only reduces the default probability but also increases the final debt payoff $\{\exp(u_t^i), P\}$. Moreover, because $\{\exp(u), P\}$ strictly increases in u over a non-trivial interval and u_t^i travels over that interval for a non-trivial amount of time, the debt value must also strictly increase in u .

Lastly, we derive the following lemma, which implies there exists a so-called lower dominance region if α is sufficiently small. Here, we use $V(u; z \equiv -\infty)$ to denote the debt value of any individual bank given that the creditors of this bank never choose to run. We can use this simplified notation because the debt value in this case does not depend on the liquidation price $\{p_t\}$. We also use $V(t, u; z, p, \alpha)$ to explicitly indicate the dependence of the debt value on the parameter α .

Lemma 3.3.4. *There is some constant $\bar{\alpha} \in (0, 1)$ such that*

$$V(u; z \equiv -\infty) \geq \min\{\bar{\alpha}F(u), P\}, \quad \forall u. \quad (3.12)$$

Further, we can find a constant $\underline{z} \in \mathbb{R}$ such that for any $\alpha \in (0, \bar{\alpha})$ and $\{z_t, p_t\}$,

$$V(t, u; z, p, \alpha) < P, \quad \forall t, u \quad \text{such that} \quad u < \underline{z}. \quad (3.13)$$

Proof. See Appendix 5.5. □

In fact, finding $\bar{\alpha}$ that satisfies condition (3.12) is straightforward because the left-hand

side in (3.12) does not depend on $\bar{\alpha}$.² We can also show the second assertion via Lemmas 3.3.1 and 3.3.3 together with condition (3.12). Importantly, this lemma says each creditor can rationally eliminate any candidate strategy $\{z_t\}$ such that $z_t < \underline{z}$ for some t , because $V(t, u; z, p) < P$ if $u < \underline{z}$ for any $\{z_t, p_t\}$. In other words, $[0, \infty) \times (-\infty, \underline{z})$ is a lower dominance region in our economy.

3.4 Partial Equilibrium in the Primary Market

In this section, we show a unique partial equilibrium exists in the primary market, given any liquidation-price path $\{p_t\}$. Because all the banks are ex-ante identical, we can focus on analyzing a partial equilibrium within any single bank, say, bank i . Also, if there is no confusion, we use the term “equilibrium” instead of “partial equilibrium”.

3.4.1 Existence

This section shows at least one equilibrium exists in the primary market. The basic idea is as follows. We first consider the largest rollover strategy $z_t^1 \equiv \infty$ and then construct a sequence of strategies $\{z_t^1, z_t^2, \dots\}$ that satisfies (i) $z_t^{n+1} \leq z_t^n$ for each (n, t) and (ii) for each n ,

$$V(t, u; z_t^n, p) \begin{cases} < P & \text{if } u < z_t^{n+1} \\ = P & \text{if } u = z_t^{n+1} \\ > P & \text{if } u > z_t^{n+1}. \end{cases} \quad (3.14)$$

Here, inequality (3.15) implies the strategy $\{z_t^{n+1}\}$ is individually optimal for each creditor of bank i if all other creditors within the bank use $\{z_t^n\}$ as their own strategy. Put another way, each creditor is indifferent between withdrawing and rolling over her funding if the bank’s current fundamental u_t^i is equal to z_t^{n+1} at time t , given all other creditors use $\{z_t^n\}$ as their own strategy. In this regard, we call $\{z_t^{n+1}\}$ the best response to $\{z_t^n\}$. But, because Lemma

2. Although the paper does not provide an explicit form of $\bar{\alpha}$, we can easily check numerically if condition (3.12) holds for any $\bar{\alpha}$.

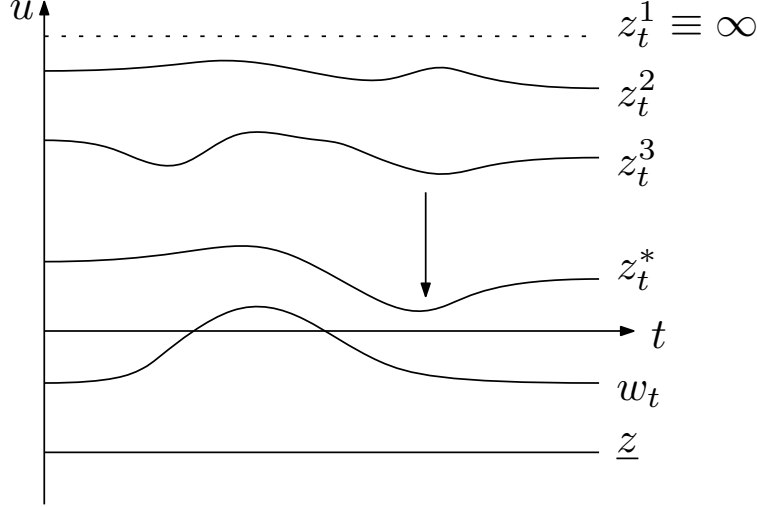


Figure 3.1: Illustration of the proof of the equilibrium existence.

3.3.4 implies $z_t^n \geq \underline{z}$ for each (n, t) , the decreasing sequence $\{z_t^1, z_t^2, \dots\}$ must converge to some limit z_t^* for each t . The strategy $\{z_t^*\}$ must be then an equilibrium because this limiting strategy is a fixed point of the best response. Figure 3.1 illustrates the above arguments and the proof of Theorem 3.4.1 includes the details.

Theorem 3.4.1. *For any $\{p_t\}$, the primary market obtains at least one equilibrium.*

Proof. See Appendix 5.6. □

3.4.2 Uniqueness

This section shows the uniqueness of equilibrium in the primary market. To this aim, we first discuss the following lemma, implying if the liquidation price $\{p_t\}$ is exogenously given and thus no feedback effect arises, the coordination game within a single bank exhibits weak strategic complementarity.

Lemma 3.4.2. *For any $\epsilon > 0$ and $\{z_t, p_t\}$, we have*

$$V(t, u + \epsilon; z + \epsilon, p) > V(t, u; z, p), \quad \forall t, u,$$

where $z + \epsilon$ denotes the rollover strategy $\{z_t + \epsilon\}$.

Proof. See Appendix 5.7. □

The underlying idea behind this lemma is as follows. Suppose (i) all the creditors of bank i has increased their strategy from $\{z_t\}$ to $\{z_t + \epsilon\}$ and (ii) the initial fundamental of bank i has increased from u_0^i to $u_0^i + \epsilon$ as well. Then, the default probability of bank i at any point of time in the future remains unchanged, because the relative position of any realized fundamental path $\{u_t^i\}$ to the threshold curve $\{z_t\}$ remains the same if both u_t^i and z_t have increased by the same amount of ϵ for each t . However, the increment in u_t^i raises the final debt payoff as well as the idiosyncratic component of the liquidation value of the bank. As a result, the debt value $V(t, u; z, p)$ must strictly increase when both u_t and z_t increase by the same amount.

We now show the equilibrium uniqueness by using a so-called translation-invariance method, which is similar to the arguments in Frankel and Pauzner (2000) and Frankel, Pauzner, and Morris (2003). By way of contradiction, suppose there is another equilibrium $\{w_t\}$. Then, we have already seen $z \leq w_t \leq z_t^*$ for each t in the previous section. We now shift the curve $\{w_t\}$ upward by $\epsilon > 0$, where ϵ is the smallest number such that $w_t + \epsilon \geq z_t^*$ for each t , as in Figure 3.2.³ Let (t^*, u^*) denote the point where the curve $\{w_t + \epsilon\}$ touches the other curve $\{z_t^*\}$. Then, applying Lemma 3.3.2 to $z_t^a = z_t^*$ and $z_t^b = w_t + \epsilon$, we have

$$V(t^*, u^*; z^*) \geq V(t^*, u^*; w + \epsilon).$$

However, Lemma 3.4.2 shows

$$V(t^*, u^* - \epsilon^*; w) < V(t^*, u^*; w + \epsilon).$$

But these two inequalities cannot hold simultaneously because $V(t^*, u^*; z^*) = P = V(t^*, u^* - \epsilon^*; w)$ by the definition of equilibrium. Hence, a unique equilibrium obtains.

3. The proof of Theorem 3.4.3 handles a special case in which such a smallest number ϵ is not attained.

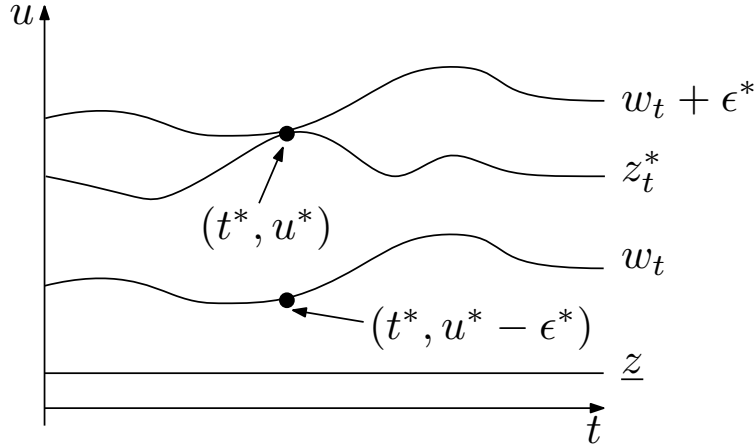


Figure 3.2: Illustration of the proof of the equilibrium uniqueness.

Theorem 3.4.3. *Given any $\{p_t\}$, the primary market has a unique partial equilibrium.*

Proof. See Appendix 5.8. □

3.5 Equilibrium of the Whole Market

This section aggregates both primary and secondary markets and then shows the whole economy obtains at least one equilibrium. Whether another equilibrium exists is unknown at the moment. In fact, if a certain economy exhibits a strong reinforcing feedback effect, that economy tends to have multiple equilibria; see Angeletos and Werning (2006) and Liu (2016) for instance. In our economy, when creditors choose to run more aggressively, they not only increase the default probability of their own banks but also push down the liquidation price, which generates a feedback effect to the primary market. In this regard, our economy may have multiple equilibria. But, as mentioned above, whether multiple equilibria indeed emerge is uncertain at this moment.

To show the existence of equilibrium, recall given any rollover strategy $\{z_t\}$, we can calculate the default rate dq_t/dt at each point of time. Given this supply flow of failed assets, a unique (partial equilibrium) liquidation price $\{p_t\}$ arises in the secondary market. We hereafter use $p(t; z)$ to denote the dependence of the liquidation price on the strategy $\{z_t\}$. We also use $V(t, u; z)$ to indicate $V(t, u; z, p(\cdot; z))$ for notational convenience. In this regard,

our goal is to find an equilibrium strategy $\{z_t^*\}$ such that $V(t, u; z^*)$ satisfies inequality (2.7) with $p(t; z^*)$ in place of p_t .

The basic idea of the proof is very similar to the proof argument in Section 3.4.1. We begin with the largest rollover strategy $z_t^1 \equiv \infty$ and then construct a sequence of strategies $\{z_t^1, z_t^2, \dots\}$ that satisfies (i) $z_t^{n+1} \leq z_t^n$ for each (n, t) and (ii) for each n ,

$$V(t, u; z^n) \begin{cases} < P & \text{if } u < z_t^{n+1} \\ = P & \text{if } u = z_t^{n+1} \\ > P & \text{if } u > z_t^{n+1}. \end{cases} \quad (3.15)$$

Of course, the sequence $\{z_t^1, z_t^2, \dots\}$ is not the same as the sequence of the strategies obtained in Section 3.4.1, because the liquidation price $p(t; z^n)$ is now endogenously determined. In fact, we can show $p(t; z^{n+1}) \geq p(t; z^n)$ for each (t, n) , which makes sense because a more aggressive strategy pushes down the liquidation price more. We can then similarly show the decreasing sequence $\{z_t^1, z_t^2, \dots\}$ must converge to some limit z_t^* for each t and this limit $\{z_t^*\}$ must be an equilibrium of the whole economy.

Lastly, even though the paper is silent about the uniqueness result, we can show if there is another equilibrium $\{w_t\}$, then $w_t \leq z_t^*$ for each t . In this regard, we can call $\{z_t^*\}$ the upper extremal equilibrium. But, unfortunately, whether a lower extremal equilibrium exists is not known at the moment, either, because the above proof argument does not seem to work well if we start with $z_t \equiv -\infty$ for some reason.

Theorem 3.5.1. *Our economy obtains at least one equilibrium, $\{z_t^*\}$, described above. If there is another equilibrium w_t , then $w_t \leq z_t^*$ for each t .*

Proof. See Appendix 5.9. □

3.6 Policy Implications

This section discusses policy implications of the model. After the demise of Lehman Brothers in 2008, the Federal Reserve (Fed) employed several unconventional monetary policy tools in addition to using the interest-rate control policy. In particular, liquidity provision to non-depository banks have received a lot of attention; see Shleifer and Vishny (2011) and Fleming (2012). In this regard, evaluating the effectiveness of this policy would be meaningful. We also discuss the implications of a debt-maturity extension policy.

Our model implies if the government provides emergency funding to only a tiny number of banks, such a subsidy can actually trigger earlier runs on those banks. To see why, imagine the government injects some emergency funding to only a single bank, say, bank i , to deter the bankruptcy of this bank. More specifically, we decrease the parameter θ for bank i only, given the liquidation-price path $\{p_t\}$. Surprisingly, this liquidity injection can either make bank i 's creditors either better or worse off because the liquidation price p_t declines over time. The underlying intuition for this phenomenon is similar to the reason why our economy may exhibit strategic substitutability. By providing emergency cash to bank i , the government can extend the lifespan of the bank but cannot affect the liquidation price, because bank i is infinitesimally small. Then, as mentioned above, the extended lifespan of the bank can either benefit or harm its creditors. Specifically, the bank's creditors can receive the coupons for a longer period; but if the bank happens to default in a far future, the creditors will receive a very low price for their collaterals. As a result, the creditors are willing to withdraw their funding earlier if the latter effect dominates the former effect.

To avoid this unintended negative consequence, the government can either directly inject liquidity to the secondary market or provide emergency funds to a large number of banks. Both of the policies can increase the liquidation price at each point of time in the future. Specifically, regarding the latter policy, if a sizable number of banks receive the emergency funds, the aggregate default rate will decrease and thus the liquidation price will be pushed up. If this positive effect outweighs the negative effect described above, the government can

effectively mitigate a banking crisis.

In addition, extending debt maturities of only a tiny number of banks can similarly exacerbate a coordination problem within those banks, because such a policy merely deters the defaults of those banks without improving their fundamentals or the aggregate liquidation price. In the literature, HX (2012) also find the negative consequences of injection of emergency funds and maturity extensions. Such an outcome arises in their model when the fundamental volatility is high. In our model, that counterintuitive outcome occurs because the liquidation price declines over time.

CHAPTER 4

CONCLUSION

This paper develops a dynamic debt-run model with heterogeneous banks to study a rollover-risk spillover effect driven by an endogenously declining liquidation price. To this end, the paper adopts a general equilibrium approach to study how the primary debt market interacts with the secondary market. In this economy, specifically, rollover strategies of creditors determine the supply of failed assets, whereas asset-acquisition decisions of potential buyers in the secondary market pin down the demand for those assets. A liquidation price is then pinned down as a market-clearing price. Indeed, potential buyers are assumed to have different asset-management skill levels and highly efficient buyers are scarce. As a result, a less and less efficient buyer becomes a marginal buyer over time, causing a price impact on the liquidation price. Rollover risks propagate to other banks because of this liquidation-driven pecuniary externality. The paper further shows injecting emergency liquidity to or extending debt maturities of only a tiny number of banks may cause earlier withdrawals on those banks. The paper invents a new method to show the existence of equilibrium analytically.

CHAPTER 5

APPENDIX

Throughout this section, we use $L(t, u; p) = \alpha F(u) - \bar{\gamma} + p_t$ to avoid the abuse of notation. We also use $V(t, u)$ to denote $V(t, u; z, p)$ if there is no confusion.

5.1 Proof of the Existence and Uniqueness of a Debt-Value Function

We consider the following general HJB equation:

$$0 = f(t, u) + \lambda \max\{P - J(t, u), 0\} - d(t, u)J(t, u) + \nu J_u(t, u) + \frac{\sigma^2}{2} J_{uu}(t, u) + J_t(t, u), \quad (5.1)$$

for some measurable functions $f(t, u)$ and $d(t, u)$ such that

$$|f(t, u)| \leq Ae^u + B, \quad \forall t, u, \quad \text{and} \quad \rho + \phi \leq d(t, u), \quad \forall t, u,$$

where $A, B \geq 0$ are some constants. We then look for a solution to this equation satisfying the following growth condition:

$$|u(t, u)| \leq Ce^u + D, \quad (5.2)$$

for some constants $C, D \geq 0$. Note that $f(t, u)$ and $c(t, u)$ need not be continuous functions; thus HJB equation (2.3) can be considered as a special example of the above general HJB equation.

In literature, Crandall, Kocan, and Świech (2000, hereafter CKŚ) show the existence of a unique solution for HJB equations with discontinuous data except for the second-order coefficient. (In fact, Wang (1992) also considers a similar problem but under a less general setup.) But because CKŚ (2000) consider only a bounded domain, we need to extend their result by using a so-called penalization method; see Theorem 6 in Section 2.3 in Evans

(2010).

To begin with, we will use the viscosity solution concept for HJB equation (5.1); see Section 1 in CKŚ (2000) for its definition. Also, the existence of a solution to HJB equation (5.1) directly comes from the Feynman-Kac formula for that equation, which obviously satisfies the above growth condition; see Fleming and Soner (2006). To show the uniqueness, we use the following lemma, a so-called maximum principle.

Lemma 5.1.1. *Suppose $J(t, u)$ is a viscosity subsolution to the following HJB equation:*

$$\mathcal{L}J(t, u) := \lambda \max\{-J(t, u), 0\} - d(t, u)J(t, u) + \nu J_u(t, u) + \frac{\sigma^2}{2} J_{uu}(t, u) + J_t(t, u) = 0. \quad (5.3)$$

Or equivalently, we can say

$$\mathcal{L}J(t, u) \geq 0.$$

We also suppose that $J(t, u)$ satisfies the growth condition (5.2). Then, we have $J(t, u) \leq 0$ for each (t, u) .

Proof. For any $\epsilon > 0$, define

$$K(t, u) = J(t, u) - \epsilon(e^{a_1 u} + e^{a_2 u} + e^{(\rho+\phi)t}),$$

where

$$a_1 = \frac{-\nu + \sqrt{\nu^2 + 2(\rho + \phi)\sigma^2}}{\sigma^2} \quad \text{and} \quad a_2 = \frac{-\nu - \sqrt{\nu^2 + 2(\rho + \phi)\sigma^2}}{\sigma^2}.$$

Note that a_1 and a_2 are the solutions to

$$\frac{\sigma^2}{2} a^2 + \nu a - (\rho + \phi) = 0.$$

Here it is easy to check $a_1 > 1$ and $a_2 < 0$ using condition (3.1). Then, it is also straightfor-

ward to see

$$\mathcal{L}K(t, u) \geq \mathcal{L}J(t, u) + (d(t, u) - \rho - \phi)\epsilon(e^{a_1 u} + e^{a_2 u} + e^{(\rho+\phi)t}) \geq 0, \quad \forall t, u. \quad (5.4)$$

We now consider a bounded domain $Q(R) = \{(t, u) : 0 \leq t \leq 2R/(\rho + \phi), |u| \leq R\}$ for some $R > 0$. The parabolic boundary of $Q(R)$ is defined as $\partial Q(R) = \{(t, u) \in Q(R) : |u| = R \text{ or } t = 2R/(\rho + \phi)\}$. Then, by using the growth condition for $J(t, u)$ and the fact that $a_1 > 1$ and $a_2 < 0$, we can show there exists some $R^* > 0$ (which depends on ϵ) such that for each $R > R^*$,

$$K(t, u) \leq 0, \quad \forall (t, u) \in \partial Q(R). \quad (5.5)$$

This inequality combined with inequality (5.4) then implies that for each $R > R^*$,

$$K(t, u) \leq 0, \quad \forall (t, u) \in Q(R),$$

due to Proposition 2.6 in CKŚ (2000). Letting $\epsilon \rightarrow 0$, we conclude $J(t, u) \leq 0, \forall (t, u) \in (0, \infty) \times \mathbb{R}$. \square

We now use this lemma to show the uniqueness of a solution to HJB equation (5.1).

Theorem 5.1.2. *HJB equation (5.1) has a unique viscosity solution satisfying the growth condition (5.2).*

Proof. Suppose $V(t, u)$ and $W(t, u)$ are two solutions to HJB equation (5.1), both of which satisfy the growth condition. Then, using the fact that $\mathcal{L}V(t, u) = \mathcal{L}W(t, u) = 0$ and

$$\max\{P - V(t, u), 0\} - \max\{P - W(t, u), 0\} \leq \max\{W(t, u) - V(t, u), 0\},$$

we have $\mathcal{L}(V - W)(t, u) \geq 0$. Then, because the function $V - W$ also satisfies the growth condition, Lemma 5.1.1 implies $V(t, u) \leq W(t, u)$ for each (t, u) . Switching the roles of V

and W , we conclude $V(t, u) \geq W(t, u)$ for each (t, u) as well. \square

5.2 Proof of Lemma 3.3.1

Proof. We first consider the case in which $-\infty < z_t < \infty$ for each t . We then suppose u is sufficiently small, that is, $u \approx -\infty$. Then, the fact

$$F(-\infty) = \frac{c}{\rho + \phi} < P$$

implies the debt value $V(t, -\infty)$ satisfies the following first-order HJB equation:

$$0 = c + \theta\lambda PL(t, -\infty; p) + \lambda \max\{P - V, 0\} - (\rho + \phi + \theta\lambda P)V + V_t. \quad (5.6)$$

But, observe

$$J(t) := \int_t^\infty e^{-(\rho + \phi + \theta\lambda P + \lambda)(s-t)} (c + \theta\lambda PL(s, -\infty; p) + \lambda P) ds \leq \frac{c + \theta\lambda P \frac{\alpha c}{\rho + \phi} + \lambda P}{\rho + \phi + \theta\lambda P + \lambda} < P,$$

where we have used $c/P < \rho + \phi$. But then, we can easily see $V(t, -\infty; z, p) = J(t)$ solves HJB equation (5.6), which verifies $V(t, -\infty) < P$.

Now we suppose u is sufficiently large, that is, $u \approx \infty$. In this case, the debt value $V(t, \infty)$ satisfies

$$0 = c + \phi P + \lambda \max\{P - V, 0\} - (\rho + \phi)V + V_t. \quad (5.7)$$

We can then show $V(t, \infty) > P$ by observing the solution to this equation is given by

$$V(t, \infty) \equiv \frac{c + \phi P}{\rho + \phi} > P, \quad (5.8)$$

where we have used the fact $c/P > \rho$.

We now consider an extreme rollover strategy given by $z_t \equiv -\infty$. First, when $u \approx -\infty$,

by removing the term $\theta\lambda P(L(t, \infty; p) - V)$ from equation (5.6), we can easily see

$$V(t, -\infty) \equiv \frac{c + \lambda P}{\rho + \phi + \lambda} < P.$$

Next, when $u \approx \infty$, the debt value $V(t, \infty)$ satisfies the equation the same as (5.7) and thus, $V(t, \infty) > P$ as we have already shown.

Lastly, we consider the other extreme rollover strategy given by $z_t \equiv \infty$. When $u \approx -\infty$, the debt value $V(t, -\infty)$ satisfies the equation the same as (5.6) and thus, $V(t, -\infty) < P$ as we have already shown. When $u \approx \infty$, we note that $\min\{\alpha F(\infty) - \bar{\gamma} + p_t, P\} = P$. Thus, the debt value $V(t, \infty)$ satisfies the following HJB equation:

$$0 = c + \phi P + \theta\lambda P^2 + \lambda \max\{P - V, 0\} - (\rho + \phi + \theta\lambda)V + V_t.$$

We can then show $V(t, \infty) > P$ by observing the solution to this equation is given by

$$V(t, \infty) \equiv \frac{c + \phi P + \theta\lambda P^2}{\rho + \phi + \theta\lambda P} > P,$$

where we have used the fact $c/P > \rho$. □

5.3 Proof of Lemma 3.3.2

Proof. We first consider the case in which condition (3.8) holds. By subtracting HJB equation (2.3) corresponding to z_t^a from the same HJB equation but corresponding to z_t^b , we have

$$\begin{aligned} 0 = & \theta\lambda P 1_{u < z_t^a} (\min\{L(t, u; p^b), P\} - \min\{L(t, u; p^a), P\}) + \\ & \theta\lambda P 1_{z_t^a \leq u < z_t^b} (\min\{L(t, u; p^b), P\} - V^b) + \lambda \max\{P - V^b, 0\} - \lambda \max\{P - V^a, 0\} - \\ & (\rho + \phi + \theta\lambda P 1_{u < z_t^a})G + \nu G_u + \frac{\sigma^a}{2} G_{uu} + G_t, \end{aligned}$$

where $V^b = V(t, u; z^b, p^b)$, $V^a = V(t, u; z^a, p^a)$, and $G(t, u) = V^b(t, u) - V^a(t, u)$. But using the fact

$$\max\{P - V^b, 0\} - \max\{P - V^a, 0\} \leq \max\{-V^b + V^a, 0\},$$

together with $L(t, u; p^b) \leq L(t, u; p^a)$ and condition (3.8), we have

$$0 \leq \lambda \max\{-G, 0\} - (\rho + \phi + \theta \lambda P 1_{u < z_t^a})G + \nu G_u + \frac{\sigma^a}{2} G_{uu} + G_t.$$

Thus, from Lemma 5.1.1, we conclude $G(t, u) \leq 0$ for each (t, u) .

Next, we assume condition (3.9) holds. Then, similarly as above, we have

$$\begin{aligned} 0 = & \theta \lambda P 1_{u < z_t^b} (\min\{L(t, u; p^b), P\} - \min\{L(t, u; p^a), P\}) + \\ & \theta \lambda P 1_{z_t^a \leq u < z_t^b} (\min\{L(t, u; p^a), P\} - V^a) + \lambda \max\{P - V^b, 0\} - \lambda \max\{P - V^a, 0\} - \\ & (\rho + \phi + \theta \lambda P 1_{u < z_t^b})G + \nu G_u + \frac{\sigma^a}{2} G_{uu}, \end{aligned}$$

which again leads to $G(t, u) \leq 0$ for each (t, u) . □

5.4 Proof of Lemma 3.3.3

Proof. We first consider the case where $-\infty < z_t < \infty$ for each t . To show $V(t, u; z, p)$ is strictly increasing in u , it suffices to show for any δ such that $0 < \delta < \epsilon$,

$$V(t, u; z, p) < V(t, u + \delta; z, p), \quad \forall t, u.$$

Note $V^\delta(t, u) := V(t, u + \delta; z, p)$ satisfies

$$\begin{aligned} 0 = & c + \phi \min\{e^{u+\delta}, P\} + \theta \lambda P 1_{u < z_t - \delta} (\min\{L(t, u + \delta; p), P\} - V^\delta) + \lambda \max\{P - V^\delta, 0\} - \\ & (\rho + \phi)V^\delta + \nu V_u^\delta + \frac{\sigma^2}{2} V_{uu}^\delta + V_t^\delta. \end{aligned}$$

By subtracting HJB equation (2.3) corresponding to $V^\delta(t, u)$ from the same HJB equation but corresponding to $V(t, u; z, p)$, we have

$$\begin{aligned}
0 = & \phi \left(\min\{e^u, P\} - \min\{e^{u+\delta}, P\} \right) + \\
& \theta \lambda P 1_{u < z_t} \left(\min\{L(t, u; p), P\} - \min\{L(t, u + \delta; p), P\} \right) + \\
& \theta \lambda P 1_{z_t - \delta \leq u < z_t} \left(\min\{L(t, u + \delta; p), P\} - V^\delta \right) + \lambda \max\{P - V, 0\} - \lambda \max\{P - V^\delta, 0\} - \\
& (\rho + \phi + \theta \lambda P 1_{u < z_t})G + \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t,
\end{aligned}$$

where $G(t, u) = V(t, u; z, p) - V(t, u + \delta; z, p)$. Here, condition (3.11) implies

$$\min\{L(t, u + \delta; p), P\} - V^\delta(t, u) \leq 0, \quad \forall (t, u) \text{ such that } z_t - \delta < u < z_t.$$

Then, together with the fact

$$\max\{P - V, 0\} - \max\{P - V^\delta, 0\} \leq \max\{V^\delta - V, 0\},$$

we can derive

$$0 \leq f(u) + \lambda \max\{-G, 0\} - (\rho + \phi + \theta \lambda P 1_{u < z_t})G + \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t, \quad (5.9)$$

where $f(u) = \phi(\min\{e^u, P\} - \min\{e^{u+\delta}, P\})$. In fact, this inequality shows $G(t, u) \leq 0$ for each (t, u) . But, recall our goal is to show $G(t, u) < 0$ for each (t, u) . We show this claim by using the fact $f(u)$ is strictly negative over a non-trivial interval.

Specifically, there are some constant $\eta < 0$ and interval $(a, b) \subseteq \mathbb{R}$ such that

$$f(u) \leq \eta, \quad \forall u \in (a, b).$$

We then consider a function $W(t, u)$ that satisfies the following HJB equation:

$$0 = \eta 1_{a < u < b} + \lambda \max\{-W, 0\} - (\rho + \phi + \theta \lambda P 1_{u < z_t})W + \nu W_u + \frac{\sigma^2}{2} W_{uu} + W_t. \quad (5.10)$$

The solution to this equation is explicitly given by

$$W(t, u) = E_t \left[\int_t^\infty e^{-\int_t^s (\rho + \phi + \lambda + \theta \lambda P 1_{u_r < z_r}) dr} \eta 1_{u_s \in (a, b)} ds \right],$$

which is strictly less than 0 because almost every realized path u_t travels over the interval (a, b) for a non-trivial amount of time due to the assumption $\sigma > 0$. Then, using (5.9) and (5.10), we have

$$0 \leq \lambda \max\{-H(t, u), 0\} - (\rho + \phi + \theta \lambda P 1_{u < z_t})H(t, u) + \nu H_u + \frac{\sigma^2}{2} H_{uu} + H_t,$$

where $H(t, u) = G(t, u) - W(t, u)$. Lastly, we have $G(t, u) < 0$ for each (t, u) due to Lemma 5.1.1. □

5.5 Proof of Lemma 3.3.4

Proof. Lemma 3.3.3 shows the debt value $V(u; z \equiv -\infty)$ is strictly increasing in u . Lemma 3.3.1 then further implies there is a point \underline{z} such that

$$V(u; z \equiv -\infty) \begin{cases} < P & \text{if } u < \underline{z} \\ = P & \text{if } u = \underline{z} \\ > P & \text{if } u > \underline{z}. \end{cases} \quad (5.11)$$

We can then find $\bar{\alpha} \in (0, 1)$ such that

$$\bar{\alpha} F(\underline{z}) \leq V(-\infty; z \equiv -\infty) = \frac{c + \lambda P}{\rho + \phi + \lambda} < P.$$

But, because $F(u)$ is also increasing in u , inequality (3.12) holds.

Moreover, once condition (3.12) holds, we can easily see condition (3.9) Lemma 3.3.2 also holds for each $\alpha < \bar{\alpha}$, when $z_t^a \equiv -\infty$ and $z_t^b = z_t$. As a result, we have

$$V(u; z \equiv -\infty) \geq V(t, u; z, p), \quad \forall t, u.$$

Combining inequality (5.11), we have $V(t, u; z, p) < P$ whenever $u < \underline{z}$. □

5.6 Proof of Theorem 3.4.1

Proof. We start with the strategy $z_t^1 \equiv \infty$. Applying Lemmas 3.3.1 and 3.3.3 to $\{z_t^1\}$, we can obtain $\{z_t^2\}$ such that (i) $z_t^2 \leq z_t^1$ for each t and (ii)

$$V(t, u; z^1, p) \begin{cases} < P & \text{if } u < z_t^2 \\ = P & \text{if } u = z_t^2 \\ > P & \text{if } u > z_t^2. \end{cases} \quad (5.12)$$

These two properties imply condition (3.8) in Lemma 3.3.2 holds with $\{z_t^2\}$ and $\{z_t^1\}$ in place of $\{z_t^a\}$ and $\{z_t^b\}$, respectively. Thus, we have $V(t, u; z^2, p) \geq V(t, u; z^1, p)$ for each (t, u) . This fact then, combined with inequality (5.12), implies $V(t, u; z^2, p) \geq P$ for each (t, u) such that $z_t^2 \leq u$. Hence, by applying Lemma 3.3.3 to $\epsilon = 1$ and $z_t = z_t^2$, we can show $V(t, u; z^2, p)$ is strictly increasing in u .

Next, applying the above arguments to $\{z_t^2\}$ in place of $\{z_t^1\}$, we can find another $\{z_t^3\}$ such that (i) $z_t^3 \leq z_t^2$ for each t and (ii) inequality (3.14) holds with $n = 2$. Then, similarly as above, we can show condition (3.8) in Lemma 3.3.2 holds with $\{z_t^3\}$ and $\{z_t^2\}$ in place of $\{z_t^a\}$ and $\{z_t^b\}$, respectively, which implies $V(t, u; z^3, p) \geq V(t, u; z^2, p)$ for each (t, u) . This fact together with inequality (3.14) with $n = 2$ implies $V(t, u; z^3) \geq P$ for each (t, u) . Lemma 3.3.3 then shows $V(t, u; z^3, p)$ is strictly increasing in u .

Now, repeatedly applying these arguments to $\{z_t^3\}$ and so on, we can construct a sequence

of strategies $\{z_t^1, z_t^2, \dots\}$ such that (i) $z_t^{n+1} \leq z_t^n$ for each (n, t) and (ii) inequality (3.14) holds for each n . But, the decreasing sequence $\{z_t^1, z_t^2, \dots\}$ must converge to some limit z_t^* for each t because Corollary 3.3.4 assures $\underline{z} \leq z_t^n$ for each (n, t) . The strategy $\{z_t^*\}$ must be then an equilibrium because this limiting strategy is a fixed point of the best response.

In fact, to obtain a completely rigorous proof, we need to show

$$\lim_{n \rightarrow \infty} V(t, u; z^n, p) = V(t, u; z^*, p), \quad \forall t, u. \quad (5.13)$$

We prove this technical claim as follows. We let $W(t, u) = \lim_{n \rightarrow \infty} V(t, u; z^n, p)$ and suppose an arbitrary number $R > 0$ is given. Then, note that (i) $V(t, u; z^n, p)$ is bounded uniformly in n and (ii) all the data in HJB equation (2.3) corresponding to $\{z_t^n, p_t\}$ are bounded uniformly in n . As a result, assertion (i) in Theorem 9.2 in CKŚ (2000) implies $V(t, u; z^n, p)$ is Hölder continuous with exponent $1 + \eta$ uniformly in n . That is, there are some constants $0 < \eta < 1$ and $K > 0$ such that

$$|V(s, w; z^n, p) - V(t, u; z^n, p)| \leq K(|s - t|^{\frac{1+\eta}{2}} + |w - u|), \quad \forall (s, w) \times (t, u) \in Q(R) \times Q(R), \quad \forall n,$$

where K depends on R . Then, the Arzelà-Ascoli theorem implies the sequence $\{V(t, u; z^n, p)\}_n$ has a uniformly convergent subsequence in $Q(R)$. Obviously, the limit of this subsequence coincides with $W(t, u)$. But, recall the original sequence $\{V(t, u; z^n, p)\}_n$ monotonically converges to $W(t, u)$ for each (t, u) . The Dini theorem then implies $V(t, u; z^n, p)$ uniformly converges to $W(t, u)$ in $Q(R)$. We can then use Theorem 6.1 in CKŚ (2000) to show $W(t, u)$ satisfies HJB equation (2.3) with the rollover strategy $\{z_t^*\}$ in place of $\{z_t\}$ for $(t, u) \in Q(R)$. But, because we can choose R arbitrarily large, we can conclude $W(t, u) = V(t, u; z^*, p)$ for each $(t, u) \in [0, \infty) \times \mathbb{R}$. \square

5.7 Proof of Lemma 3.4.2

Proof. Let $V(t, u) = V(t, u; z, p)$, $V^\epsilon(t, u) = V(t, u + \epsilon; z + \epsilon, p)$, and $G(t, u) = V(t, u) - V^\epsilon(t, u)$. Then, $V^\epsilon(t, u)$ satisfies the following HJB equation:

$$0 = c + \phi \min\{e^{u+\epsilon}, P\} + \theta \lambda P 1_{u < z_t} (\min\{L(t, u + \epsilon; p), P\} - V^\epsilon) + \lambda \max\{P - V^\epsilon, 0\} - (\rho + \phi)V^\epsilon + \nu V_u^\epsilon + \frac{\sigma^2}{2} V_{uu}^\epsilon + V_t^\epsilon.$$

The function $G(t, u)$ then satisfies

$$0 = \phi(\min\{e^u, P\} - \min\{e^{u+\epsilon}, P\}) + \theta \lambda P 1_{u < z_t} (\min\{L(t, u; p), P\} - \min\{L(t, u + \epsilon; p), P\}) + \lambda \max\{P - V, 0\} - \lambda \max\{P - V^\epsilon, 0\} - (\rho + \phi + \theta \lambda P 1_{u < z_t})G + \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t.$$

But then, using the fact (i) $\min\{e^u, P\} \leq \min\{e^{u+\epsilon}, P\}$ with inequality strict over a non-trivial interval, (ii) $L(t, u; p) \leq L(t, u + \epsilon; p)$, and (iii) $\max\{P - V, 0\} - \max\{P - V^\epsilon, 0\} \leq \max\{-G, 0\}$, we can show

$$G(t, u) < 0, \quad \forall t, u,$$

as in the proof of Lemma 3.3.3. □

5.8 Proof of Theorem 3.4.3

Proof. Suppose there is another equilibrium $\{w_t\}$. Then, we have already seen $\underline{z} \leq w_t \leq z_t^*$ for each t . We now define

$$\epsilon = \inf\{\delta \geq 0 : w_t + \delta \geq z_t^*, \forall t\}.$$

We first show ϵ is finite. To this aim, we consider the debt-value function $V(t, u; z_t \equiv \infty, p_t \equiv \bar{\gamma})$. Then, there must be some constant \bar{z} such that

$$V(t, u; z_t \equiv \infty, p_t \equiv \bar{\gamma}) \begin{cases} < P & \text{if } u < \bar{z} \\ = P & \text{if } u = \bar{z} \\ > P & \text{if } u > \bar{z}. \end{cases}$$

Then, from Lemma 3.3.2, we have $z_t^n \leq \bar{z}$ for each (t, n) , where $\{z_t^n\}$ is the sequence of strategies, described in Section 3.4.1, converging to $\{z_t^*\}$. Thus, $z_t^* \leq \bar{z}$ for each t as well, implying ϵ is finite.

In a general case in which there is a point (t^*, u^*) at which $w_t + \epsilon$ indeed touches z_t^* , we have derived a desired contradiction in the body-text. We now consider a special case in which $w_t + \epsilon$ converges to z_t^* as t goes to ∞ without touching the curve $\{z_t^*\}$. To handle this case, we note that for any $\{z_t, p_t\}$, $\lim_{t \rightarrow \infty} V(t, u; z, p)$ solves the following stationary HJB equation:

$$0 = c + \phi(\min\{e^u, P\} - J(u)) + \theta \lambda P 1_{u < z_\infty} (\min\{\alpha F(u) - \bar{\gamma} + p_\infty, P\} - J(u)) + \lambda \max\{P - J(u), P\} - \rho J(u) + \nu J_u(u) + \frac{\sigma^2}{2} J_{uu}(u),$$

where $z_\infty = \lim_{t \rightarrow \infty} z_t$ and $p_\infty = \lim_{t \rightarrow \infty} p_t$. But, similarly as in Lemma 3.4.2, we can show

$$J(u + \epsilon; w_\infty + \epsilon, p_\infty) > J(u; w_\infty, p_\infty), \quad \forall u.$$

This result leads to a contradiction because

$$J(w_\infty; w_\infty, p_\infty) = \lim_{t \rightarrow \infty} V(t, w_\infty; w, p) = P = \lim_{t \rightarrow \infty} V(t, z_\infty^*; z^*, p) = J(w_\infty + \epsilon; w_\infty + \epsilon, p_\infty),$$

by the definition of equilibrium. □

5.9 Proof of Theorem 3.5.1

Proof. To prove this theorem, we only need to show $p(t; z) \geq p(t; w)$ for any pair of rollover strategies $\{z_t, w_t\}$ such that $z_t \leq w_t$ for each t . The reason is that using this property, we can proceed in the same way as in the proof of Theorem 3.4.1 by replacing p_t in that proof with either $p(t; z^n)$ or $p(t; z^*)$ appropriately.

To show this claim, we let $m^z(t, u) = m(t, u; z)$ and $m^w(t, u) = m(t, u; w)$. Subtracting equation (2.6) corresponding to $\{w_t\}$ from the same equation but corresponding to $\{z_t\}$, we have

$$n_t(t, u) = \theta\lambda P1_{z_t < u < w_t} m^w(t, u) - \phi n(t, u) - \theta\lambda P1_{u < z_t} n(t, u) - \nu n_u(t, u) + \sigma^2/2 n_{uu}(t, u),$$

where $n(t, u) = m^z(t, u) - m^w(t, u)$. Here, because the non-homogeneous term,

$$\theta\lambda P1_{z_t < u < w_t} m^w(t, u),$$

is positive, the probabilistic formula for $n(t, u)$ implies $n(t, u) \geq 0$ for each (t, u) . We then note the cumulative supply of failed assets under the strategies $\{z_t\}$ and $\{w_t\}$ are respectively given by

$$q(t; z) = 1 - \int_0^t \phi m^z(s, u) du \quad \text{and} \quad q(t; w) = 1 - \int_0^t \phi m^w(s, u) du,$$

which implies $q(t; z) \leq q(t; w)$ for each t . As a result,

$$p(t; z) = \rho \int_t^\infty e^{-\rho(s-t)} \gamma(q(s; z)) ds \geq \rho \int_t^\infty e^{-\rho(s-t)} \gamma(q(s; w)) ds = p(t; w), \quad \forall t,$$

because $\gamma(\cdot)$ is a decreasing function. □

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